What is Probability?

- Probability reasons about a sample, knowing the population.
- The goal of statistics is to estimate the population based on a sample.
- Both provide invaluable tools to modern machine learning.
Plan

- Facts about sets (to get our brains in gear).
- Definitions and facts about probabilities.
- Random variables and joint distributions.
- Characteristics of distributions (mean, variance, entropy).
- Some asymptotic results (a “high level” perspective).

Goals: get some intuition about probability, learn how to formulate a simple proof, lay out some useful identities for use as a reference.

Non-goal: supplant an entire semester long course in probability.
A set is just a collection of elements denoted e.g.,
\[ S = \{s_1, s_2, s_3\}, \quad R = \{r : \text{some condition holds on } r\}. \]

- **Intersection**: the elements that are in both sets:
  \[ A \cap B = \{x : x \in A \text{ and } x \in B\} \]

- **Union**: the elements that are in either set, or both:
  \[ A \cup B = \{x : x \in A \text{ or } x \in B\} \]

- **Complementation**: all the elements that aren’t in the set:
  \[ A^C = \{x : x \notin A\}. \]
Properties of Set Operations

▶ **Commutativity:** $A \cup B = B \cup A$
▶ **Associativity:** $A \cup (B \cup C) = (A \cup B) \cup C$.
▶ Likewise for intersection.
▶ Proof?
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**Distributive properties**: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
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- **Distributive properties**: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
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- **Proof?** Show each side of the equality contains the other.
- **DeMorgan’s Law** ...see book.
Disjointness and Partitions

A sequence of sets $A_1, A_2, \ldots$ is called **pairwise disjoint** or **mutually exclusive** if for all $i \neq j$, $A_i \cap A_j = \emptyset$.

If the sequence is pairwise disjoint and $\bigcup_{i=1}^{\infty} A_i = S$, then the sequence forms a **partition** of $S$.

Partitions are useful in probability theory and in life:

\[
B \cap S = B \cap \left( \bigcup_{i=1}^{\infty} A_i \right) \quad \text{(def of partition)}
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Note that the sets $B \cap A_i$ are also pairwise disjoint (proof?).
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## Probability Terminology

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$\mathcal{F} = 2^\Omega = \{\{1\}, \{2\} \ldots \{1, 2\} \ldots \{1, 2, 3\} \ldots \{1, 2, 3, 4, 5, 6\}, \{\}\}$

$P(\{1\}) = P(\{2\}) = \ldots = \frac{1}{6}$ (i.e., a fair die)
$P(\{1, 3, 5\}) = \frac{1}{2}$ (i.e., half chance of odd result)
$P(\{1, 2, 3, 4, 5, 6\}) = 1$ (i.e., result is “almost surely” one of the faces).
Axioms for Probability

A set of conditions imposed on probability measures (due to Kolmogorov)

- $P(A) \geq 0, \forall A \in \mathcal{F}$
- $P(\Omega) = 1$
- $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$ where \(\{A_i\}_{i=1}^{\infty} \in \mathcal{F}\) are pairwise disjoint.

These quickly lead to:
- $P(A^C) = 1 - P(A)$ (since $P(A) + P(A^C) = P(A \cup A^C) = P(\Omega) = 1$).
- $P(A) \leq 1$ (since $P(A^C) \geq 0$).
- $P(\emptyset) = 0$ (since $P(\Omega) = 1$).
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Recall that $A, A^C$ form a partition of $\Omega$:

$$B = B \cap \Omega = B \cap (A \cup A^C) = (B \cap A) \cup (B \cap A^C)$$

Very important difference between disjoint and non-disjoint unions.

Same idea yields the so-called "union bound" aka Boole's inequality.
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Conditional Probabilities

For events $A, B \in \mathcal{F}$ with $P(B) > 0$, we may write the conditional probability of $A$ given $B$:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Interpretation: the outcome is definitely in $B$, so treat $B$ as the entire sample space and find the probability that the outcome is also in $A$. 

