UNIT 5C
Merge Sort

Course Announcements

• Exam information
  2:30 Exam: Sections A, B, C, D, E go to Rashid (GHC 4401) and Sections F, G go to PH 125C.
  Bring your CMU id!

• A sample exam and extra exercises available in the Schedule page and Resources page.

• Sunday office hours dedicated to exam review.
Arrays (Review)

• Ruby uses “list” and “array” interchangeably
  – We will see later in the course that there are in fact subtle differences between the two data structures

• An array is an ordered collection of data
  – [“cherry”, “apple”, “banana”]
  – [8, “cherry”, -58.6, true] not necessarily the same type of elements
  – [] empty array
  – [ [“Max”, 4], [“John”, 1], [“Mary”, 3] ] can be nested

Arrays (continued)

• Examples for creating an array
  – a = Array.new # assigns [ ] to variable a
  – a = Array.new(3,0) # assigns [0, 0, 0] to variable a
  – a = Array(1..5) # assigns [1,2,3,4,5] to variable a

• Examples for accessing elements of an array
  scores = [78, 93, 80, 68]
scores[0] = 78
scores[3] = 68
More examples

• scores = [78, 93, 80, 68]
  scores[1..3] = [93, 80, 68]
  scores[0..2] = [78, 93, 80]
• year = [ [“Max”, 4], [“John”, 1], [“Mary”, 3] ]
  year[0] = [“Max”, 4]
  year[0][1] = 4
  year[0][0] = “Max”

More examples

• scores = [78, 93, 80, 68]
  scores.length = 4
  scores.first = 78
  scores.last = 68
  (scores.first) * 2 = 156
  scores.include?(85) = false
• year = [ [“Max”, 4], [“John”, 1], [“Mary”, 3] ]
  year.length = 3
  year.first = [“Max”, 4]
Adding Elements to an Array

- scores = [78, 93, 80, 68]
- The assignment
  
  scores = scores << 85

updates the scores

scores = [78, 93, 80, 68, 85]

Example: Append

- year = [ [“Max”, 4], [“John”, 1], [“Mary”, 3] ]
- What is the value of the array year after the following assignment?
  
  year = year << [“Jane”, 2]

year = [ [“Max”, 4], [“John”, 1], [“Mary”, 3], [“Jane”, 1] ]
Divide and Conquer

• In the military: strategy to gain or maintain power
• In computation:
  – Divide the problem into “simpler” versions of itself.
  – Conquer each problem using the same process (usually recursively).
  – Combine the results of the “simpler” versions to form your final solution.
• Examples: Towers of Hanoi, fractals, Binary Search, Merge Sort

Merge Sort

• Input: Array A of n elements.
• Result: Returns a new array containing the same elements in non-decreasing order.
• General algorithm for merge sort:
  1. Sort the first half using merge sort. (recursive!)
  2. Sort the second half using merge sort. (recursive!)
  3. Merge the two sorted halves to obtain the final sorted array.
Divide (Split)

Conquer (Merge)
Example 1: Merge

array a  | array b  | array c
---------|---------|---------
0 1 2 3  | 0 1 2 3 | 0 1 2 3 4 5 6 7
12 44 58 62 | 29 31 74 80 | 12

Example 1: Merge (cont’d)

array a  | array b  | array c
---------|---------|---------
0 1 2 3  | 0 1 2 3 | 0 1 2 3 4 5 6 7
12 44 58 62 | 29 31 74 80 | 12 29 31 44 58
**Example 2: Merge**

<table>
<thead>
<tr>
<th>array a</th>
<th>array b</th>
<th>array c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 1 2 3 58 67 74 90</td>
<td>0 1 2 3 19 26 31 44</td>
<td>0 1 2 3 19</td>
</tr>
<tr>
<td>0 1 2 3 58 67 74 90</td>
<td>0 1 2 3 19 26 31 44</td>
<td>0 1 2 3 19 26</td>
</tr>
<tr>
<td>0 1 2 3 58 67 74 90</td>
<td>0 1 2 3 19 26 31 44</td>
<td>0 1 2 3 19 26 31</td>
</tr>
<tr>
<td>0 1 2 3 58 67 74 90</td>
<td>0 1 2 3 19 26 31 44</td>
<td>0 1 2 3 19 26 31 44</td>
</tr>
</tbody>
</table>

**Merge**

- **Input:** Two arrays a and b.
  - Each array must be sorted already in non-decreasing order.
- **Result:** Returns a new array containing the same elements merged together into a new array in non-decreasing order.
- We'll need two variables to keep track of where we are in arrays a and b: `index_a` and `index_b`.
  1. Set `index_a` equal to 0.
  2. Set `index_b` equal to 0.
  3. Create an empty array c.
Merge (cont’d)

4. While \( \text{index}_a < \) the length of array \( a \) and \( \text{index}_b < \) the length of array \( b \), do the following:
   a. If \( a[\text{index}_a] \leq b[\text{index}_b] \), then do the following:
      i. append \( a[\text{index}_a] \) on to the end of array \( c \)
      ii. add 1 to \( \text{index}_a \)
   Otherwise, do the following:
      i. append \( b[\text{index}_b] \) on to the end of array \( c \)
      ii. add 1 to \( \text{index}_b \)

5. If \( \text{index}_a < \) the length of array \( a \), then:
   append all remaining elements of array \( a \) on to the end of array \( c \)

