1. (20 pts) The following question deals with recursion and recursive algorithms.

(a) (6 pts) The function \( f \) is defined recursively as follows:

\[
\begin{align*}
  f(n) &= \begin{cases} 
    1 & \text{if } n = 0 \text{ or } n = 1 \\
    f(n-1) + 2 \times f(n-2) & \text{if } n > 1
  \end{cases}
\end{align*}
\]

Compute \( f(4) \) by drawing a recursion tree showing all of the computation required and then use your tree to compute the answer.

Recursion Tree: \( f(4) = \) _____________
(b) (4 pts) You are asked to guess a secret number between 1 and 1 billion by asking yes/no questions of the form “Is the number less than X?” (e.g. “Is the number less than 1 million?”)

What algorithm would you use to efficiently find the secret number? __________________________

Using your algorithm, what is the minimum number of questions you need to ask so you are guaranteed to find the secret number?  ____________
HINT: 1 billion bytes is approximately equal to 1 GB.

(c) (6 pts) Complete the following Ruby function recursively so that it computes \(3^n\) for \(n \geq 0\). You may assume \(n\) is an integer. Do not use a loop in your answer.

```ruby
def power3(n)
    if _____________________________ then
        return __________________________________________________
    else
        return __________________________________________________
    end
end
```

(d) (4 pts) On a computer system, a user can draw a circle using the function

```ruby
def draw_circle(center_x, center_y, radius)
end
```

where \(center_x, center_y,\) and \(radius\) are given in pixels.

In the 200 X 200 window shown below, draw what would be displayed by the following recursive function if it is initially called with \(x = 100, y = 100,\) and \(r = 100\). (Grid lines are provided for you to help you draw your picture.)

```ruby
def recursive_draw(x, y, r)
    if r > 0 then
        draw_circle(x, y, r)
        recursive_draw(x, y, r-40)
    end
end
```
2. (20 pts) This problem focuses on the representation of data in a computer. The following tables may be helpful in this question:

<table>
<thead>
<tr>
<th>Bin</th>
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<th>512</th>
<th>256</th>
<th>128</th>
<th>64</th>
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<th>16</th>
<th>8</th>
<th>4</th>
<th>2</th>
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</tr>
</thead>
<tbody>
<tr>
<td>Hex</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
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<td>E</td>
<td>F</td>
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</tbody>
</table>

(a) (2 pts) Compute the decimal value of the byte 11111111 if it is interpreted as an unsigned integer. ________________________________

HINT: \( \sum_{i=0}^{n} 2^i = 2^{n+1} - 1 \).

(b) (2 pts) Compute the decimal value of the byte 11111111 if it is interpreted as a signed 2’s complement integer. ________________________________

(c) (2 pts) Express the byte 11011001 in hexadecimal. ________________________________

(d) (2 pts) The ASCII character ‘Y’ is represented in binary using 7 bits as 1011001. The character is to be sent via satellite using odd parity. What eighth bit is sent along with this byte: 1 or 0? ____________

(e) (2 pts) Suppose that exactly one bit is corrupted during transmission of the eight bits from part (d) and is “flipped” (either from 0 to 1 or 1 to 0). Which of the following is true? Select the appropriate letter and write it here: ____________

(A) The receiver cannot detect the error.
(B) The receiver can detect the error but cannot determine which bit is wrong.
(C) The receiver can detect the error and can correct the bit that is wrong.

(f) (4 pts) In an HTML file for a webpage, the designer used the following font tag that changes the color of the font based on the 6 digit hexadecimal value 804020.

<font color=“#804020”>This is a colorful sentence.</font>

The 6 digit hexadecimal value specifies the amount of red, green and blue for the font’s color, respectively. Express the amount of green in the font as an integer between 0 and 255, inclusive. Show your work.
(g) (2 pts) Does the MP3 sound file format use a lossless or lossy compression algorithm?

