Repeat After Me.

This part of the quiz is intended to test your ability to study material that has been labeled as likely to appear on exams.

1. [15 points]

   a. True or False: \( \Pr[X \geq E[X]] > 0 \)
      
      True: Not everybody can be strictly less than the average.

   b. True or False: \( E[100X] = 100E[X] \)
      
      True: linearity of expectations.

   c. True or False: For two events \( A \) and \( B \), \( \Pr[A \cup B] \leq \Pr[A] + \Pr[B] \).
      
      True: The RHS counts everything in the LHS, and even counts the intersection twice.

   d. True or False: For two independent events, \( \Pr[A \cup B] = \Pr[A] + \Pr[B] \)
      
      False. Flip a penny and a nickel – the chance that both come up heads is \( \frac{1}{4} \), while the RHS is \( \frac{1}{2} + \frac{1}{2} \).

   e. True or False: \( E[X/Y] = E[X]/E[Y] \) for \( X \) independent of \( Y \)
      
      False. Let \( X \) always be 1, and \( Y \in \{1, 2\} \) with equal probability, then
      
      \[
      E[X/Y] = \frac{1}{2} \left( \frac{1}{1} + \frac{1}{2} \right) \neq \frac{1}{\frac{1}{2}(1 + 2)} = \frac{E[X]}{E[Y]}
      \]

2. [5 points]

   Given two events \( A \) and \( B \), write down a mathematical definition of \( A \) and \( B \) being independent.

   Either of the following would do:

   \[
   \Pr[A \cap B] = \Pr[A] \times \Pr[B] \\
   \Pr[B | A] = \Pr[B] \\
   \Pr[A | B] = \Pr[A]
   \]
3. [5 points]
Suppose we have a biased coin that comes up heads with probability \( p \) for some \( 0 < p < 1 \). What is the probability of getting exactly \( i \) heads if we flip this biased coin \( n \) times?

\[
\binom{n}{i} p^i (1 - p)^{n-i}.
\]

4. [5 points]
You flip two coins, an unbiased nickel \( N \) with the two sides labeled 1 and 2, and a biased penny \( P \) with sides labeled 0 and 3 – where the side labeled 0 appears with probability \( \frac{3}{4} \).

(2+3 points) What is the sample space? What are the probabilities of each of the outcomes?

The sample space is

\[
\begin{align*}
(1, 0) & \text{ with probability } \frac{3}{8}, \\
(1, 3) & \text{ with probability } \frac{1}{8}, \\
(2, 0) & \text{ with probability } \frac{3}{8}, \\
(2, 3) & \text{ with probability } \frac{1}{8}.
\end{align*}
\]

5. [15 points]

Given that \( A \) and \( B \) are countable, which of the following are countable:

<table>
<thead>
<tr>
<th>( A \cup B )</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A \cap B )</td>
<td>T</td>
</tr>
<tr>
<td>( {(a, b) \mid a \in A \text{ and } b \in B} )</td>
<td>T</td>
</tr>
<tr>
<td>( \mathcal{P}(A) ) (the power set of ( A ))</td>
<td>F</td>
</tr>
<tr>
<td>The set of functions from ( A ) to ( B )</td>
<td>F</td>
</tr>
</tbody>
</table>
Reading Solutions.

This section tests whether you read the solutions that we hand out.

6. [10 points]
Suppose there are 25 different types of coupons, and suppose that each time one obtains a coupon, it is equally likely to be any one of the 25 types. Compute the expected number of different types that are contained in a set of 10 coupons. Hint: Use indicator variables and linearity of expectation.

Solution: We enumerate the coupons from 1 to 25. Let $X_i$ be the indicator random variable for the event that the $i$th coupon is among the 10 coupons. I.e., think of $X_i$ as raising its hand if the $i$th coupon appears. So, the important thing here is that the number of different coupons is just the number of hands you see, which is $\sum_{i=1}^{25} X_i$. Now,

$$E[X_i] = Pr\{\text{at least one type } i \text{ coupon is in the set of 10}\}$$
$$= 1 - Pr\{\text{no type } i \text{ coupons are in the set of 10}\}$$
$$= 1 - \left(\frac{24}{25}\right)^{10}$$

Hence, the expected number of coupons is:

$$E\left(\sum_{i=1}^{25} X_i\right) = \sum_{i=1}^{25} E(X_i) \quad \text{(by linearity of expectation)}$$
$$= 25 \left[1 - \left(\frac{24}{25}\right)^{10}\right]$$
$$\approx 8.38.$$
7. [10 points]
Draw, using only NAND, AND, OR, NOT and XOR gates, a simple 1-bit carry-adder. The inputs are two bits $a$ and $b$, and the outputs are the sum $(a + b)$ and the carry generated.

Figure 1: Half adder
**Basic Techniques.**

This part will test your ability to apply techniques that we have explicitly identified in lecture. You need to have practiced each technique enough to be able to handle small variations in the problems.

8. [15 points]

Bonzo throws 6 chocolates into two bins labeled $A$ and $B$, with each chocolate having an equal probability of landing in either bin. Odette then chooses the bin with more chocolates (picking $A$ if both bins are equally loaded).

What is the expected number of chocolates she gets?

