15-251: Great Theoretical Ideas In Computer Science

Recitation 6 Solutions

Coloring

(a) How many ways are there to \( k \)-color a tree with \( v \) vertices?

\[ k \cdot (k-1)^{v-1}. \]

Color one node (\( k \) choices). Then, color in nodes that are neighbors of an already-colored node, one at a time (\( k-1 \) choices at each step, since exactly 1 neighbor is already colored). This will color all nodes, since trees are connected, and no uncolored node will border two colored nodes at any point because it is a tree and this would imply a cycle, which trees don't have.

Graph-iti, Probably

(a) Let \( G = (V, E) \) be a connected graph and randomly choose a subgraph \( G' = (V', E') \) of \( G \) such that each vertex in \( V \) is in \( V' \) with probability \( p \), and each edge is in \( E' \) if and only if both of its incident vertices are in \( V' \).

Construct a set of vertices \( H \) as follows: for each edge \( e \) in \( E' \), randomly choose one of its incident vertices to not be in \( H \). \( H \) is the set of vertices that were never chosen. Find a value of \( p \) that \( |H| \geq \frac{n^2}{4m} \) where \( n = |V| \) and \( m = |E| \), respectively.

First, note that every edge in \( E' \) will remove at most 1 node from \( V' \) to get to \( H \).
So \( E[|H|] \geq E[|V'|] - |E'| \). By linearity of expectation,
\[ E[|H|] \geq E[|V'|] - E[|E'|]. \]

Note that \( E[|V'|] \) is just \( pn \) (number of vertices in the original graph times probability of choosing a vertex), and \( E[|E'|] \) is just \( p^2 m \). So we have
\[ E[|H|] \geq pn - p^2 m. \]
If we let \( p = \frac{n}{2m} \), then we get exactly our equation to be proven:
\[ E[|H|] \geq \frac{n^2}{2m} - \frac{n^2}{4m} = \frac{n^2}{4m}. \]
(Note that \( \frac{n}{2m} < 1 \) because the graph is connected, meaning that there are at least \( n-1 \) edges.)

Hypercubes and Ultracubes

(a) An \( n \)-cube is a cube in \( n \) dimensions. A cube in one dimension is a line segment; in two dimensions, it’s a square, in three, a normal cube, and in general, to go to the next dimension, a copy of the cube is made and all corresponding vertices are connected. If we consider the cube to be composed of the vertices and edges only, show that every \( n \)-cube has a Hamiltonian cycle.
A Very Average Tree

(a) What is the average number of spanning trees for simple labeled graphs with $n$ vertices?

Let $s(G)$ be the number of spanning trees of a graph $G$, and let $g(T)$ be the number of graphs having tree $T$ as a spanning tree. Then

$$\sum_G s(G) = \sum_T g(T)$$

(you can count the number of spanning trees of a graph and sum across graphs, or count the number of graphs for a tree and sum across trees). There are $2\binom{n}{2}$ possible graphs on $n$ vertices (because there are $\binom{n}{2}$ possible edges), so

$$\frac{\sum_G s(G)}{2\binom{n}{2}} = \frac{\sum_T g(T)}{2\binom{n}{2}} = \text{our average.}$$

Note that $|E(T)| = n - 1$ (the number of edges in the tree). Note that to count the graphs that have $T$ as a spanning tree, we simply count the number of ways to pick edges not in $T$: $2\binom{n}{2} - (n - 1) = g(T)$ for all $T$. Since by Cayley’s formula, there are $n^{n-2}$ trees on $n$ vertices, the average is

$$\frac{n^{n-2} \cdot 2\binom{n}{2} - (n - 1)}{2\binom{n}{2}} = \frac{n^{n-2}}{2^{n-1}}.$$

A Little Time Complexity

Assume all functions are from the positive integers to the positive integers.

(a) Prove or disprove: “For all $f, g$, at least one of $f \in O(g)$, $g \in O(f)$ is true.”

False. Counterexample:

$$f(n) = \begin{cases} n!, & n \text{ is even} \\ (n-1)!, & n \text{ is odd} \end{cases}, \quad g(n) = \begin{cases} (n-1)!, & n \text{ is even} \\ n!, & n \text{ is odd} \end{cases}$$

(b) Prove or disprove: “For all $f, g$, $f(n) \in \Theta(g(n))$ implies that $f(n)^3 \in \Omega(g(n)^2)$.”
True.
\( f(n) \in \Theta(g(n)) \Rightarrow f(n) \in \Omega(g(n)) \)
So there exist \( c, N \) such that \( f(n) \geq c \cdot g(n) \) for all \( n \geq N \).
Cubing both sides, \( f(n)^3 \geq c^3 g(n)^3 \geq c^3 g(n)^2 \) since \( g(n) \) is a positive integer.
So there exist \( c', N' \) such that \( f(n)^3 \in \Omega(g(n)^2) \): let \( c' = c^3 \) and \( N' = N \).

(c) Prove or disprove: “For all \( f, g \), \( f(n) \in \Theta(g(n)) \) implies that \( f(n) \notin \Theta(g(n)^2) \)

False. Counterexample: \( f(n) = g(n) = 1 \). \( g(n)^2 = 1 \), therefore \( f(n) \in \Theta(g(n)^2) \).

(d) Prove or disprove: “For all \( f \), if \( f(n) \in \Omega(n^k) \) for all \( k \), then \( f(n) \in \Omega(2^n) \).

False. Counterexample: \( f(n) = \lfloor 1.5^n \rfloor \). This grows faster than any polynomial function but slower than \( 2^n \).