This rapidly leads to:

$$P(A|B)P(B) = P(A \cap B) \text{ aka the "chain rule for probabilities." (why?)}$$

When $A_1, A_2, ...$ are a partition of $\Omega$:

$$P(B) = \sum_{i=1}^{\infty} P(B \cap A_i) = \sum_{i=1}^{\infty} P(B|A_i)P(A_i)$$

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Conditional Probability Example

Suppose we throw a fair die:
\( Ω = \{1, 2, 3, 4, 5, 6\}, \ F = 2^Ω, \ P(\{i\}) = \frac{1}{6}, \ i = 1 \ldots 6 \)
\( A = \{1, 2, 3, 4\} \) i.e., “result is less than 5,”
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Note that in general,
\( P(A|B) \neq P(B|A) \) however we may quantify their relationship.
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Note that in general, \( P(A|B) \neq P(B|A) \) however we may quantify their relationship.
Bayes’ Rule

Using the chain rule we may see:

\[ P(A|B)P(B) = P(A \cap B) = P(B|A)P(A) \]

Rearranging this yields **Bayes’ rule**:

\[ P(B|A) = \frac{P(A|B)P(B)}{P(A)} \]

Often this is written as:

\[ P(B_i|A) = \frac{P(A|B_i)P(B_i)}{\sum_i P(A|B_i)P(B_i)} \]

Where \( B_i \) are a partition of \( \Omega \) (note the bottom is just the law of total probability).
Independence

Two events \( A, B \) are called \textbf{independent} if \( P(A \cap B) = P(A)P(B) \).
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Two events \( A, B \) are called **conditionally independent given** \( C \)
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When $P(A) > 0$ we may write $P(B|A, C) = P(B|C)$ e.g., “the weather tomorrow is independent of the weather yesterday, knowing the weather today.”
A random variable is a function $X : \Omega \rightarrow \mathbb{R}^d$
eq, e.g.,

- Roll some dice, $X = \text{sum of the numbers}$.
- Indicators of events: $X(\omega) = 1_A(\omega)$. e.g., toss a coin, $X = 1$ if it came up heads, 0 otherwise. Note relationship between the set theoretic constructions, and binary RVs.
- Give a few monkeys a typewriter, $X = \text{fraction of overlap with complete works of Shakespeare}$.
- Throw a dart at a board, $X \in \mathbb{R}^2$ are the coordinates which are hit.
Distributions

- By considering random variables, we may think of probability measures as functions on the real numbers.
- Then, the probability measure associated with the RV is completely characterized by its **cumulative distribution function (CDF)**: $F_X(x) = P(X \leq x)$.
- If two RVs have the same CDF we call them **identically distributed**.
- We say $X \sim F_X$ or $X \sim f_X$ (where $f_X$ is coming soon) to indicate that $X$ has the distribution specified by $F_X$ (resp, $f_X$).
If $X$ takes on only a countable number of values, then we may characterize it by a **probability mass function (PMF)** which describes the probability of each value: $f_X(x) = P(X = x)$.
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We have: $\sum_x f_X(x) = 1$ (why?)
Discrete Distributions

- If $X$ takes on only a countable number of values, then we may characterize it by a **probability mass function (PMF)** which describes the probability of each value: $f_X(x) = P(X = x)$.
- We have: $\sum_x f_X(x) = 1$ (why?) — since each $\omega$ maps to one $x$, and $P(\Omega) = 1$.
- e.g., general discrete PMF: $f_X(x_i) = \theta_i$, $\sum_i \theta_i = 1$, $\theta_i \geq 0$. 
- e.g., bernoulli distribution: $X \in \{0, 1\}$, $f_X(x) = \theta^x (1 - \theta)^{1-x}$.
Discrete Distributions

- If $X$ takes on only a countable number of values, then we may characterize it by a **probability mass function (PMF)** which describes the probability of each value: $f_X(x) = P(X = x)$.
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- e.g., Bernoulli distribution: $X \in \{0, 1\}$, $f_X(x) = \theta^x (1 - \theta)^{1-x}$
- A general model of binary outcomes (coin flips etc.).
Rather than specifying each probability for each event, we may consider a more restrictive parametric form, which will be easier to specify and manipulate (but sometimes less general).

\[ X \in \mathbb{N}^d, \sum_{i=1}^d x_i = n, \quad f_X(x) = \frac{n!}{x_1! x_2! \cdots x_d!} \prod_{i=1}^d \theta x_i. \]

Sometimes used in text processing (dimensions correspond to words, \( n \) is the length of a document).

What have we lost in going from a general form to a multinomial?
Rather than specifying each probability for each event, we may consider a more restrictive parametric form, which will be easier to specify and manipulate (but sometimes less general).

