15-251
Cooking for Computer Scientists
I understand that making pancakes can be a dangerous activity and that, by doing so, I am taking a risk that I may be injured.

I hereby assume all the risk described above, even if Luis von Ahn, his TAs or agents, through negligence or otherwise, otherwise be deemed liable. I hereby release, waive, discharge covenant not to sue Luis von Ahn, his TAs or any agents, participants, sponsoring agencies, sponsors, or others associated with the event, and, if applicable, owners of premises used to conduct the pancake cooking event, from any and all liability arising out of my participation, even if the liability arises out of negligence that may not be foreseeable at this time.

Please don’t burn yourself...
Stuffs

Office Hours

No recitations this Monday

Emailing Luis/Anupam
The chefs at our place are sloppy: when they prepare pancakes, they come out all different sizes.

When the waiter delivers them to a customer, he rearranges them (so that smallest is on top, and so on, down to the largest at the bottom).

He does this by grabbing several from the top and flipping them over, repeating this (varying the number he flips) as many times as necessary.
Developing A Notation: Turning pancakes into numbers
How do we sort this stack?
How many flips do we need?
4 Flips Are Sufficient
Best Way to Sort

X = Smallest number of flips required to sort:

? \leq X \leq 4

Upper Bound

Lower Bound
Four Flips Are Necessary

If we could do it in three flips:
Flip 1 has to put 5 on bottom
Flip 2 must bring 4 to top (if it didn’t, we would spend more than 3)
$4 \leq X \leq 4$

Lower Bound $\geq$ Upper Bound

$X = 4$
5\textsuperscript{th} Pancake Number

\[ P_5 = \text{MAX over } s \text{ stacks of 5 of MIN } \# \text{ of flips to sort } s \]

Number of flips required to sort the worst case stack of 5 pancakes

<table>
<thead>
<tr>
<th>( X_1 )</th>
<th>( X_2 )</th>
<th>( X_3 )</th>
<th>( X_{119} )</th>
<th>( X_{120} )</th>
</tr>
</thead>
<tbody>
<tr>
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</tbody>
</table>
5\textsuperscript{th} Pancake Number

\[ 4 \leq P_5 \leq ? \]
\( P_n = \text{MAX over } s \in \text{stacks of } n \text{ pancakes of MIN \# of flips to sort } s \)

\( P_n = \text{The number of flips required to sort the worst-case stack of } n \text{ pancakes} \)
What is $P_n$ for small $n$?

Can you do $n = 0, 1, 2, 3$?
### Initial Values of $P_n$

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_n$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>
P_3 = 3

requires 3 Flips, hence \( P_3 \geq 3 \)

ANY stack of 3 can be done by getting the big one to the bottom (\( \leq 2 \) flips), and then using \( \leq 1 \) flips to handle the top two.
$P_n = \text{Number of flips required to sort the worst case stack of } n \text{ pancakes}$

$? \leq P_n \leq ?$

- **Lower Bound**
- **Upper Bound**
Bracketing:
What are the best lower and upper bounds that I can prove?

\[ \leq f(x) \leq \]
$? \leq P_n \leq ?$

Try to find upper and lower bounds on $P_n$, for $n > 3$
Bring-to-top Method

Bring biggest to top
Place it on bottom

Bring next largest to top
Place second from bottom

And so on…
Upper Bound On $P_n$:
Bring-to-top Method For n Pancakes

If $n=1$, no work required — we are done!
Otherwise, flip pancake $n$ to top and then flip it to position $n$

Now use:

Bring To Top Method For n-1 Pancakes

Total Cost: at most $2(n-1) = 2n - 2$ flips
Better Upper Bound On $P_n$:
Bring-to-top Method For $n$ Pancakes

If $n=2$, at most one flip and we are done!
Otherwise, flip pancake $n$ to top and then flip it to position $n$

Now use: Bring To Top Method For $n-1$ Pancakes

Total Cost: at most $2(n-2) + 1 = 2n - 3$ flips
? \leq P_n \leq 2n-3
Bring-to-top not always optimal for a particular stack

Bring-to-top takes 5 flips, but we can do it in 4 flips
What other bounds can you prove on $P_n$?
Breaking Apart Argument

Suppose a stack $S$ has a pair of adjacent pancakes that will not be adjacent in the sorted stack.

Any sequence of flips that sorts stack $S$ must have one flip that inserts the spatula between that pair and breaks them apart.

Furthermore, this is true of the “pair” formed by the bottom pancake of $S$ and the plate.
n \leq P_n

Suppose n is even

S contains n pairs that will need to be broken apart during any sequence that sorts it

Detail: This construction only works when n > 2
Suppose $n$ is odd

$S$ contains $n$ pairs that will need to be broken apart during any sequence that sorts it.

Detail: This construction only works when $n > 3$.
\[ n \leq P_n \leq 2n - 3 \text{ for } n > 3 \]

Bring-to-top is within a factor of 2 of optimal!
From ANY stack to sorted stack in $\leq P_n$

From sorted stack to ANY stack in $\leq P_n$?

Reverse the sequences we use to sort

Hence, from ANY stack to ANY stack in $\leq 2P_n$
Can you find a faster way than $2P_n$ flips to go from ANY to ANY?
ANY Stack $S$ to ANY stack $T$ in $\leq P_n$

$S$: 4,3,5,1,2  \quad T$: 5,2,4,3,1

1,2,3,4,5  \quad 3,5,1,2,4

"new T"

Rename the pancakes in $S$ to be 1,2,3,..,n
Rewrite $T$ using the new naming scheme that you used for $S$

The sequence of flips that brings the sorted stack to the "new T" will bring $S$ to $T$
The Known Pancake Numbers

<table>
<thead>
<tr>
<th>n</th>
<th>$P_n$</th>
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<tbody>
<tr>
<td>1</td>
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<td>14</td>
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<td>13</td>
<td>15</td>
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</tbody>
</table>
$P_{14}$ is Unknown

$1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot 13 \cdot 14 = 14!$ orderings of 14 pancakes

$14! = 87,178,291,200$
Is This Really Computer Science?
$(17/16)n \leq P_n \leq (5n+5)/3$

\[(15/14)n \leq P_n \leq (5n+5)/3\]

How many different stacks of $n$ pancakes are there?

$n! = 1 \times 2 \times 3 \times \ldots \times n$
Pancake Network: Definition For n! Nodes

For each node, assign it the name of one of the n! stacks of n pancakes

Put a wire between two nodes if they are one flip apart
Network For \( n = 3 \)
Network For n=4
What is the maximum distance between two nodes in the pancake network?
Pancake Network: Reliability

If up to n-2 nodes get hit by lightning, the network remains connected, even though each node is connected to only n-1 others.

The Pancake Network is optimally reliable for its number of edges and nodes.
Mutation Distance

Head Cabbage
(Brassica oleracea capitata)

Turnip
(Brassica rapa)
One “Simple” Problem

A host of problems and applications at the frontiers of science
High Level Point

Computer Science is not merely about computers and programming, it is about mathematically modeling our world, and about finding better and better ways to solve problems.

Today’s lecture is a microcosm of this exercise.
Definitions of:
- n<sup>th</sup> pancake number
- lower bound
- upper bound

Proof of:
- ANY to ANY in ≤ P<sub>n</sub>

Important Technique:
- Bracketing