Otherwise:
   append all remaining elements of array \( b \) on to the end of array \( c \)

6. Return array \( c \) as the result.

(Once we finish step 4, we’ve added all of the elements of either array \( a \) or array \( b \) to array \( c \). The other array still has some elements left that need to be added to array \( c \).)
Merge in Ruby

```ruby
def merge(a, b)
  index_a = 0
  index_b = 0
  c = []
  while index_a < a.length and index_b < b.length do
    if a[index_a] <= b[index_b] then
      c << a[index_a]
      index_a = index_a + 1
    else
      c << b[index_b]
      index_b = index_b + 1
    end
  end
  return c
end
```

Merge in Ruby (cont’d)

```ruby
if (index_a < a.length) then
  for i in (index_a..a.length-1) do
    c << a[i]
  end
else
  for i in (index_b..b.length-1) do
    c << b[i]
  end
end
return c
end
```
Merge Sort: Base Case

- General algorithm for merge sort:
  1. Sort the first half using merge sort. (recursive!)
  2. Sort the second half using merge sort. (recursive!)
  3. Merge the two sorted halves to obtain the final sorted array.
- What is the base case?
  If the list has only 1 element, it is already sorted so just return the list as the result.

Merge Sort: Halfway Point

- General algorithm for merge sort:
  1. Sort the first half using merge sort. (recursive!)
  2. Sort the second half using merge sort. (recursive!)
  3. Merge the two sorted halves to obtain the final sorted array.
- How do we determine the halfway point where we want to split the array list?
  First half: \(0..\text{list.length}/2-1\)
  Second half: \(\text{list.length}/2..\text{list.length}-1\)
Merge Sort in Ruby

def msort(list)
    return list if list.length == 1 # base case
    halfway = list.length/2
    list1 = list[0..halfway-1]
    list2 = list[halfway..list.length-1]
    newlist1 = msort(list1)       # recursive!
    newlist2 = msort(list2) # recursive!
    newlist = merge(newlist1, newlist2)
    return newlist
end

Analyzing Efficiency

- If you merge two lists of size i/2 into one new list of size i, what is the maximum number of appends that you must do?
- Clearly, each element must be appended to the new list at some point, so the total number of appends is i.
- If you have a set of pairs of lists that need to be merged (two pairs at a time), and the total number of elements in all of the lists combined is n, the total number of appends will be n.
How many group merges?

• How many group merges does it take to go from n groups of size 1 to 1 group of size n?
• Example: Merge sort on 32 elements.
  – Break down to groups of size 1 (base case).
  – Merge 32 lists of size 1 into 16 lists of size 2.
  – Merge 16 lists of size 2 into 8 lists of size 4.
  – Merge 8 lists of size 4 into 4 lists of size 8.
  – Merge 4 lists of size 8 into 2 lists of size 16.
  – Merge 2 lists of size 16 into 1 list of size 32.
• In general: $\log_2 n$ group merges must occur.

Putting it all together

Total number of elements per level is always n.

It takes $\log_2 n$ iterations to go from $n$ groups of size 1 to a single group of size n.

It takes n appends to merge all pairs to the next higher level.
Big O

• In the worst case, merge sort requires $O(n \log n)$ time to sort an array with $n$ elements.

<table>
<thead>
<tr>
<th>Number of operations</th>
<th>Order of Complexity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n \log_2 n$</td>
<td>$O(n \log n)$</td>
</tr>
<tr>
<td>$4n \log_{10} n$</td>
<td>$O(n \log n)$</td>
</tr>
<tr>
<td>$n \log_2 n + 2n$</td>
<td>$O(n \log n)$</td>
</tr>
</tbody>
</table>

Number of Operations (not drawn to scale)

For an $n \log_2 n$ algorithm, the performance is better than a quadratic algorithm but just a little worse than a linear algorithm.
Comparing Insertion Sort to Merge Sort  
(Worst Case)

\[
\begin{array}{|c|c|c|}
\hline
n & \text{isort } (n(n+1)/2) & \text{msort } (n \log_2 n) \\
\hline
8 & 36 & 24 \\
16 & 136 & 64 \\
32 & 528 & 160 \\
2^{10} & 524,800 & 10,240 \\
2^{20} & 549,756,338,176 & 20,971,520 \\
\hline
\end{array}
\]

For array sizes less than 100, there’s not much difference between these sorts, but for larger array sizes, there is a clear advantage to merge sort.

Sorting and Searching

- Recall that if we wanted to use binary search, the array must be sorted.
  - What if we sort the array first using merge sort?
    - Merge sort \( O(n \log n) \) (worst case)
    - Binary search \( O(\log n) \) (worst case)
    - Total time: \( O(n \log n) + O(\log n) = O(n \log n) \) (worst case)
Comparing Big O Functions

- $O(2^n)$
- $O(n^2)$
- $O(n \log n)$
- $O(n)$
- $O(\log n)$
- $O(1)$

$n$ (amount of data)

Number of Operations

Merge Sort: Iteratively
(optional)

- If you are interested, the textbook discusses an iterative version of merge sort which you can read on your own.
- This version uses an alternate version of the merge function that is not shown in the textbook but is given in the RubyLabs gem.
Quick Sort

• Uses the technique of divide-and-conquer
  1. Pick a pivot
  2. Divide the array into two subarrays, those that are smaller and those that are greater
  3. Put the pivot in the middle, between the two sorted arrays