- e.g., multinomial distribution:
  \[ X \in \mathbb{N}^d, \quad \sum_{i=1}^d x_i = n, \quad f_X(x) = \frac{n!}{x_1!x_2!\cdots x_d!} \prod_{i=1}^d \theta_i^{x_i}. \]

- Sometimes used in text processing (dimensions correspond to words, \( n \) is the length of a document).

- What have we lost in going from a general form to a multinomial?
Continuous Distributions

- When the CDF is continuous we may consider its derivative
  \[ f_x(x) = \frac{d}{dx} F_X(x). \]
- This is called the **probability density function (PDF)**.
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The probability of an interval \((a, b)\) is given by
\[ P(a < X < b) = \int_a^b f_X(x) \, dx. \]

The probability of any specific point \(c\) is zero: \(P(X = c) = 0\) (why?).

e.g., Uniform distribution:
\[ f_X(x) = \frac{1}{b - a} \cdot 1_{(a, b)}(x) \]
e.g., Gaussian aka “normal:”
\[ f_X(x) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left\{ -\frac{(x - \mu)^2}{2\sigma^2} \right\} \]

Note that both families give probabilities for every interval on the real line, yet are specified by only two numbers.
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- Note that both families give probabilities for every interval on the real line, yet are specified by only two numbers.
Multiple Random Variables

We may consider multiple functions of the same sample space, e.g., $X(ω) = 1_A(ω)$, $Y(ω) = 1_B(ω)$:

May represent the joint distribution as a table:

<table>
<thead>
<tr>
<th></th>
<th>$X=0$</th>
<th>$X=1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y=0$</td>
<td>0.25</td>
<td>0.15</td>
</tr>
<tr>
<td>$Y=1$</td>
<td>0.35</td>
<td>0.25</td>
</tr>
</tbody>
</table>

We write the joint PMF or PDF as $f_{X,Y}(x,y)$.
Multiple Random Variables

Two random variables are called **independent** when the joint PDF factorizes:

\[ f_{X,Y}(x, y) = f_X(x)f_Y(y) \]

When RVs are independent and identically distributed this is usually abbreviated to “i.i.d.”

Relationship to independent events: \( X, Y \) ind. iff
\( \{\omega : X(\omega) \leq x\}, \{\omega : Y(\omega) \leq y\} \) are independent events for all \( x, y \).
We have similar constructions as we did in abstract prob. spaces:

- **Marginalizing**: \( f_X(x) = \int_Y f_{X,Y}(x, y) \, dy \).
  Similar idea to the law of total probability (identical for a discrete distribution).

- **Conditioning**: \( f_{X|Y}(x, y) = \frac{f_{X,Y}(x, y)}{f_Y(y)} = \frac{f_{X,Y}(x, y)}{\int_X f_{X,Y}(x, y) \, dx} \).
  Similar to previous definition.

<table>
<thead>
<tr>
<th>Old?</th>
<th>Blood pressure?</th>
<th>Heart Attack?</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0.22</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0.01</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0.15</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0.01</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0.18</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

How to compute \( P(\text{heart attack}|\text{old}) \)?
Characteristics of Distributions

We may consider the \textbf{expectation} (or “mean”) of a distribution:

\[
E(X) = \begin{cases} 
\sum_x xf_X(x) & \text{X is discrete} \\
\int_{-\infty}^{\infty} xf_X(x) \, dx & \text{X is continuous}
\end{cases}
\]
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\int_{-\infty}^{\infty} x f_X(x) \, dx & \text{X is continuous}
\end{cases}
\]

Expectation is linear:

\[
E(aX + bY + c) = \sum_{x,y} (ax + by + c)f_{X,Y}(x,y)
\]

\[
= a \sum_{x,y} xf_{X,Y}(x,y) + b \sum_{x,y} yf_{X,Y}(x,y) + c \sum_{x,y} f_{X,Y}(x,y)
\]

\[
= a \sum_x x \sum_y f_{X,Y}(x,y) + b \sum_y y \sum_x f_{X,Y}(x,y) + c
\]

\[
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\]

\[
= aE(X) + bE(Y) + c
\]
Characteristics of Distributions

Questions:

1. $E[EX] =$
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2. \( E(X \cdot Y) = E(X)E(Y) \)?
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Questions:

1. \( E[EX] = \sum_x (EX)f_X(x) = (EX) \sum_x f_X(x) = EX \)

2. \( E(X \cdot Y) = E(X)E(Y) \) ?
   
