It’s My Turn

Jill Clayburgh∗ Ryan O’Donnell†

September 19, 2013

Abstract

I popped a DVD into my DVD player and—thank goodness—the encoding on it was Region 1. *It’s My Turn* was the DVD. I fast-forwarded to the proof of the “Snake Lemma”... my favorite part of the whole movie...

Homological algebra

Figure 1 below shows the key diagram for the Snake Lemma.

![Diagram](http://i.stack.imgur.com/fvbn8.png)

Figure 1: The Snake Lemma’s key diagram

Don’t panic, you will not be required to draw that. Instead, you should directly include that image (drawn by Mr. Andrew Stacey) as found at [http://i.stack.imgur.com/fvbn8.png](http://i.stack.imgur.com/fvbn8.png). You should also scale it to 50% of its size.

∗jclay@math.org
†odonell@cs.cmu.edu
On the other hand, for the following diagram you’re on your own:

![Diagram](image)

This section doesn’t really contain any homological algebra; perhaps that’s why it’s unnumbered. Moving on.

# 1 More math

Let’s now get some practice with some **MORC** math. The topics to be covered:

- Some nonsense.
- \( T^y p e^S e \cdot t \cdot i \cdot n \cdot g. \)
- Citations.

## 1.1 Some nonsense

Chan et al. probably didn’t make the following definition in [CLRS13]:

**Definition 1.1.** We define the following set:

\[
A[1] = \left\{ x \mid \int_0^2 t^2 dt \leq \left( \frac{6}{2} \right) \right\}.
\]

I certainly admit this is taking things to an extreme, but when I see some expression like \( \sqrt{a} + \sqrt{b} + \sqrt{c} \) I get antsy because the vertical positioning of the square-root signs is different. Don’t you think it looks nicer as \( \sqrt{a} + \sqrt{b} + \sqrt{c} \), or am I deluding myself?

## 1.2 Typesetting

In contrast to Subsection 1.1, herein we will prove a theorem. It is a kind of anticoncentration result, which can also essentially be deduced from the Berry–Esseen Theorem.

**Theorem 1.2.** Assume \( a_0, a_1, \ldots, a_n \in \mathbb{R} \) satisfy

\[
\sum_{j=1}^n a_j^2 = 1, \quad \max_{1 \leq j \leq n} \{ |a_j| \} = \epsilon.
\]

Let \( x_1, \ldots, x_n \) be i.i.d. random variables, each being +1 with probability \( \frac{1}{2} \) and −1 with probability \( \frac{1}{2} \). Then if \( X = a_0 + a_1 x_1 + \ldots + a_n x_n \) it holds that \( \Pr[|X| \leq \epsilon] \leq 2.74 \epsilon. \)
Proof. The proof is a streamlining of one due to Petrov [Pet95, Theorem 2.14]. It will be convenient to rescale the \( a_j \)'s so that \( \epsilon = 1 \); we then want to show

\[
\Pr[|X| \leq 1] \leq \frac{2.74}{\sigma},
\]

where \( \sigma := \sqrt{\sum_{j=1}^{n} a_j^2} \). Define the functions \( f, g : \mathbb{R} \to \mathbb{R}_{\geq 0} \) by

\[
f(x) = \frac{2(1 - \cos x)}{x^2}, \quad g(t) = \begin{cases} 1 - |t| & \text{if } |t| \leq 1, \\ 0 & \text{else.} \end{cases}
\]

(The function \( f \) has a removable discontinuity at \( 0 \).) Integration by parts shows that \( f \) is the inverse Fourier transform of \( h_i \); i.e.,

\[
f(x) = \int_{-\infty}^{\infty} e^{-itx} g(t) \, dt.
\]

By considering the first two terms of the Taylor series for \( \cos x \) we see that \( f(x) \geq \frac{11}{12} \) on \([-1, 1]\); hence \( \frac{12}{11} f(x) \geq 1_{x \in [-1, 1]} \) for all \( x \in \mathbb{R} \). We therefore have

\[
\Pr[|X| \leq 1] \leq \mathbb{E} \left[ \frac{12}{11} f(X) \right]
\]

\[
= \frac{12}{11} \mathbb{E} \left[ \int_{-\infty}^{\infty} e^{-itX} g(t) \, dt \right]
\]

\[
= \frac{12}{11} \int_{-\infty}^{\infty} e^{-ita_0} g(t) \mathbb{E} \left[ e^{-itX'} \right] \, dt \quad \text{(writing } X' = X - a_0 \text{)}
\]

\[
= \frac{12}{11} \left| \int_{-\infty}^{\infty} e^{-ita_0} g(t) \mathbb{E} \left[ e^{-itX'} \right] \, dt \right| \quad \text{(the quantity is already real and nonnegative)}
\]

\[
\leq \frac{12}{11} \int_{-\infty}^{\infty} |e^{-ita_0}| \cdot g(t) \cdot \mathbb{E} \left[ e^{-itX'} \right] \, dt
\]

\[
\leq \frac{12}{11} \int_{-1}^{1} \mathbb{E} \left[ e^{-itX'} \right] \, dt,
\]

where the last inequality used the fact that \( |e^{-ita_0}| \leq 1 \), \( g(t) = 0 \) outside \([-1, 1]\), and \( g(t) \leq 1 \) otherwise. But

\[
\mathbb{E} \left[ e^{-itX'} \right] = \mathbb{E} \left[ \exp \left( -it \sum_{j=1}^{n} a_j x_j \right) \right]
\]

\[
= \mathbb{E} \left[ \prod_{j=1}^{n} \exp(-ita_jx_j) \right]
\]

\[
= \prod_{j=1}^{n} \mathbb{E}[\exp(-ita_jx_j)] \quad \text{(independence)}
\]

\[
= \prod_{j=1}^{n} \left( \frac{1}{2} \exp(-ita_j) + \frac{1}{2} \exp(+ita_j) \right)
\]

\[
= \prod_{j=1}^{n} \cos(a_j t).
\]
Substituting (2) into (1) and using \( \cos x \leq \exp(-\frac{1}{2}x^2) \) for \( x \in [-1, 1] \) (which can be seen from the Taylor expansion), we get

\[
\Pr[|X| \leq 1] \leq \frac{12}{n!} \int_{-1}^{1} \prod_{j=1}^{n} \exp \left( -\frac{1}{2} a_j^2 t_j^2 \right) \, dt
\]

\[
= \frac{12}{n!} \int_{-1}^{1} \exp \left( -\frac{1}{2} \sigma^2 t^2 \right) \, dt
\]

\[
\leq \frac{12}{n!} \int_{-\infty}^{\infty} \exp \left( -\frac{1}{2} \sigma^2 t^2 \right) \, dt
\]

\[
= \frac{12\sqrt{2\pi}}{n!} \int_{-\infty}^{\infty} \exp \left( -\frac{1}{2} u^2 \right) \, du \quad \text{(changing variables: } t = u/\sigma)\]

\[
= \frac{12\sqrt{2\pi}}{n!} \leq \frac{2.74}{\sigma}.
\]

\[\square\]

1.3 Citations

Here are 10 citations of papers I want to read: [DR04, DS05, Enf70, GT09, Guy89, Hat12, Hoe48, Kle66, KS88, NP00].

2 Conclusions

In conclusion, \( \heartsuit \).

Acknowledgments

The authors would like to thank Bob Tarjan, Les Valiant, Sasha Razborov, and Avi Wigderson for their careful proofreading of this manuscript.

References


