Social choice theory

• A mathematical theory that deals with aggregation of individual preferences
• Origins in ancient Greece
• Formal foundations: 18th Century (Condorcet and Borda)
• 19th Century: Charles Dodgson
• 20th Century: Nobel prizes to Arrow and Sen
The voting model

- Set of voters $N = \{1, \ldots, n\}$
- Set of alternatives $A$, $|A| = m$
- Each voter has a ranking over the alternatives
- Preference profile = collection of all voters’ rankings

\[
\begin{array}{ccc}
1 & 2 & 3 \\
\hline
a & c & b \\
b & a & c \\
c & b & a \\
\end{array}
\]
Voting rules

• **Voting rule** = function from preference profiles to alternatives that specifies the winner of the election

• **Plurality**
  - Each voter awards one point to top alternative
  - Alternative with most points wins
  - Used in almost all political elections
More voting rules

• Borda count
  o Each voter awards $m - k$ points to alternative ranked $k$’th
  o Alternative with most points wins
  o Proposed in the 18th Century by the chevalier de Borda
  o Used for elections to the national assembly of Slovenia
  o Similar to rule used in the Eurovision song contest

Lordi, Eurovision 2006 winners
More voting rules

• Positional scoring rules
  o Defined by vector \((s_1, \ldots, s_m)\)
  o Plurality = \((1,0,\ldots,0)\), Borda = \((m-1,m-2,\ldots,0)\)

• \(x\) beats \(y\) in a pairwise election if the majority of voters prefer \(x\) to \(y\)

• Plurality with runoff
  o First round: two alternatives with highest plurality scores survive
  o Second round: pairwise election between these two alternatives
More voting rules

• Single Transferable vote (STV)
  o $m - 1$ rounds
  o In each round, alternative with least plurality votes is eliminated
  o Alternative left standing is the winner
  o Used in Ireland, Malta, Australia, and New Zealand (and Cambridge, MA)
### STV: Example

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15-251 Fall 2015: Lecture 15
Social choice axioms

- How do we choose among the different voting rules? Via desirable properties!
- **Majority consistency** = if a majority of voters rank alternative \( x \) first, then \( x \) should be the winner
- **Poll 1**: Which rule is **not** majority consistent?
  1. Plurality
  2. Plurality with runoff
  3. Borda count
  4. STV
Marquis de Condorcet

- 18th Century French Mathematician, philosopher, political scientist
- One of the leaders of the French revolution
- After the revolution became a fugitive
- His cover was blown and he died mysteriously in prison
Condorcet winner

• Recall: $x$ beats $y$ in a pairwise election if a majority of voters rank $x$ above $y$

• Condorcet winner beats every other alternative in pairwise election

• Condorcet paradox = cycle in majority preferences

\[
\begin{array}{ccc}
1 & 2 & 3 \\
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**Condorcet consistency**

- **Condorcet consistency** = select a Condorcet winner if one exists
- **Poll 2**: Which rule is Condorcet consistent?
  1. Plurality
  2. Borda count
  3. Both
  4. Neither
Condorcet consistency

• **Poll 3**: What is the relation between majority consistency and Condorcet consistency?
  3. Equivalent
  4. Incomparable
More voting rules

• **Copeland**: Alternative’s score is 
  \#alternatives it beats in pairwise elections

• Why does Copeland satisfy the Condorcet criterion?
  
  o If \( x \) is a Condorcet winner, score \( = m - 1 \)
  
  o Otherwise, score \( < m - 1 \)
Dodgson’s Rule

• Dodgson score of $x$ = the number of swaps between adjacent alternatives needed to make $x$ a Condorcet winner

• Dodgson’s rule: select alternative that minimizes Dodgson score

• The problem of computing the Dodgson score is NP-complete!
Awesome example

- Plurality: $a$
- Borda: $b$
- Condorcet winner: $c$
- STV: $d$
- Plurality with runoff: $e$

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Manipulation

- Using Borda count
- Top profile: \( b \) wins
- Bottom profile: \( a \) wins
- By changing his vote, voter 3 achieves a better outcome!
- Borda responded: “My scheme is intended only for honest men!”
Strategyproofness

• A voting rule is strategyproof (SP) if a voter can never benefit from lying about his preferences.

• **Poll 4:** What is the largest value of $m$ for which plurality is SP?
  1. $m = 1$
  2. $m = 2$
  3. $m = 3$
  4. $m = \infty$
Strategyproofness

• A voting rule is **dictatorial** if there is a voter who always gets his most preferred alternative

• A voting rule is **constant** if the same alternative is always chosen

• **Poll 5:** Are constant functions and dictatorships SP?
  1. Only dictatorships
  2. Only constant functions
  3. Both
  4. Neither
Gibbard-Satterthwaite

- A voting rule is onto if any alternative can win
- Theorem (Gibbard-Satterthwaite): If \( m \geq 3 \) then any voting rule that is SP and onto is dictatorial
- In other words, any voting rule that is onto and nondictatorial is manipulable
Complexity of manipulation

- Manipulation is always possible in theory
- But can we design voting rules where it is difficult in practice?
- Are there “reasonable” voting rules where manipulation is a hard computational problem? [Bartholdi et al., SC&W 1989]
The computational problem

• **R-Manipulation** problem:
  - Given votes of nonmanipulators and a preferred candidate \( p \)
  - Can manipulator cast vote that makes \( p \) uniquely win under R?

• Example: Borda, \( p = a \)

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A greedy algorithm

• Rank $p$ in first place
• While there are unranked alternatives:
  o If there is an alternative that can be placed in next spot without preventing $p$ from winning, place this alternative
  o Otherwise return false
## Example: Borda

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**When does the alg work?**

- **Fact:** The greedy algorithm is a polynomial-time algorithm for \( R\)-\textsc{MANIPULATION} for \( R \in \{\text{plurality, Borda count, plurality with runoff, Copeland,}\ldots\} \)

- **Theorem [Bartholdi and Orlin, 1991]:** the \( \text{STV-}\text{MANIPULATION} \) problem is NP-complete!
Is social choice practical?

• UK referendum: Choose between plurality and STV as a method for electing MPs
• Academics agreed STV is better...
• ... but STV seen as beneficial to the hated Nick Clegg
• Hard to change political elections!
Computational social choice

• However:
  o in human computation systems...
  o in multiagent systems...
  the designer is free to employ any voting rule!

• Computational social choice focuses on positive results through computational thinking
What we have learned

• Definitions:
  o Plurality, Borda count, plurality with runoff, STV, Copeland
  o Majority consistency
  o Condorcet winner, Condorcet consistency
  o Strategyproofness
  o The Gibbard-Satterthwaite Thm

• Principles:
  o NP-hardness can be good!