   Not in general, although when \( f_{X,Y} = f_X f_Y \):
   
   \[
   E(X \cdot Y) = \sum_{x,y} xyf_X(x)f_Y(y) = \sum_x xf_X(x) \sum_y yf_Y(y) = EX \cdot EY
   \]
Characteristics of Distributions

We may consider the **variance** of a distribution:

\[
\text{Var}(X) = E(X - EX)^2
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This may give an idea of how “spread out” a distribution is.
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A useful alternate form is:

\[
E(X - EX)^2 = E[X^2 - 2XE(X) + (EX)^2]
\]
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= E(X^2) - 2E(X)E(X) + (EX)^2
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Variance of a coin toss?
Characteristics of Distributions

Variance is non-linear but the following holds:

\[ \text{Var}(aX) = E((aX - E(aX))^2) = E((aX - aEX)^2) = a^2 E(X - EX)^2 = a^2 \text{Var}(X) \]
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\[ \text{Var}(X+c) = E(X+c - E(X+c))^2 = E(X - EX + c - c)^2 = E(X - EX)^2 = \text{Var}(X) \]
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\[ \text{Var}(X + Y) = E((X - EX + Y - EY)^2) \]
\[ = \begin{align*}
& E((X - EX)^2) + E((Y - EY)^2) \\
& + 2 E((X - EX)(Y - EY)) \\
& = \underbrace{\text{Var}(X)} + \underbrace{\text{Var}(Y)} + \underbrace{2 \text{Cov}(X, Y)}
\end{align*} \]
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\[ = \underbrace{E(X - EX)^2} + \underbrace{E(Y - EY)^2} + 2 \underbrace{E(X - EX)(Y - EY)}_{\text{Cov}(X,Y)} \]

So when \( X, Y \) are independent we have:

\[ \text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) \]

(why?)
Putting it all together

Say we have $X_1 \ldots X_n$ i.i.d., where $E X_i = \mu$ and $\text{Var}(X_i) = \sigma^2$.
We want to know the expectation and variance of $\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$.

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\[
\text{Var}(\bar{X}_n) = \text{Var}\left(\frac{1}{n} \sum_{i=1}^{n} X_i\right) = \frac{1}{n^2} \text{Var}(X_i) = \frac{\sigma^2}{n}
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Entropy of a Distribution

**Entropy** is a measure of uniformity in a distribution.

\[ H(X) = - \sum_x f_X(x) \log_2 f_X(x) \]

Imagine you had to transmit a sample from \( f_X \), so you construct the optimal encoding scheme:

Entropy gives the mean depth in the tree (= mean number of bits).
Law of Large Numbers (LLN)

Recall our variable \( \bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \).
We may wonder about its behavior as \( n \to \infty \).
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Using Chebyshev’s inequality:

$$P(|\bar{X}_n - \mu| \geq \epsilon) \leq \frac{\sigma^2}{n\epsilon^2} \to 0$$

For any fixed $\epsilon$, as $n \to \infty$. 
Law of Large Numbers (LLN)

Recall our variable $\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$. We may wonder about its behavior as $n \to \infty$.

The weak law of large numbers: $\lim_{n \to \infty} P( |\bar{X}_n - \mu| < \epsilon ) = 1$

In English: choose $\epsilon$ and a probability that $|\bar{X}_n - \mu| < \epsilon$, I can find you an $n$ so your probability is achieved.

The strong law of large numbers: $P( \lim_{n \to \infty} \bar{X}_n = \mu ) = 1$

In English: the mean converges to the expectation “almost surely” as $n$ increases.

Two different versions, each holds under different conditions, but i.i.d. and finite variance is enough for either.
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Central Limit Theorem (CLT)

The distribution of $\bar{X}_n$ also converges weakly to a Gaussian,

$$\lim_{n \to \infty} F_{\bar{X}_n}(x) = \Phi\left( \frac{x - \mu}{\sqrt{n\sigma}} \right)$$

Simulated $n$ dice rolls and took average, 5000 times:
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Two kinds of convergence went into this picture (why 5000?):

1. True distribution converges to a Gaussian (CLT).
2. Empirical distribution converges to true distribution (Glivenko-Cantelli).
Ideas like these are crucial to machine learning:

- We want to minimize error on a whole population (e.g., classify text documents as well as possible)
- We minimize error on a training set of size $n$.
- What happens as $n \to \infty$?
- How does the complexity of the model, or the dimension of the problem affect convergence?