The Mayer-Vietoris Sequence in HoTT

E Cavallo
Carnegie Mellon University

with Bob Harper, Dan Licata,
Carlo Angiuli, Ed Morehouse

November 10, 2014
- Axioms for Cohomology
- A Model in HoTT
- Mayer-Vietoris?
- Cubes
- Mayer-Vietoris

}\quad \text{Shulman and IAS}

}\quad \text{Licata}

All results are formalized in Agda!
Axioms for Cohomology
A Model in HoTT
Mayer-Vietoris?
Cubes
Mayer-Vietoris
Cohomology Theory

A cohomology theory is:

- family of contravariant functors $C^n : \text{Type}_* \to \text{AbGrp}$ for $n : \mathbb{Z}$
- satisfying certain axioms (Eilenberg-Steenrod Axioms)

Think homotopy groups: associate a group $C^n(X)$ to each dimension $n$ of a space $X$.

Note! Types will always be pointed, and functions basepoint-preserving.

$$\text{Type}_* \equiv \sum_{A : \text{Type}} A \quad (A, a_0) \to (B, b_0) \equiv \sum_{f : A \to B} f \cdot a_0 = b_0$$
Cohomology Axioms in HoTT

Eilenberg-Streenrod Axioms

1. Suspension Axiom
2. Exactness Axiom
3. Additivity Axiom (?)
1. Suspension Axiom: $C^n(X) = C^{n+1}(\Sigma X)$

data $\Sigma X$ where
north : $\Sigma X$
south : $\Sigma X$
merid : $X \rightarrow$ north = south
Eilenberg-Steenrod Axioms (2/3)

For $f : X \rightarrow Y$, there is the cofiber space:

\[ \text{data Cof}(f) \text{ where} \]
\[ \text{cfbase} : \text{Cof}(f) \]
\[ \text{cfcod} : Y \rightarrow \text{Cof}(f) \]
\[ \text{cfglue} : (x : X) \rightarrow \text{cfbase} = \text{cfcod}(f(x)) \]
Eilenberg-Steenrod Axioms (2/3)

Thus for each \( f : X \rightarrow Y \) a sequence

\[
X \xrightarrow{f} Y \xrightarrow{\text{cfcod}} \text{Cof}(f)
\]

2. Exactness Axiom:

For \( f : X \rightarrow Y \), an exact sequence:

\[
C^n(\text{Cof}(f)) \xrightarrow{\text{cfcod}^*} C^n(Y) \xrightarrow{f^*} C^n(X)
\]

“The image of \( \text{cfcod}^* \) is the kernel of \( f^* \).”

That is, for \( v : C^n(Y) \), \( f^* v = e \) if and only if there merely exists \( u : C^n(\text{Cof}(f)) \) such that \( \text{cfcod}^* u = v \).
Exactness Axiom

Extending the short exact sequence:

\[ X \xrightarrow{f} Y \rightarrow \cdots \]

\[ X \quad \text{f} \quad Y \rightarrow \text{Cof}(f) \rightarrow \text{Cof(cfcod}_f) \rightarrow \cdots \]

\[ \text{cfcod}_f \]

\[ \text{cfcod}_{\text{cfcod}_f} \]

\[ \text{cfcod}_{\text{cfcod}_{\text{cfcod}_f}} \]

\[ \text{cfcod}(\cdots) \rightarrow \cdots \]
Exactness Axiom

Extending the short exact sequence:

\[
\begin{array}{c}
X \xrightarrow{f} Y \xrightarrow{\cdot} \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\cdot \quad \text{Cof}(f) \quad \text{extglue} \quad \Sigma X \\
\downarrow \quad \downarrow \quad \downarrow \\
\cdot \quad \Sigma Y \quad \Sigma f \\
\end{array}
\]

\[
\begin{array}{c}
\Sigma f \quad \Sigma \text{cfcod}_f \\
\downarrow \\
\cdot \\
\end{array}
\]
3. Additivity Axiom (?):

