Announcements

Homework 4 will be out today. Due 4/4/06. Please start early.

Midterm stats: A range → 40+, B range → 30+

40+ → 13 students
30+ → 9 students
Below 30 → 6 students
Shape from Shading

Lecture #13

Image Intensity and 3D Geometry

- Shading as a cue for shape reconstruction
- What is the relation between intensity and shape?
  - Reflectance Map
Reflectance Map - RECAP

- Relates image irradiance $I(x,y)$ to surface orientation $(p,q)$ for given source direction and surface reflectance.
- Lambertian case:
  
  $k$ : source brightness
  
  $\rho$ : surface albedo (reflectance)
  
  $c$ : constant (optical system)

Image irradiance:

$$I = \frac{\rho}{\pi} k c \cos \theta_i = \frac{\rho}{\pi} k c n \cdot s$$

Let $\frac{\rho}{\pi} k c = 1$ then $I = \cos \theta_i = n \cdot s$

Reflectance Map - RECAP

- Lambertian case

$$I = \cos \theta_i = n \cdot s = \frac{(pp_i + qq_i + 1)}{\sqrt{p^2 + q^2 + 1\sqrt{p_s^2 + q_s^2 + 1}}} = R(p, q)$$
Reflectance Map - RECAP

- Lambertian case

\[
R(p, q) = \frac{p^2 + q^2}{p^2 + q^2 + 1}
\]

\[
\theta_z = 90^\circ
\]

\[
(p_s q_s + 1 = 0)
\]

Note: \( R(p, q) \) is maximum when \( (p, q) = (p_s, q_s) \)

Shape from a Single Image?

- Given a single image of an object with known surface reflectance taken under a known light source, can we recover the shape of the object?
- Given \( R(p, q) \) (\( (p_s, q_s) \) and surface reflectance) can we determine \( (p, q) \) uniquely for each image point?

NO
Solution

- Take more images
  - Photometric stereo (previous class)

- Add more constraints
  - Shape-from-shading (this class)

Photometric Stereo
Photometric Stereo

Lambertian case:

\[ I = \frac{\rho}{\pi} k \cos \theta = \rho \mathbf{n} \cdot \mathbf{s} \quad \left( \frac{k \cos \theta}{\pi} = 1 \right) \]

Image irradiance:

\[ I_1 = \rho \mathbf{n} \cdot \mathbf{s}_1 \]
\[ I_2 = \rho \mathbf{n} \cdot \mathbf{s}_2 \]
\[ I_3 = \rho \mathbf{n} \cdot \mathbf{s}_3 \]

• We can write this in matrix form:

\[
\begin{bmatrix}
  I_1 \\
  I_2 \\
  I_3 
\end{bmatrix} = \rho \begin{bmatrix}
  s_1^T \\
  s_2^T \\
  s_3^T
\end{bmatrix} \mathbf{n}
\]

Solution

• Take more images
  – Photometric stereo (previous class)

• Add more constraints
  – Shape-from-shading (this class)
Human Perception

- Our brain often perceives shape from shading.
- Mostly, it makes many assumptions to do so.
- For example:
  
  Light is coming from above (sun).
  
  Biased by occluding contours.

by V. Ramachandran

See Ramachandran’s work on Shape from Shading by Humans

http://psy.ucsd.edu/chip/ramabio.html
Stereographic Projection

\[(p,q)\)-space (gradient space)

\[(f,g)\)-space

Problem 
\[(p,q)\] can be infinite when \(\theta = 90^\circ\)

\[f = \frac{2p}{1 + \sqrt{1 + p^2 + q^2}} \quad g = \frac{2q}{1 + \sqrt{1 + p^2 + q^2}}\]

Redefine reflectance map as \(R(f,g)\)

Occluding Boundaries

\[\text{n} \perp \text{e}, \quad \text{n} \perp \text{v} \quad \therefore \quad \text{n} = \text{e} \times \text{v} \] 
\text{e and v are known}

The \(\text{n}\) values on the occluding boundary can be used as the boundary condition for shape-from-shading
Image Irradiance Constraint

- Image irradiance should match the reflectance map

\[
e_i = \iint_{\text{image}} (I(x,y) - R(f,g))^2 \, dx \, dy
\]

(minimize errors in image irradiance in the image)

Smoothness Constraint

- Used to constrain shape-from-shading
- Relates orientations \((f,g)\) of neighboring surface points

\[
e_s = \iint_{\text{image}} \left( f_x^2 + f_y^2 + g_x^2 + g_y^2 \right) \, dx \, dy
\]

\((f,g)\): surface orientation under stereographic projection

\[
f_x = \frac{\partial f}{\partial x}, f_y = \frac{\partial f}{\partial y}, g_x = \frac{\partial g}{\partial x}, g_y = \frac{\partial g}{\partial y}
\]

(penalize rapid changes in surface orientation \(f\) and \(g\) over the image)
Shape-from-Shading

- Find surface orientations \((f, g)\) at all image points that minimize

\[
e = e_s + \lambda e_i
\]

Minimize

\[
e = \iint_{\text{image}} \left( f_x^2 + f_y^2 \right) + \left( g_x^2 + g_y^2 \right) + \lambda \left( I(x, y) - R(f, g) \right)^2 \, dx \, dy
\]

