1 Odd Parts (30)

Background
Define the “odd part” operation $\text{odd} : \Sigma^+ \rightarrow \Sigma^+$ by

$$\text{odd}(x_1x_2 \ldots x_n) = x_1x_3 \ldots x_{2n/2-1}$$

$\text{odd}$ naturally extends to languages.
For any modulus $m \geq 2$ and remainder $0 \leq r < m$, let

$$L_{r,m} = \{ x \in \{a,b\}^* \mid \#_ax = r \pmod{m} \}$$

and set $K_{r,m} = \text{odd}(L_{r,m})$.

Task

A. Show that $\text{odd}(L)$ is regular whenever $L$ is regular.
B. Show that $K_{r,m}$ is cofinite.
C. Determine the length $\kappa_{r,m}$ of a longest word in the complement of $K_{r,m}$.
D. Directly construct a DFA for $K_{r,m}$ (building an NFA first might help).

Comment
It seems difficult to determine the cardinality of the complement of $K_{r,m}$. Extra credit.

2 Primitivity and Minimality (30)

Background
Let $w$ be an non-empty word. Then $z$ is the root of $w$ if $w \in z^*$ but there is no shorter word with this property. A word is primitive if it is its own root. For example, $ab$ is the root of $abababab$. The primitive words of length 3 over $\{a,b\}$ are $aab$, $aba$, $abb$, $baa$, $bab$, $bba$. 
Given a binary word \( u = u_0u_1 \ldots u_{n-1} \) define the corresponding subset of \( S(u) \subseteq \{0, 1, \ldots, n-1\} \) by \( i \in S(u) \iff u_i = 1 \) (in other words, think of \( u \) as a bitvector for membership). Define a DFA \( M(u) \) over the one-letter alphabet \( \{a\} \) as follows:

\[
M(u) = \langle \{0, 1, \ldots, n-1\}, \{a\}, \delta; 0, S(u) \rangle
\]

where \( n = |u| \) and \( \delta(p, a) = p + 1 \mod n \).

**Task**

- A. Show that \( M(u) \) is minimal if, and only if, \( u \) is primitive (reason about the behavior of the states in \( M(u) \)).
- B. Give an alternative proof by “running” a minimization algorithm on \( M(u) \).
- C. Find a way to characterize minimal DFAs over a one-letter alphabet.

**Comment**

For the last part come up with an elegant description of the automata and make sure to prove that your description is correct.

If you like extra work (for extra credit), figure out how to count minimal DFAs of size \( n \) over a one-letter alphabet.

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### 3 Presburger and Skolem Arithmetic (40)

**Background**

Presburger arithmetic is the fragment of arithmetic that has addition but no multiplication; Skolem arithmetic is the fragment that has multiplication but no addition. Both turn out to have decidable first-order theory (in stark contrast to general arithmetic).

It is a folklore result that addition is rational and even synchronous. More precisely, there is a synchronous relation \( \alpha(x, y, z) \) that expresses \( x + y = z \) where the numbers are given in reverse binary. There is no corresponding synchronous (or even rational) relation \( \mu(x, y, z) \) for multiplication. However, we can choose a different encoding from \( \mathbb{N}_+ \) to \( 2^* \) that makes multiplication rational. Alas, this encoding is slightly strange and mangles addition badly.

**Task**

- A. Show that the addition predicate \( \alpha \) is rational for numbers in reverse binary.
- B. Argue that \( \alpha \) is actually synchronous.
- C. Show that multiplication is not rational for numbers in reverse binary.
- D. Concoct an encoding of the positive natural numbers that makes multiplication rational. Hint: think about primes.
- E. Explain why your encoding does not work for addition.
Comment  For part (A) you can either write a rational expression or draw a diagram. This is a bit of a pain; whatever you do, try to make it look nice.

For part (C) you may freely use pumping lemmings and the like.

For part (E) no formal proof is necessary, just a reasonable explanation.