For suitable (?) $I$ and $Z : I \to \text{Type}_*$,

$$C^n(\bigvee_{i : I} Z_i) = \prod_{i : I} C^n(Z_i)$$

data $X \vee Y$ where

winl : $X \to X \vee Y$

winr : $Y \to X \vee Y$

wglue : winl $x_0 = \text{winr} \ y_0$
Axioms for Cohomology
A Model in HoTT
Mayer-Vietoris?
Cubes
Mayer-Vietoris
A Model in HoTT

\( K(G, n) \), for \( G : \text{AbGrp} \) and \( n : \mathbb{N} \), is the \( n \)th \textit{Eilenberg-MacLane space}, which satisfies

\[
\begin{align*}
\pi_k(K(G, n)) &= \begin{cases} 
G, & k = n \\
0, & k \neq n 
\end{cases} \\
\Omega K(G, n + 1) &= K(G, n)
\end{align*}
\]

Formalized in Agda by Dan Licata:
\( K(G, 1) \) is a HIT, \( K(G, n + 1) \equiv \| \Sigma^n K(G, 1) \|_{n+1} \).

These are classically known to be representing spaces for cohomology theories.
A Model in HoTT

Fix $G : \text{AbGrp}$. Define

$$C^n(X) \equiv || X \to K(G, n)||_0 \quad f^*(|g|_0) \equiv |g \circ f|_0$$

(with $C^n(X) \equiv 1$ for $n < 0$.)

Group structure on $C^n(X)$ inherited from $K(G, n) = \Omega K(G, n+1)$.

Analogous to $\pi_n(X) \equiv || S^n \to X||_0$; the property $K(G, n) = \Omega K(G, n+1)$ is dual to $\Sigma S^n = S^{n+1}$.
Axioms in the Model (1/3)

1. Suspension Axiom: $C^n(X) = C^{n+1}(\Sigma X)$

\[
C^{n+1}(\Sigma X) = \| \Sigma X \to K(G, n+1) \|_0 \\
= \| X \to \Omega K(G, n+1) \|_0 \\
= \| X \to K(G, n) \|_0 \\
= C^n(X)
\]
Axioms in the Model (2/3)

2. Exactness Axiom:

For $f : X \to Y$, an exact sequence:

$$C^n(\text{Cof}(f)) \xrightarrow{\text{cf} \text{cod}^*} C^n(Y) \xrightarrow{f^*} C^n(X)$$

For $|g|_0 : C^n(Y)$, have $|g \circ f|_0 = e$ iff there is $|h|_0 : C^n(\text{Cof}(f))$ such that $|g|_0 = |h \circ \text{cf} \text{cod}|_0$.

Recall the definition of the cofiber space...
A function $\text{Cof}(f) \to K(G, n)$ is (approximately) a function $Y \to K(G, n)$ which maps the “subset” $f[X]$ to the basepoint.
Axioms in the Model (3/3)

3. Additivity Axiom (?):

For $X : I \to \text{Type}_*$, $C^n(\bigvee_{i : I} X_i) = \prod_{i : I} C^n(X_i)$.

$$C^n(\bigvee_{i : I} X_i) = \left\| \bigvee_{i : I} X_i \to K(G, n) \right\|_0 = \left\| \prod_{i : I} (X_i \to K(G, n)) \right\|_0$$

$$\prod_{i : I} C^n(X_i) = \prod_{i : I} \left\| X_i \to K(G, n) \right\|_0$$

Does $\Pi$ commute with truncation? (Not often)
- Axioms for Cohomology
- A Model in HoTT
- Mayer-Vietoris?
- Cubes
- Mayer-Vietoris
Mayer-Vietoris Sequence

Cohomology of spheres is easy in our model:

\[ C^n(S^k) = \| S^k \to K(G, n) \|_0 = \pi_k(K(G, n)) = \begin{cases} G, & n = k \\ 0, & n \neq k \end{cases} \]

Many spaces can be built from spheres using pushouts. What is the cohomology of a homotopy pushout?

