Randomness and the Universe

Does the universe have “true” randomness?

Newtonian physics:

Quantum physics:
Randomness is an essential tool in **modeling and analyzing nature**.

It also plays a key role in **computer science**.

**Randomness and Computer Science**

**Statistics via Sampling**

Population: 300m  Random sample size: 2000

**Theorem:**
Randomized Algorithms

**Dimer Problem:**
Given a region, in how many different ways can you tile it with 2x1 rectangles (dominoes)?

e.g. 
1024 tilings

Captures thermodynamic properties of matter.

---

Distributed Computing

---

Nash Equilibria in Games

**The Chicken Game**

```
<table>
<thead>
<tr>
<th></th>
<th>Swerve</th>
<th>Straight</th>
</tr>
</thead>
<tbody>
<tr>
<td>Swerve</td>
<td>1 1</td>
<td>0 2</td>
</tr>
<tr>
<td>Straight</td>
<td>2 0</td>
<td>-3 -3</td>
</tr>
</tbody>
</table>
```

**Theorem (Nash):**
**Cryptography**

Adversary

Eavesdropper

“Meets me at 5.”

encryption

“loru23n8uladjkfb!#@”

decryption

“Meet me at 5.”

Shannon:

**Error-Correcting Codes**

Alice

“bit.ly/vrxUBN”

noisy channel

Bob

Each symbol can be corrupted with a certain probability. How can Alice still get the message across?

**Communication Complexity**

Want to check if the contents of two databases are exactly the same. How many bits need to be communicated?
Quantum Computing

Probability Theory: The CS Approach

The Big Picture

The Non-CS Approach

Real World \(\rightarrow\) Mathematical Model

(random)

experiment/process

probability space
The Big Picture

Real World  ➔  Mathematical Model

Flip a coin.

Ω

H 1/2
T 1/2

Ω = “sample space”
= set of all possible outcomes

Pr : Ω → [0, 1] prob. distribution

∑_{\ell \in \Omega} Pr[\ell] = 1  (why?)

The Big Picture

Real World  ➔  Mathematical Model

Flip a coin.

unit pie, area = 1

Pr[outcome] = area of outcome
= area of outcome
area of pie

The Big Picture

Real World  ➔  Mathematical Model

Flip two coins.

Ω

HH 1/4
HT 1/4
TH 1/4
TT 1/4

HH
HT
TH
TT
Flip a coin.
If it is Heads, throw a 3-sided die.
If it is Tails, throw a 4-sided die.

The CS Approach

flip ← Bernoulli(1/2)
if flip = 1: # i.e. Heads
die ← RandInt(3)
else:
die ← RandInt(4)
Probability Tree

```
flip ← Bernoulli(1/2)
if flip = H:
die ← RandInt(3)
else:
die ← RandInt(4)
```

Outcomes:
- (H,1)
- (H,2)
- (H,3)
- (T,1)
- (T,2)
- (T,3)
- (T,4)

```
Prob:
1/6 1/6 1/6 1/8 1/8 1/8 1/8
```

Events

Flip a coin. If it is Heads, throw a 3-sided die. If it is Tails, throw a 4-sided die.

```
flip ← Bernoulli(1/2)
if flip = H:
die ← RandInt(3)
else:
die ← RandInt(4)
```

What is the probability
- "event" die roll is ≥ 3?

```
E = die roll is 3 or higher
```

Extend Pr to:

```
Pr : \mathcal{P}(\Omega) → [0, 1]
```
Conditional Probability

Real World → Code → Probability Tree

Flip a coin. If it is Heads, throw a 3-sided die. If it is Tails, throw a 4-sided die.

What is the probability of flipping Heads given the die roll is ≥ 3?

conditioning on partial information

Conditional Probability

Revising probabilities based on ‘partial information’.

‘partial information’ = event E

Conditioning on E = Assuming/promising E has happened

Conditional Probability

Outcomes: (H,1) (H,2) (H,3) (T,1) (T,2) (T,3) (T,4)

Prob: 1/6 1/6 1/6 1/8 1/8 1/8 1/8

Prob | E: 0 0 2/5 0 0 3/10 3/10

E = die roll is 3 or higher
**Conditioning**

\[ \Omega \]

