Some motivating real-world examples

**matching machines and jobs**

Job 1

Job 2

⋮

Job n

Some motivating real-world examples

**matching professors and courses**

15-110

15-112

15-122

15-150

15-251

⋮
Some motivating real-world examples

**matching rooms and courses**

<table>
<thead>
<tr>
<th>Room</th>
<th>Course</th>
</tr>
</thead>
<tbody>
<tr>
<td>GHC 4401</td>
<td>15-110</td>
</tr>
<tr>
<td>DH 2210</td>
<td>15-112</td>
</tr>
<tr>
<td>GHC 5222</td>
<td>15-122</td>
</tr>
<tr>
<td>WEH 7500</td>
<td>15-150</td>
</tr>
<tr>
<td>DH 2315</td>
<td>15-251</td>
</tr>
</tbody>
</table>


Some motivating real-world examples

**matching kidney donors and patients**

![Kidney and Patient Image]

How do you solve a problem like this?

1. Formulate the problem

2. **Ask:** Is there a trivial algorithm? Find and analyze.

3. **Ask:** Is there a better algorithm? Find and analyze.
Remember the CS life lesson

First step: Formulate the problem

Purpose:
- Get rid of all the distractions, identify the crux.
- Get a clean mathematical model that is easier to reason about.
- Solutions often generalize to other settings.

Bipartite Graphs

$G = (V, E)$ is bipartite if:
Bipartite Graphs

Given a graph $G = (V, E)$, we could ask, is it bipartite?

Poll

Is this graph bipartite?

- Yes
- No
- Beats me

Important Characterization

An obstruction for being bipartite:

Contains a cycle of odd length.

Is this the only type of obstruction?

Theorem:
Bipartite Graphs

Often we write the bipartition explicitly:

\[ G = (X, Y, E) \]

Bipartite Graphs

Great at modeling relations between two classes of objects.

**Examples:**

- **X = machines, Y = jobs**
  
  An edge \( \{x, y\} \) means \( x \) is capable of doing \( y \).

- **X = professors, Y = courses**
  
  An edge \( \{x, y\} \) means \( x \) can teach \( y \).

- **X = students, Y = internship jobs**
  
  An edge \( \{x, y\} \) means \( x \) and \( y \) are interested in each other.

Matchings in bipartite graphs

Often, we are interested in finding a **matching** in a graph

A **matching**:
Often, we are interested in finding a **matching** in a graph.

**Maximum matching:**

![Maximum matching diagram](image)

**Maximal matching:**

![Maximal matching diagram](image)

**Perfect matching:**

![Perfect matching diagram](image)
Poll

How many different perfect matchings does the graph have (in terms of n)?

\[ |X| = |Y| = n \]

Important Note

We can define matchings for non-bipartite graphs as well.

Maximum matching problem

The problem we want to solve is:

**Maximum matching problem**

**Input:** A graph \( G = (V, E) \).

**Output:** A maximum matching in \( G \).
Bipartite maximum matching problem

Actually, we want to solve the following restriction:

**Bipartite maximum matching problem**

**Input:** A bipartite graph $G = (X, Y, E)$.

**Output:** A maximum matching in $G$.

---

How do you solve a problem like this?

1. **Formulate the problem**

2. **Ask:** Is there a trivial algorithm? Find and analyze.

3. **Ask:** Is there a better algorithm? Find and analyze.

---

Bipartite maximum matching problem

Is there a (trivial) algorithm to solve this problem?
How do you solve a problem like this?

1. Formulate the problem

2. **Ask**: Is there a trivial algorithm? Find and analyze.

3. **Ask**: Is there a better algorithm? Find and analyze.

---

**Bipartite maximum matching problem**

A good first attempt:

What if we picked edges “greedily”?

```
1   5
2   6
3   7
4   8
```

Is there a way to get out of this **local optimum**?
Important Definition: Augmenting paths

Let \( M \) be some matching.

An **alternating path** with respect to \( M \) is a path in \( G \) such that:

An **augmenting path** with respect to \( M \) is an alternating path such that:

\[
\begin{align*}
\text{Augmenting path:} & \quad 4-8-2-5-1-7 \\
\text{can obtain a bigger matching.}
\end{align*}
\]
Important Definition: Augmenting paths

Augmenting path: 4-8

Matching = red edges

Augmenting paths and maximum matchings

Augmenting path ⇒ can obtain a bigger matching.

In fact, it turns out:

no augmenting path ⇒ maximum matching.

Theorem:

Proof:

If there is an augmenting path with respect to \( M \), we saw that \( M \) is not maximum.

Want to show:

If \( M \) not maximum, there is an augmenting path w.r.t. \( M \).

Let \( M^* \) be a maximum matching. \( |M^*| > |M| \).

Let \( S \) be the set of edges contained in \( M^* \) or \( M \) but not both.

\[ S = (M^* \cup M) - (M \cap M^*) \]
Augmenting paths and maximum matchings

**Proof (continued):**

\[
S = (M^* \cup M) - (M \cap M^*)
\]

(\text{will find an augmenting path in } S)

\text{red} = M^* \quad \text{blue} = M

---

**Theorem:**

A matching $M$ is maximum if and only if there is no augmenting path with respect to $M$.

**Summary of proof:**
Algorithm to find maximum matching

**Theorem:**
A matching $M$ is maximum if and only if there is no augmenting path with respect to $M$.

**Algorithm to find max matching:**