Great Theoretical Ideas in Computer Science for 15-251 Some
CMU Student
Crush Script

http://1982087.com/crush.html
The Mathematics Of 1950’s Dating: Who wins The Battle of The Sexes?

Lecture 10 (February 12, 2009)
WARNING: This lecture contains mathematical content that may be shocking to some students.
Dating Scenario

There are $n$ boys and $n$ girls

Each girl has her own ranked preference list of all the boys

Each boy has his own ranked preference list of the girls

The lists have no ties

Question: How do we pair them off?
More Than One Notion of What Constitutes A “Good” Pairing

Maximizing total satisfaction

Hong Kong and to an extent the USA

Maximizing the minimum satisfaction

Western Europe

Minimizing maximum difference in mate ranks

Sweden

Maximizing people who get their first choice

Barbie and Ken Land
We will ignore the issue of what is “best”!
Rogue Couples

Suppose we pair off all the boys and girls.

Now suppose that some boy and some girl prefer each other to the people to whom they are paired.

They will be called a **rogue couple**.
Why be with them when we can be with each other?
What use is fairness, if it is not stable?

Any list of criteria for a good pairing must include **stability**. (A pairing is doomed if it contains a rogue couple.)
Stable Pairings

A pairing of boys and girls is called stable if it contains no rogue couples
A pairing of boys and girls is called **stable** if it contains no rogue couples.
The study of stability will be the subject of the entire lecture

We will:

- Analyze various mathematical properties of an algorithm that looks a lot like 1950’s dating
- Discover the **naked mathematical truth** about which sex has the romantic edge
- Learn how the world’s largest, most successful dating service operates
Given a set of preference lists, how do we find a stable pairing?

Wait! We don’t even know that such a pairing always exists!
Better Question:

Does every set of preference lists have a stable pairing?
Idea: Allow the pairs to keep breaking up and reforming until they become stable
Can you argue that the couples will not continue breaking up and reforming forever?
An Instructive Variant: Bisexual Dating
Insight

Any proof that heterosexual couples do not break up and re-form forever must contain a step that fails in the bisexual case.

If you have a proof idea that works equally well in the hetero and bisexual versions, then your idea is not adequate to show the couples eventually stop.
The Traditional Marriage Algorithm

Worshipping Males

Female

String
The Traditional Marriage Algorithm

For each day that some boy gets a “No” do:

**Morning**

- Each girl stands on her balcony
- Each boy proposes to the best girl whom he has not yet crossed off

**Afternoon (for girls with at least one suitor)**

- To today’s best: “Maybe, return tomorrow”
- To any others: “No, I will never marry you”

**Evening**

- Any rejected boy crosses the girl off his list

If no boys get a “No”, each girl marries boy to whom she just said “maybe”
Does Traditional Marriage Algorithm always produce a stable pairing?

Wait! There is a more primary question!
Does TMA Always Terminate?

It might encounter a situation where algorithm does not specify what to do next (e.g. “core dump error”)

It might keep on going for an infinite number of days
Improvement Lemma:
If a girl has a boy on a string, then she will always have someone at least as good on a string (or for a husband)

She would only let go of him in order to “maybe” someone better

She would only let go of that guy for someone even better

She would only let go of that guy for someone even better

AND SO ON…
Corollary: Each girl will marry her absolute favorite of the boys who visit her during the TMA
Lemma: No boy can be rejected by all the girls

Proof (by contradiction):

Suppose boy $b$ is rejected by all the girls

At that point:

Each girl must have a suitor other than $b$

(By Improvement Lemma, once a girl has a suitor she will always have at least one)

The $n$ girls have $n$ suitors, and $b$ is not among them. Thus, there are at least $n+1$ boys

Contradiction
Theorem: The TMA always terminates in at most $n^2$ days

A “master list” of all $n$ of the boys lists starts with a total of $n \times n = n^2$ girls on it.

Each day at least one boy gets a “No”, so at least one girl gets crossed off the master list.

Therefore, the number of days is bounded by the original size of the master list.
Great! We know that TMA will terminate and produce a pairing. But is it stable?
Theorem: The pairing $T$ produced by TMA is stable.

I rejected you when you came to my balcony. Now I’ve got someone better.
Opinion Poll

Who is better off in traditional dating, the boys or the girls?
Forget TMA For a Moment…

How should we define what we mean when we say “the optimal girl for boy b”?

Flawed Attempt: “The girl at the top of b’s list”
The Optimal Girl

A boy’s optimal girl is the highest ranked girl for whom there is some stable pairing in which the boy gets her.

She is the best girl he can conceivably get in a stable world. Presumably, she might be better than the girl he gets in the stable pairing output by TMA.
The Pessimal Girl

A boy’s **pessimal girl** is the lowest ranked girl for whom there is some stable pairing in which the boy gets her.

She is the **worst** girl he can conceivably get in a stable world.
Dating Heaven and Hell

A pairing is **male-optimal** if every boy gets his **optimal** mate. This is the best of all possible stable worlds for every boy simultaneously.

A pairing is **male-pessimal** if every boy gets his **pessimal** mate. This is the worst of all possible stable worlds for every boy simultaneously.
A pairing is **female-optimal** if *every* girl gets her **optimal** mate. This is the best of all possible stable worlds for every girl simultaneously.

A pairing is **female-pessimal** if *every* girl gets her **pessimal** mate. This is the worst of all possible stable worlds for every girl simultaneously.
The Naked Mathematical Truth!

The Traditional Marriage Algorithm always produces a male-optimal, female-pessimal pairing.
Theorem: TMA produces a male-optimal pairing

Suppose, for a contradiction, that some boy gets rejected by his optimal girl during TMA.

Let $t$ be the earliest time at which this happened.

At time $t$, boy $b$ got rejected by his optimal girl $g$ because she said “maybe” to a preferred $b^*$. 

By the definition of $t$, $b^*$ had not yet been rejected by his optimal girl.

Therefore, $b^*$ likes $g$ at least as much as his optimal.
Some boy $b$ got rejected by his optimal girl $g$ because she said “maybe” to a preferred $b^*$. $b^*$ likes $g$ at least as much as his optimal girl.

There must exist a stable pairing $S$ in which $b$ and $g$ are married.

$b^*$ wants $g$ more than his wife in $S$: $g$ is at least as good as his best and he does not have her in stable pairing $S$.

$g$ wants $b^*$ more than her husband in $S$: $b$ is her husband in $S$ and she rejects him for $b^*$ in TMA.

Contradiction.
Theorem: The TMA pairing, T, is female-pessimal

We know it is male-optimal. Suppose there is a stable pairing S where some girl g does worse than in T.

Let b be her mate in T.

Let b* be her mate in S.

By assumption, g likes b better than her mate in S.

b likes g better than his mate in S (we already know that g is his optimal girl).

Therefore, S is not stable.

Contradiction.
The largest, most successful dating service in the world uses a computer to run TMA!
Definition of:
• Stable Pairing
• Traditional Marriage Algorithm

Proof that:
• TMA Produces a Stable Pairing
• TMA Produces a Male-Optimal, Female-Pessimal Pairing

Here’s What You Need to Know...