(h) (4 pts) Based on the following Huffman tree:

What word is represented by the following binary string based on the Huffman tree:

111000010010101

Suppose we want to encode words made using the nine letters from the tree above using a fixed-width encoding with the fewest bits possible for each letter. How many bits are required to encode each letter?
3. (20 pts) This problem focuses on the structuring of data in a computer.

(a) (7 pts) A hash table of size 11 is used to store integer keys. The hash table is implemented as an array of “buckets” with a hash function of $h(key) = key \mod 11$. Store the following keys into the table using the given hash function.

18, 34, 98, 29

Using Ruby, show how this hash table of 4 integers would be represented as a two-dimensional array.

```
table = ______________________________________________________________
```

In general, if $n$ data elements are evenly distributed within the hash table and the hash table is sufficiently large so that each bucket contains only a few elements, then a search for an element in the hash table takes how much time using big O notation? _____________

(b) (3 pts) You are given a sequence of 1 million integers that can be stored in an array or a linked list.

Which structure is faster if you want to access the 800,000th integer? _________________________

Which structure is faster if you want to insert a new integer between the first and second integers in the sequence? _________________________

Which structure requires less memory in a computer? _________________________

(c) (5 pts) Show how to compute the following RPN expression using a stack, showing the contents of the stack each time something is pushed or popped. The first three stacks are shown for you.

$9 \ 5 \ 2 \ + \ 6 \ * \ 3 \ 1 \ - \ / \ +$

Stacks:

```

  2

  5   5

9   9   9
```
(d) (5 pts) A college campus has 6 buildings, connected with 7 direct walking paths between buildings. The matrix below represents a graph for this campus, where the weights indicate the amount of time it takes to walk down each walkway in minutes.

\[
\begin{array}{cccccc}
0 & \infty & 3 & \infty & \infty & 7 \\
\infty & 0 & \infty & 4 & 5 & 1 \\
3 & \infty & 0 & \infty & \infty & 6 \\
\infty & 4 & \infty & 0 & 2 & \infty \\
\infty & 5 & \infty & 2 & 0 & \infty \\
7 & 1 & 6 & \infty & \infty & 0 \\
\end{array}
\]

Complete the graph below by adding in the appropriate weighted edges based on the representation given above for the college campus. (If edges cross in your graph, you may assume that there is a bridge so students can’t change paths midway between buildings.) Be sure to complete your graph so it is unambiguous for maximum credit.
4. (20 pts) This question deals with binary trees.

(a) (4 pts) Draw the binary search tree that results by inserting the following integers into the tree in the order shown.

73 92 31 56 80 42 66

(b) (4 pts) If you have a binary search tree with 1000 nodes, what is the minimum number of levels this tree can have? ________________
what is the maximum number of levels this tree can have? ________________

(c) (4 pts) Given the following max-heap, redraw the max-heap after the root is removed.

85
/  \\
39  71
/  /  /  \\
28  17  43  20
/  \\
4  12

(d) (4 pts) A heap of 1000 nodes is stored in an array, level by level, starting with the root at index 0. For the heap node stored at index 4 in the array,
at what index is its left child stored? ________________
at what index is its right child stored? ________________

(e) (4 pts) For a max-heap, the maximum is always stored in the root. Describe a simple algorithm in English for finding the maximum in a binary search tree.

_______________________________________________________________________________________
_______________________________________________________________________________________
_______________________________________________________________________________________
5. (20 pts) The following question involves Boolean logic and abstraction.

(a) (9 pts) Let \( S = (A \land C) \lor (B \land \neg A) \), where \( A, B \) and \( C \) are Boolean variables. Fill in the truth table below to compute \( S \).

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(b) (6 pts) The Boolean value \( S \) can be computed by an electronic circuit. Draw this circuit at the gate level of abstraction.

(c) (2 pts) A Boolean function contains 7 different Boolean variables. How many different possible logical assignments are there for this function? In other words, how many rows would your truth table need to define this function?  ____________

(d) (3 pts) Use DeMorgan’s Law to convert the following while loop to an until loop in Ruby.

```ruby
while x > 0 or y == 100  until   ______________________________
  ...
end  end
```