1. Let $X_1, \ldots, X_n \sim P$ where $P$ has density $p$ and $0 \leq X_i \leq 1$. Find the asymptotic bias of $\hat{p}_h(0)$ where $\hat{p}_h$ is the kernel density estimator. Assume that $p \in \Sigma(2, L)$ and assume a Gaussian kernel. Note: because 0 is a boundary point, the bias is different than the bias we computed in class.

2. Let $X_1, \ldots, X_n \sim P$ where $P$ has density $p$ and $X_i \in \mathbb{R}$. Let

$$\hat{p}_h(x) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{h} K \left( \frac{x - X_i}{h} \right)$$

be the usual kernel density estimator. Assume that $h = h_n$ is such that $h_n \to 0$ and $nh_n \to \infty$ as $n \to \infty$.

(a) Assuming only that $p(x)$ is continuous in a neighborhood of $x$, show that

$$\hat{p}_h(x) \xrightarrow{P} p(x).$$

(b) Let $p_h(x) = \mathbb{E}(\hat{p}_h(x))$. It can be shown that

$$\frac{\hat{p}_h(x) - p_h(x)}{se_n(x)} \xrightarrow{\text{d}} N(0, 1)$$

where $se_n(x) = \sqrt{\text{Var}(\hat{p}_h(x))}$. Note that $se_n(x)$ is the standard error of the mean as $\hat{p}_h(x) = \frac{1}{n} \sum_{i=1}^{n} Z_i$ where

$$Z_i = \frac{1}{h} K \left( \frac{x - X_i}{h} \right).$$

Let $s^2$ be the sample variance of $Z_1, \ldots, Z_n$:

$$s^2 = \frac{1}{n} \sum_{i=1}^{n} (Z_i - \bar{Z}_n)^2.$$

Let $\hat{se}_n(x) = s/\sqrt{n}$. Show that

$$\frac{\hat{se}_n(x)}{se_n(x)} \xrightarrow{P} 1$$

and that

$$\frac{\hat{p}_h(x) - p_h(x)}{\hat{se}_n(x)} \xrightarrow{\text{d}} N(0, 1).$$
Note: keep in mind that \( h = h_n \) is changing with \( n \). You may make extra assumptions if needed.

(c) Suppose that \( h_n \) takes the form of the optimal bandwidth, e.g., \( h_n = Cn^{-1/5} \) for \( p \in \Sigma(2, L) \). Show that

\[
\frac{\hat{p}_h(x) - p(x)}{se_n(x)} \to N(b(x), 1)
\]

for some function \( b(x) \).

(d) Continuing from part (c), suppose we use the confidence interval

\[
C_n = \left[ \hat{p}_h(x) - z_{\alpha/2} \hat{se}_n(x), \hat{p}_h(x) + z_{\alpha/2} \hat{se}_n(x) \right].
\]

Explain why \( P(p(x) \in C_n) \) does not converge to \( 1 - \alpha \) as \( n \to \infty \). (In fact, no one really knows how to compute a confidence interval for a density estimator.)

3. Let \( \hat{m} \) be a linear smoother so that \( \hat{m}(x) = \sum_i \ell_i(x)Y_i \) and \( \hat{Y} = LY \). Prove the leave-one-out identity:

\[
\frac{1}{n} \sum_{i=1}^{n} (Y_i - \hat{m}_{(-i)}(X_i))^2 = \frac{1}{n} \sum_{i=1}^{n} \left( \frac{Y_i - \hat{m}(X_i)}{1 - L_{ii}} \right)^2 .
\]

4. Let \( \mathcal{H} \) be a Hilbert space of functions. Suppose that the evaluation functionals \( \delta_x f = f(x) \) are continuous. Show that \( \mathcal{H} \) is a reproducing kernel Hilbert space and find the kernel.