Consider the event $E_i$ that the bin with more chocolates has $i$ chocolates in it; hence

$$\Pr[E_i] = \binom{6}{i}/2^6.$$  

Let $X$ be the random variable indicating the number of chocolates that Odette gets; let $Y = 6 - X$ be the number of chocolates that are left over. Note that for each sample point $x \in E_i$, $X(x) = \max(i, 6 - i)$, and $Y(x) = \min(i, 6 - i)$.

By linearity of expectation, the expected number of chocolates Odette gets is $E[X] = 6 - E[Y]$; so it suffices to calculate $E[Y]$, which is a bit easier, since the numbers are smaller. (It is fine to calculate $E[X]$ directly – the method is very similar to the one for $E[Y]$.)

The expected number of chocolates left over are

$$E[Y] = \sum_{i=0}^{6} \binom{6}{i}/2^6 \times \min(i, 6 - i)$$

$$= 2^{-6} \left( \binom{6}{0} + \binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} + \binom{6}{5} + \binom{6}{6} \right)$$

$$= 2^{-6} (0 + 6 + 30 + 60 + 30 + 6 + 0) = \frac{132}{64}.$$  

this is $6 - \frac{132}{64} = \frac{252}{4} = 3.9375$.

**Common Mistakes:** Many of you assumed that the expected value should be an integer – that is not true!! Another mistake was to fix a bin and look at the number of chocolates that fell into it (which is 3); but Odette is not taking a random bin – she is taking the heavier bin.
A Moment’s Thought!
This section tests your ability to think a little bit more insightfully. You must give complete explanations of your answers.

9. [20 points]
Professors Gupta and Rudich each independently pick \( k \) problems uniformly at random from a set of \( k^2 \) problems to place on an exam. How many problems do we expect will be picked by both of them? How many problems do we expect will be picked by exactly one of them?

Let \( G_i \) be the indicator variable for the event that the \( i \)-th problem is picked by Professor Gupta, and \( R_i \) be the indicator variable for the event that the \( i \)-th problem is picked by Professor Rudich. Hence, if \( B_i \) is the indicator variable for the event that the \( i \)-th problem is picked by both, then \( B_i = R_i \times G_i \). Since \( G_i \) and \( R_i \) are independent random variables,

\[
E[B_i] = E[G_i \times R_i] = E[G_i] \times E[R_i].
\]

Now, if the \( i \)-th problem is picked by Prof Gupta, then the number of ways for him to pick the other \( k-1 \) problems is \( \binom{k^2-1}{k-1} \). Since the total number of ways to pick \( k \) problems out of \( k^2 \) is \( \binom{k^2}{k} \), and each set of \( k \) problems is equally likely,

\[
E[G_i] = \Pr[i\text{-th problem is picked by Gupta}] = \frac{\binom{k^2-1}{k-1}}{\binom{k^2}{k}} = \frac{1}{k}.
\]

Similarly, \( E[R_i] = \frac{1}{k} \), giving us that \( E[B_i] = \frac{1}{k^2} \). Finally, if the random variable \( B \) denotes the number of problems picked by both, then \( B = \sum_{i=1}^{k^2} B_i \), and by the linearity of expectation

\[
E[B] = \sum_{i=1}^{k^2} E[B_i] = k^2 \times \frac{1}{k^2} = 1.
\]

Problems picked by only one: If \( Z_i \) is the indicator random variable for the event that the \( i \)-th problem is picked by only one of them, then

\[
Z_i = G_i(1-R_i) + R_i(1-G_i) = G_i + R_i - 2R_iG_i = G_i + R_i - 2B_i,
\]

and hence \( E[Z_i] = 2/k - 2/k^2 \). Thus the expected number of problems picked by exactly one of them is \( \sum_{i=1}^{k^2} E[Z_i] = k^2(2/k - 2/k^2) = 2(k-1) \) using the linearity of expectation.

Common Mistakes: You must explain why \( E[G_i] \) or \( E[B_i] \) are \( 1/k \) — just saying that it is \( k/k^2 \) is not enough.
While we don’t always expect you to write quite as detailed solutions as the one above in an exam, we do want you to tell us when you are using linearity of expectations; in that case, you must make also clear what the random variables are. Also, you cannot talk about \( \Pr[X] \) if \( X \) is an r.v., just as you cannot talk about \( E[A] \) if \( A \) is an event.
10. [10 points EXTRA CREDIT]

You are given a biased coin with some unknown chance $0 < p < 1$ of coming up heads. Describe how you can simulate a fair coin using this biased coin. You may not use any other external form of randomness.

Flip the coin twice. If you get $HT$ (which happens with probability $p(1 - p)$), say “Heads”; if you get $TH$ (which happens with the same probability $(1 - p)p$), say “Tails”. Else, you must have got $HH$ or $TT$, in which case you repeat the experiment.

Note that you dont need to know the value of $p$; all you use is that $HT$ and $TH$ are equally likely. Also, note that while there is no guarantee that you will stop after a fixed number of steps, the chance that you will require more than $k$ steps is $q^k$, where $q = p^2 + (1 - p)^2 < 1$ (since $0 < p < 1$). As $k \to \infty$, this probability tends to 0, and hence you will stop with probability 1.

(This trick is usually attributed to John von Neumann.)