16-711 Kinematics, Dynamic Systems, and Control
Spring term, 2000
Problem Set 1
Due: beginning of lecture, Thursday, February 3.

15384 grads can skip the problems: 1a and 1f. But why bother, they should be easy? In the future there will be skippable challenging problems!

1. Fundamental concepts

   (a) For each planar mechanism, find the number of DOFs. You may assume the elephant is rigid, except for the trunk which may take on the shape of any smooth curve of the correct length.

   (b) Consider the group \((G, +)\), where \(G = \{0, 1, 2\}\) and the operation \(+\) gives the value \(x + y\) indicated in the table below:

\[
\begin{array}{c|ccc}
   x = 0 & 0 & 1 & 2 \\
   1 & 1 & 2 & 0 \\
   2 & 2 & 0 & 1 \\
\end{array}
\]
What is the identity? What is the inverse of 2? Is the group commutative?

(c) The circle is a \( d \)-manifold for what value of \( d \)?

(d) Prove the circle is a differentiable manifold. That is, given that the circle is a topological manifold, choose coordinate patches that cover the circle and prove they are compatible.

(e) Sciavicco and Siciliano’s Figure 2.7 shows by example that \( SO(3) \) (the group of spatial rotations) is not commutative. Give an example showing that \( SE(2) \) (the group of planar displacements) is not commutative.

(f) For each of the “primed” frames, indicate whether the planar motion is a displacement, a rotation, and/or a translation.

2. (From Murray, Li, and Sastry, p73.) Let \( R \in SO(3) \) be a rotation matrix generated by rotating about a unit vector \( \omega \) by \( \theta \) radians. That is, \( R \) satisfies \( R = \exp(\hat{\omega}_\theta) \).

(a) Show that the eigenvalues of \( \hat{\omega} \) are 0, \( i \), and \( -i \), where \( i = \sqrt{-1} \). What are the corresponding eigenvectors?
(b) Show that the eigenvalues of $R$ are $1$, $e^{i\theta}$, and $e^{-i\theta}$. What is the eigenvector whose eigenvalue is $1$?

3. (From Murray, Li, and Sastry, p74.) Show that the set of unit quaternions satisfies the axioms of a group.

4. (From Murray, Li, and Sastry, p74.) Let $SO(2)$ be the set of all $2 \times 2$ orthogonal matrices with determinant equal to $+1$.

(a) Show that $SO(2)$ can be identified with $S^1$, the unit circle in $\mathbb{R}^2$.

(b) Let $\omega \in \mathbb{R}$ be a real number and define $\hat{\omega} \in so(2)$ as the skew-symmetric matrix

$$\hat{\omega} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}.$$ 

Show that

$$e^{\hat{\omega} \theta} = \begin{bmatrix} \cos \omega \theta & -\sin \omega \theta \\ \sin \omega \theta & \cos \omega \theta \end{bmatrix}.$$ 

Is the exponential map $\exp : so(2) \to SO(2)$ surjective? injective?

(c) Show that for $R \in SO(2)$ and $\hat{\omega} \in so(2)$, $R\hat{\omega}R^T = \hat{\omega}$. 

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