\[
\begin{array}{c}
Z \xrightarrow{g} Y \\
\downarrow f \quad \quad \quad \downarrow \\
X \xrightarrow{} X \cup_Z Y
\end{array}
\]
Mayer-Vietoris Sequence

data $X \sqcup_Z Y$ where

left : $X \to X \sqcup_Z Y$

right : $Y \to Y \sqcup_Z Y$

$\text{glue} : (z : Z) \to \text{left} (f z) = \text{right} (g z)$
Mayer-Vietoris Sequence

Classically, for $X \xleftarrow{f} Z \xrightarrow{g} Y$, a long exact sequence

$$
\cdots \rightarrow C^{n-1}(Z) \rightarrow C^{n}(X \sqcup_{Z} Y) \rightarrow C^{n}(X) \times C^{n}(Y) \rightarrow C^{n}(Z) \rightarrow \cdots
$$

Try for a short exact sequence

$$
C^{n}(\Sigma Z) \rightarrow C^{n}(X \sqcup_{Z} Y) \rightarrow C^{n}(X \vee Y)
$$

working from

$$
X \vee Y \rightarrow X \sqcup_{Z} Y \rightarrow \Sigma Z
$$
Mayer-Vietoris Sequence

Start with a map $X \lor Y \rightarrow X \sqcup_{Z} Y$:

$$\text{reglue} : X \lor Y \rightarrow X \sqcup_{Z} Y$$
$$\text{reglue} \ (\text{winl} \ x) = \text{left} \ x$$
$$\text{reglue} \ (\text{winr} \ y) = \text{right} \ y$$
$$\text{ap}_{\text{reglue}} \ \text{wglue} = \text{glue} \ z_{0}$$

What is the cofiber space of this map?
Mayer-Vietoris Sequence
Mayer-Vietoris Sequence

An equivalence $\text{Cof}(\text{reglue}) \simeq \Sigma Z$ gives us:

$$ C^n(\Sigma(X \vee Y)) \to C^n(\text{Cof}(\text{reglue})) \to C^n(X \sqcup Z Y) \to C^n(X \vee Y) $$
Mayer-Vietoris Sequence

To prove $\text{Cof}(\text{reglue}) \simeq \Sigma Z$, need maps

$$\text{into} : \text{Cof}(\text{reglue}) \to \Sigma Z \quad \text{out} : \Sigma Z \to \text{Cof}(\text{reglue})$$

and need to prove

$$\text{out-into} : (\kappa : \text{Cof}(\text{reglue})) \to \text{out} \ (\text{into} \ \kappa) = \kappa$$

(and more).

How in general to construct

$$(\kappa : \text{Cof}(\text{reglue})) \to h \kappa = k \kappa$$

for $h, k : \text{Cof}(\text{reglue}) \to C$?
Mayer-Vietoris Sequence

To prove $p : (\kappa : \text{Cof}(\text{reglue})) \to h \kappa = k \kappa$ by induction on the cofiber space, we need to give

1. $p_{\text{cfbase}} : h \text{cfbase} = k \text{cfbase}$
2. $p_{\text{cfcod}} : (\gamma : X \sqcup Z Y) \to h (\text{cfcod}\gamma) = k (\text{cfcod}\gamma)$
3. A proof that, for $w : X \vee Y$, $p_{\text{cfbase}} \equiv_{\text{cfglue}_w}^{\kappa. h\kappa = k\kappa} p_{\text{cfcod}} (\text{reglue}_w)$

Proving the third by induction on $w$ would mean constructing a dependent path in the fibration $w \cdot p_{\text{cfbase}} \equiv_{\text{cfglue}_w}^{\kappa. h\kappa = k\kappa} p_{\text{cfcod}} (\text{reglue}_w)$...

