In this assignment, you will prove metatheorems about sequent calculus, and propositions about natural numbers.

1 Contraction (20 points)

The contraction lemma for sequent calculus states that two equal propositions on the left of a sequent can be replaced by a single instance:

**Lemma 1** (Contraction). If $\Delta, A, A \to D$, then $\Delta, A \to D$.

We can prove contraction by structural induction on the derivation $\Delta, A, A \to D$, in a similar fashion to the proofs of the substitution and weakening lemmas for natural deduction.

**Task 1** (20 points). Prove contraction for the $\supset, \land$ fragment of sequent calculus. That is, show the cases of the proof where the last rule to be applied is:

- init
- $\supset R$
- $\supset L$
- $\land R$
- $\land L_1$

You don’t need to show the $\land L_2$ case, because it is very similar to the $\land L_1$ case.
2 Derivability and Admissibility (10 points)

Task 2 (10 points). For each of the following sequent calculus rules, state whether or not it is derivable, and whether or not it is admissible. Explain.

\[
\begin{align*}
\Delta, A, A \rightarrow D & \quad (1) \\
\Delta, A \rightarrow D & \quad (1) \\
\Delta, A \rightarrow D & \quad (2)
\end{align*}
\]

\[
\begin{align*}
\Delta, A \rightarrow D & \quad (3) \\
\Delta, A \rightarrow D & \quad (3)
\end{align*}
\]

3 Natural numbers (10 points)

In recitation, we saw how to define the natural numbers as a data type, and prove theorems quantified over them. We also defined the even and odd predicates on natural numbers:

\[
\begin{align*}
even(0) & : \text{nat} \\
\text{even}(n) & : \text{nat} \\
\text{odd}(n) & : \text{nat}
\end{align*}
\]

Task 3 (5 points). Derive the following judgment:

\[
\forall n : \text{nat}. (\text{even}(n) \supset \text{even}(s(s(n)))) \text{ true}
\]

Task 4 (5 points). Derive the following judgment:

\[
\forall n : \text{nat}. (\text{even}(n) \lor \text{even}(s(n))) \text{ true}
\]