Kruskal’s Minimum Spanning Tree Algorithm & Union-Find Data Structures

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AD 4.5–4.6
Greedy minimum spanning tree rules

All of these greedy rules work:

1. Starting with any root node, add the frontier edge with the smallest weight. (Prim’s Algorithm)

2. Add edges in increasing weight, skipping those whose addition would create a cycle. (Kruskal’s Algorithm)

3. Start with all edges, remove them in decreasing order of weight, skipping those whose removal would disconnect the graph. (“Reverse-Delete” Algorithm)
Prim’s Algorithm

**Prim’s Algorithm**: Starting with any root node, add the frontier edge with the smallest weight.

**Theorem.** *Prim’s algorithm produces a minimum spanning tree.*

$S = \text{set of nodes already in the tree when } e \text{ is added}$
Theorem (Cycle Property). Let $C$ be a cycle in $G$. Let $e = (u, v)$ be the edge with maximum weight on $C$. Then $e$ is not in any MST of $G$.

Suppose the theorem is false. Let $T$ be a MST that contains $e$.

Deleting $e$ from $T$ partitions vertices into 2 sets:

- $S$ (that contains $u$) and $V - S$ (that contains $v$).

Cycle $C$ must have some other edge $f$ that goes from $S$ and $V - S$.

Replacing $e$ by $f$ produces a lower cost tree, contradicting that $T$ is an MST.
1. **Cut Property**: The smallest edge crossing any cut must be in all MSTs.

2. **Cycle Property**: The largest edge on any cycle is never in any MST.
Reverse-Delete Algorithm

**Reverse-Delete Algorithm**: Remove edges in decreasing order of weight, skipping those whose removal would disconnect the graph.

**Theorem.** *Reverse-Delete algorithm produces a minimum spanning tree.*

Because removing e won't disconnect the graph, there must be another path between u and v.

Because we're removing in order of decreasing weight, e must be the largest edge on that cycle.
Kruskal’s Algorithm

Kruskal’s Algorithm: Add edges in increasing weight, skipping those whose addition would create a cycle.

Theorem. Kruskal’s algorithm produces a minimum spanning tree.

Proof. Consider the point when edge $e = (u, v)$ is added:

![Diagram of graph with nodes and edges demonstrating Kruskal's Algorithm]

$S =$ nodes to which $v$ has a path just before $e$ is added

$u$ is in $V-S$ (otherwise there would be a cycle)
Example run of Kruskal’s
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Example run of Kruskal’s
Example run of Kruskal’s algorithm.
Example run of Kruskal’s
Data Structure for Kruskal’s Algorithm

**Kruskal’s Algorithm**: Add edges in increasing weight, *skipping* those whose addition would create a cycle.

How would we check if adding an edge $\{u, v\}$ would create a cycle?
Kruskal’s Algorithm: Add edges in increasing weight, skipping those whose addition would create a cycle.

How would we check if adding an edge \{u, v\} would create a cycle?

- Would create a cycle if \( u \) and \( v \) are already in the same component.
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- We start with a component for each node.
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How would we check if adding an edge \{u, v\} would create a cycle?

- Would create a cycle if \(u\) and \(v\) are already in the same component.
- We start with a component for each node.
- Components merge when we add an edge.
Kruskal’s Algorithm: Add edges in increasing weight, skipping those whose addition would create a cycle.

How would we check if adding an edge \{u, v\} would create a cycle?

- Would create a cycle if \(u\) and \(v\) are already in the same component.
- We start with a component for each node.
- Components merge when we add an edge.
- Need a way to: check if \(u\) and \(v\) are in same component and to merge two components into one.
The Union-Find abstract data type supports the following operations that maintain a collection of sets of elements:

- **UF.create(S)** — create the data structure containing $|S|$ sets, each containing one item from $S$.

- **UF.find(i)** — return the “name” of the set containing item $i$.

- **UF.union(a,b)** — merge the sets with names $a$ and $b$ into a single set.
A Union-Find Data Structure

UF Items:

UF Sizes:

UF Sets Array:
Implementing the union & find operations

**make_union_find(S)** Create data structures on previous slide.
   Takes time proportional to the size of S.

**find(i)** Return UF.sets[i].
   Takes a constant amount of time.

**union(x,y)** Use the “size” array to decide which set is smaller.
   Assume x is smaller.
   Walk down elements i in set x, setting sets[i] = y.
   Set size[y] = size[y] + size[x].
   Make y point to start of x list and end of x list point to y.
Last step of Union operation

Update links to prepend smaller list to larger list.
Example for union(7,4):

UF Items:
Runtime of array-based Union-Find

**Theorem.** Any sequence of $k$ union operations on a collection of $n$ items takes time at most proportional to $k \log k$.

**Proof.** After $k$ unions, at most $2k$ items have been involved in a union. (Each union can touch at most 2 new items).

We upper bound the number of times set[$v$] changes for any $v$:

- Every time set[$v$] changes, the size of the set that $v$ is in at least doubles. *why?*
- So, set[$v$] can have changed at most $\log_2(2k)$ times.

At most $2k$ items have been modified at all, and each were updated at most $\log_2(2k)$ times $\implies 2k \log_2(2k)$ work.
Running time of Kruskal’s algorithm

Sorting the edges: \( \approx m \log m \) for \( m \) edges.

\[
m \leq n^2, \text{ so } \log m < \log n^2 = 2 \log n
\]

Therefore sorting takes \( \approx m \log n \) time.

At most \( 2m \) “find” operations: \( \approx 2m \) time. To check if \( u \) and \( v \) are in the same component.

At most \( n - 1 \) union operations: \( \approx n \log n \) time.

\( \Rightarrow \) Total running time of \( \approx m \log n + 2m + n \log n \).

The biggest term is \( m \log n \) since \( m \geq n \) if the graph is connected and not already a tree.
Another way to implement Union-Find
Another way to implement Union-Find

union(1, 2)
Tree-based Union-Find

make_union_find(S) Create $|S|$ trees each containing a single item and size 1. Takes time proportional to the size of $S$.

find(i) Follow the pointer from $i$ to the root of its tree.

union(x,y) If the size of set $x$ is $<$ that of $y$, make $y$ point to $x$. Takes constant time.
Runtime of tree-based Find

**Theorem.** \( \text{find}(i) \) takes time \( \approx \log n \) in a tree-based union-find data structure containing \( n \) items.

**Proof.** The depth of an item equals the number of times the set it was in was renamed.

The size of the set containing \( v \) at least doubles every time the name of the set containing \( v \) is changed.

The largest number of times the size can double is \( \log_2 n \).
Running time of Kruskal’s algorithm using tree-based union-find

Same running time as using the array-based union-find:

- Sorting the edges: $\approx m \log n$ for $m$ edges.
- At most $2m$ “find” operations: $\approx \log n$ time each.
- At most $n - 1$ union operations: $\approx n$ time.

$\Rightarrow$ Total running time of $\approx m \log n + 2m \log n + n$.

The biggest term is $m \log n$ since $m \geq n$ if the graph is connected and not already a tree.