How do we build such a path?
Idea: Represent these paths as cubes.
Cubes

For $p : x = y$, $u : f x = g x$, and $v : f y = g y$, the dependent path type $u =_{p}^{z.fz=gz} v$ is equivalent to the type of commutative squares:

$$
\begin{array}{ccc}
fx & \xrightarrow{apf \ p} & fy \\
\downarrow{u} & & \downarrow{v} \\
gx & \xrightarrow{apg \ p} & gy
\end{array}
$$
We can express the type of commutative squares of paths as
\[
\begin{array}{c}
\text{a} \quad \xrightarrow{p} \quad \text{c} \\
\downarrow \quad q \downarrow \\
\text{b} \quad \xrightarrow{r} \quad \text{d}
\end{array}
\]
as \text{Square } p \; q \; r \; s \text{ where Square is inductively defined as}
\[
data \text{ Square} : (a = b) \rightarrow (a = c) \rightarrow (b = d) \rightarrow (c = d) \rightarrow \text{Type where }
s \text{refl} : \text{Square} \; \text{refl} \; \text{refl} \; \text{refl} \; \text{refl}
\]
Cubes

- Dependent type in a family of paths is a square,
- Dependent type in a family of squares is a cube.

```
data Cube : (⋯six faces⋯) → Type where
  crefl : Cube srefl srefl srefl srefl srefl srefl srefl
```
Cubes

Propositionally unique fillers exist:

```
  a  \(q\) \(\rightarrow\) c
    \(p\)       \(q^{-1} \cdot p \cdot r\)
    \(\downarrow\)    \(\downarrow\)
  b  \(r\) \(\rightarrow\) d
```

Shifting faces around gives equivalent types:

```
  a  \(q\) \(\rightarrow\) c  \(\sim\)  a  \(q\) \(\rightarrow\) c  \(s\) \(\rightarrow\) e
    \(p\)       \(s\)       \(p\)
    \(\downarrow\)    \(\downarrow\)    \(\downarrow\)
  b  \(r\) \(\rightarrow\) d  b  \(r\) \(\rightarrow\) d
    \(t\)       \(t\)
```
The case of $\text{Cof}(\text{reglue})$

Were trying to prove $p : (\kappa : \text{Cof}(\text{reglue})) \rightarrow h\kappa = k\kappa$.

Needed, for $w : X \lor Y$, a dependent path $p_{\text{cfbase}} =_{\text{cfglue}_w} p_{\text{cfcod}}(\text{reglue}_w)$. Equivalently, a square

$$
\begin{array}{ccc}
h_{\text{cfbase}} & \overset{\text{ap}_h (\text{cfglue}_w)}{\longrightarrow} & h(\text{cfcod (reglue}_w) \\
\downarrow p_{\text{cfbase}} & & \downarrow p_{\text{cfcod (reglue}_w)} \\
k_{\text{cfbase}} & \overset{\text{ap}_k (\text{cfglue}_w)}{\longrightarrow} & k(\text{cfcod (reglue}_w)
\end{array}
$$

To give this by induction on $w : X \lor Y$...
To give this by induction on $w : X \lor Y$...

\[
\begin{align*}
\text{l-square : } (x : X) & \to h \text{ cfbase} \\
& \Downarrow p_{\text{base}} \\
& k \text{ cfbase} \\
& \Downarrow \\
& h \text{ cfbase} \\
\end{align*}
\]

\[
\begin{align*}
\text{l-square : } (x : X) & \to h \text{ cfbase} \\
& \Downarrow p_{\text{base}} \\
& k \text{ cfbase} \\
& \Downarrow \\
& h \text{ cfbase} \\
\end{align*}
\]

and a cube with (among other faces) left face l-square $x_0$ and right face r-square $y_0$. 
Given \textit{l-square} and \textit{r-square}, we can define a replacement \textit{r-square}' which automatically satisfies the cube requirement.

\[
\text{\textit{r-square}'y = } \quad h \text{ cfbase} \xrightarrow{\text{refl}} h \text{ cfbase} \xrightarrow{\ldots} h (\text{cfcod (reglue (winr y))})
\]

where \textit{base-filler} is the filling face giving the correct cube between \textit{l-square} \(x_0\) and \textit{base-filler} \(\cdot^h\) \textit{r-square} \(y_0\).

done!
Showed that when proving $p : (\kappa : \text{Cof}(\text{reglue})) \rightarrow h\kappa = k\kappa$ we can get the highest coherence condition automatically.

Rest of Mayer-Vietoris:

- Formalization: github.com/HoTT/HoTT-Agda, at cohomology.MayerVietoris
- Paper proof: www.contrib.andrew.cmu.edu/~ecavallo