Numerical Shape-from-Shading

- **Smoothness error** at image point \((i,j)\)

\[
s_{i,j} = \frac{1}{4} \left( (f_{i+1,j} - f_{i,j})^2 + (f_{i,j+1} - f_{i,j})^2 + (g_{i+1,j} - g_{i,j})^2 + (g_{i,j+1} - g_{i,j})^2 \right)
\]

Of course you can consider more neighbors (smoother results)

- **Image irradiance error** at image point \((i,j)\)

\[
r_{i,j} = \left( I_{i,j} - R(f_{i,j}, g_{i,j}) \right)^2
\]

Find \(\{f_{i,j}\}\) and \(\{g_{i,j}\}\) that minimize

\[
e = \sum_i \sum_j \left( s_{i,j} + \lambda r_{i,j} \right)
\]

(Ikeuchi & Horn 89)
**Numerical Shape-from-Shading**

Find \( \{f_{i,j}\} \) and \( \{g_{i,j}\} \) that minimize
\[
e = \sum_i \sum_j (s_{i,j} + \lambda r_{i,j})
\]

If \( f_{k,l} \) and \( g_{k,l} \) minimize \( e \), then
\[
\frac{\partial e}{\partial f_{k,l}} = 0, \quad \frac{\partial e}{\partial g_{k,l}} = 0
\]

\[
\frac{\partial e}{\partial f_{k,l}} = 2(f_{k,l} - \bar{f}_{k,l}) - 2\lambda(I_{k,l} - R(f_{k,l}, g_{k,l})) \left. \frac{\partial R}{\partial f_{k,l}} \right|_{f_{k,l}} = 0
\]

\[
\frac{\partial e}{\partial g_{k,l}} = 2(g_{k,l} - \bar{g}_{k,l}) - 2\lambda(I_{k,l} - R(f_{k,l}, g_{k,l})) \left. \frac{\partial R}{\partial g_{k,l}} \right|_{g_{k,l}} = 0
\]

where \( \bar{f}_{k,l} \) and \( \bar{g}_{k,l} \) are 4-neighbors average around image point \( (k,l) \)

\[
\bar{f}_{k,l} = \frac{1}{8}(f_{i_{1,l}} + f_{i_{2,l}} + f_{i_{3,l}} + f_{i_{4,l}})
\]

\[
\bar{g}_{k,l} = \frac{1}{8}(g_{i_{1,l}} + g_{i_{2,l}} + g_{i_{3,l}} + g_{i_{4,l}})
\]

(Ikeuchi & Horn 89)

**Numerical Shape-from-Shading**

\[
\frac{\partial e}{\partial f_{k,l}} = 2(f_{k,l} - \bar{f}_{k,l}) - 2\lambda(I_{k,l} - R(f_{k,l}, g_{k,l})) \left. \frac{\partial R}{\partial f_{k,l}} \right|_{f_{k,l}} = 0
\]

\[
\frac{\partial e}{\partial g_{k,l}} = 2(g_{k,l} - \bar{g}_{k,l}) - 2\lambda(I_{k,l} - R(f_{k,l}, g_{k,l})) \left. \frac{\partial R}{\partial g_{k,l}} \right|_{g_{k,l}} = 0
\]

Update rule

\[
f_{k,l}^{n+1} = \bar{f}_{k,l} + \lambda(I_{k,l} - R(f_{k,l}, g_{k,l})) \left. \frac{\partial R}{\partial f_{k,l}} \right|_{f_{k,l}}
\]

\[
g_{k,l}^{n+1} = \bar{g}_{k,l} + \lambda(I_{k,l} - R(f_{k,l}, g_{k,l})) \left. \frac{\partial R}{\partial g_{k,l}} \right|_{g_{k,l}}
\]

- Use known \( (f, g) \) values on the occluding boundary to constrain the solution (boundary conditions)
- Compare \( (f_{k,l}^{n+1}, g_{k,l}^{n+1}) \) with \( (f_{k,l}^{n}, g_{k,l}^{n}) \) for convergence test
- As the solution converges, increase \( \lambda \) to remove the smoothness constraint

(Ikeuchi & Horn 89)
Calculus of Variations

Minimize
\[ e = \int \int_{\text{image}} F(f, g, f_x, f_y, g_x, g_y) \, dx \, dy \]
\[ F = \left( f_x^2 + f_y^2 \right) + \left( g_x^2 + g_y^2 \right) + \lambda \left( I(x, y) - R(f, g) \right)^2 \]

Euler equations for \( F \)
\[ F_x - \frac{\partial}{\partial x} F_{f_x} - \frac{\partial}{\partial y} F_{f_y} = 0, F_g - \frac{\partial}{\partial x} F_{g_x} - \frac{\partial}{\partial y} F_{g_y} = 0 \]

Euler equations for shape-from-shading
\[ \nabla^2 f = -\lambda \left( I(x, y) - R(f, g) \right) \frac{\partial R}{\partial f}, \nabla^2 g = -\lambda \left( I(x, y) - R(f, g) \right) \frac{\partial R}{\partial g} \]

Solve this coupled pair of second-order partial differential equations with the occluding boundary conditions!

Results

by Bauchi and Horn
Results

Scanning Electron Microscope image (inverse intensity)
by Ikouchi and Horn

Next Two Classes

• Binocular Stereo
• Reading: Horn, Chapter 13.