Automata Theory: Solutions 6

Problem 1
Demonstrate that, if \( L_1 \) is a regular language on the alphabet \( \Sigma = \{a, b\} \), then the following subset of \( L_1 \) is also a regular language:

\[
L_2 = \{w : w \in L_1 \text{ and } w \text{ includes at least one } b\}.
\]

Consider the language \( L_3 \) defined by the expression \( a^* \); since we describe \( L_3 \) by a regular expression, it is a regular language. This language includes all strings that have no \( b \), and we can express \( L_2 \) as follows:

\[
L_2 = L_1 - L_3.
\]

Thus, \( L_2 \) is the difference of two regular languages, which implies that it is also regular.

Problem 2
Argue that the language \( \{a^n b^{2n} : n \geq 0\} \) is not regular.

We suppose that the language is regular and derive a contradiction. If this language is regular, there is some \( m \)-state DFA that accepts it, where \( m \) is an unknown fixed number.

Suppose that we begin from the initial state of the DFA and trace the path for each of the following strings: \( a^0, a^1, a^2, \ldots, a^m \). Since the total number of these strings is \( m + 1 \), two of them must lead to the same state; we denote these two strings \( a^i \) and \( a^j \) (see the picture).

Since the automaton accepts the string \( a^i b^{2i} \), this string must lead from the initial state to a final state, as shown in the picture. Then, the string \( a^j b^{2i} \) leads to the same final state, contradicting the fact that \( a^j b^{2i} \) is not in the language.