<table>
<thead>
<tr>
<th>Event</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>(H,1)</td>
<td>1/6</td>
</tr>
<tr>
<td>(H,2)</td>
<td>1/6</td>
</tr>
<tr>
<td>(H,3)</td>
<td>1/6</td>
</tr>
<tr>
<td>(T,1)</td>
<td>1/8</td>
</tr>
<tr>
<td>(T,2)</td>
<td>1/8</td>
</tr>
<tr>
<td>(T,3)</td>
<td>1/8</td>
</tr>
<tr>
<td>(T,4)</td>
<td>1/8</td>
</tr>
</tbody>
</table>

**E**

<table>
<thead>
<tr>
<th>Event</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>(H,3)</td>
<td>2/5</td>
</tr>
<tr>
<td>(T,3)</td>
<td>3/10</td>
</tr>
<tr>
<td>(T,4)</td>
<td>3/10</td>
</tr>
</tbody>
</table>

\[ \Pr : \Omega \to [0, 1] \]

\[ \Pr_E : E \to [0, 1] \]

**Conditional Probability \(\rightarrow\) Chain Rule**

\[ \Pr[A \mid E] = \]

(cannot condition on an event with prob. 0)

\[ \Pr[A \cap B] = \]

“For \(A\) and \(B\) to occur:
- first \(A\) must occur
- then \(B\) must occur given that \(A\) occurred”

Generalizes to more than two events.

**e.g.**

\[ \Pr[A \cap B \cap C] = \]
Conditional Probability $\rightarrow$ LTP

LTP = Law of Total Probability

$$Pr[E] =$$

---

**SUMMARY SO FAR**

**Real World $\rightarrow$ Code $\rightarrow$ Probability Tree $\rightarrow$ Mathematical Model**

**Events**

**Conditional probability:**
$$Pr[A | B] = \frac{Pr[A \cap B]}{Pr[B]}$$

**Chain rule:**
$$Pr[A \cap B] = Pr[A] \cdot Pr[B | A]$$

**Law of total probability:**
$$Pr[B] = Pr[A] \cdot Pr[B | A] + Pr[A^c] \cdot Pr[B | A^c]$$

**Independent events:**
$$Pr[A \cap B] = Pr[A] \cdot Pr[B]$$

**Union bound:**
$$Pr[A \cup B] \leq Pr[A] + Pr[B]$$

---

Random Variables
What is a Random Variable?

Typical description: $X = \text{number of Tails}$

Why?

Often we are interested in numerical outcomes
(e.g. # Tails we see if we toss $n$ coins)

but initially outcomes are best expressed non-numerically.
(e.g. an outcome is a sequence of $n$ coin tosses)

We like talking about mean values (averages), variance, etc.
What is a Random Variable?

2nd Definition:

Example:

\[
S \leftarrow \text{RandInt}(6) + \text{RandInt}(6)
\]
\[
\text{if } S = 12: \quad I \leftarrow 1
\]
\[
\text{else: } \quad I \leftarrow 0
\]

Random variables:

What is a Random Variable?

\[
S \leftarrow \text{RandInt}(6) + \text{RandInt}(6)
\]
\[
\text{if } S = 12: \quad I \leftarrow 1
\]
\[
\text{else: } \quad I \leftarrow 0
\]

Expectation of a Random Variable

Expected Value = Mean = (Weighted) Average

\[
\text{Weighted Average} = \sum_{\text{elements } e} \text{value}(e) \cdot \text{weight}(e)
\]

Example:

<table>
<thead>
<tr>
<th>Weight</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>30% Final</td>
<td>85</td>
</tr>
<tr>
<td>20% Midterm</td>
<td>75</td>
</tr>
<tr>
<td>50% Homework</td>
<td>82</td>
</tr>
</tbody>
</table>

\[
\text{Weighted Average} = 0.3 \cdot 85 + 0.2 \cdot 75 + 0.5 \cdot 82 = 81.5
\]
**Expectation of a Random Variable**

**Expected value of a random variable** $X$:

$$E[X] \overset{\text{def}}{=} \sum_{x \in \text{range}(X)} x \cdot Pr[X = x]$$

**Example**

Let $X = \text{Bernoulli}(6)$, $Y = \text{Bernoulli}(6)$, $Z = \text{Bernoulli}(6)$

Let $S = X + Y + Z$

$$E[S] = \Pr[S = 3] + 4 \cdot \Pr[S = 4] + \cdots + 18 \cdot \Pr[S = 18]$$

Lot's of arithmetic :-(

$= 10.5$

---

**Most Useful Equality in Probability Theory:**
Linearity of Expectation

Let $X = \text{Bernoulli}(6), \quad Y = \text{Bernoulli}(6), \quad Z = \text{Bernoulli}(6)$

Let $S = X + Y + Z$

$E[S] = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$