Counting III: Pascal's Triangle, Polynomials, and Vector Programs

1 + X^1 + X^2 + X^3 + ... + X^n + ... = \frac{1}{1 - X}

The Geometric Series

1 + X^1 + X^2 + X^3 + ... + X^n + ... = \frac{X^{n+1} - 1}{X - 1}

The Infinite Geometric Series

1 + x + ... + \frac{1}{1 - x} = \frac{x}{1 - x^2}

Geometric Series (Linear Form)
Suppose we multiply this out to get a single, infinite polynomial.

What is an expression for $C_n$?

If $a = b$ then

$$C_n = (n+1)(a^n)$$

If $a \neq b$ then

$$C_n = \frac{a^{n+1} - b^{n+1}}{a - b}$$
\[(1 + aX + a^2X^2 + \ldots + a^nX^n + \ldots) (1 + bX + b^2X^2 + \ldots + b^nX^n + \ldots) = \frac{1}{(1 - aX)(1 - bX)} = \sum_{n=0}^{\infty} \frac{a^{n+1} - b^{n+1}}{a - b} X^n \text{ or } \sum_{n=0}^{\infty} (n+1)a^n X^n \text{ when } a = b\]

Previously, we saw that Polynomials Count!

What is the coefficient of \(BA^2N^2\) in the expansion of \((B + A + N)^6\)?

The number of ways to rearrange the letters in the word BANANA.

The Binomial Formula

\[(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k\]
One polynomial, two representations

\[(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k\]

*“Product form” or “Generating form”*

*“Additive form” or “Expanded form”*

Power Series Representation

\[(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k\]

*“Closed form” or “Generating form”*

*“Power series” (“Taylor series”) expansion*

By playing these two representations against each other we obtain a new representation of a previous insight:

\[(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k\]

Let \(x = 1\).

\[2^n = \sum_{k=0}^{n} \binom{n}{k} \]

The number of subsets of an \(n\)-element set

By varying \(x\), we can discover new identities

\[(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k\]

Let \(x = -1\).

\[0 = \sum_{k=0}^{n} \binom{n}{k} \cdot (-1)^k\]

Equivalently,

\[\sum_{k \text{ even}} \binom{n}{k} = \sum_{k \text{ odd}} \binom{n}{k} = 2^{n-1}\]

The number of even-sized subsets of an \(n\) element set is the same as the number of odd-sized subsets.

\[(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k\]

Let \(x = -1\).

\[0 = \sum_{k=0}^{n} \binom{n}{k} \cdot (-1)^k\]

Equivalently,

\[\sum_{k \text{ even}} \binom{n}{k} = \sum_{k \text{ odd}} \binom{n}{k} = 2^{n-1}\]

We could discover new identities by substituting in different numbers for \(x\). One cool idea is to try complex roots of unity, however, the lecture is going in another direction.
Proofs that work by manipulating algebraic forms are called "algebraic" arguments. Proofs that build a 1-1 onto correspondence are called "combinatorial" arguments.

\[(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k\]

A Combinatorial Proof

Let \( O_n \) be the set of binary strings of length \( n \) with an odd number of ones.
Let \( E_n \) be the set of binary strings of length \( n \) with an even number of ones.

A combinatorial proof must construct a one-to-one correspondence between \( O_n \) and \( E_n \).

An attempt at a correspondence

Let \( f_n \) be the function that takes an \( n \)-bit string and flips all its bits.

\( f_n \) is clearly a one-to-one and onto function for odd \( n \). E.g. in \( f_5 \) we have:

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>0010011</td>
<td>1010011</td>
</tr>
<tr>
<td>1001101</td>
<td>0001101</td>
</tr>
<tr>
<td>110011</td>
<td>010011</td>
</tr>
<tr>
<td>101010</td>
<td>001010</td>
</tr>
</tbody>
</table>

Uh oh. Complementing maps evens to evens!

A correspondence that works for all \( n \)

Let \( f_n \) be the function that takes an \( n \)-bit string and flips only the first bit.

For example,

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>0010011</td>
<td>1010011</td>
</tr>
<tr>
<td>1001101</td>
<td>0001101</td>
</tr>
<tr>
<td>110011</td>
<td>010011</td>
</tr>
<tr>
<td>101010</td>
<td>001010</td>
</tr>
</tbody>
</table>

The binomial coefficients have so many representations that many fundamental mathematical identities emerge...
The Binomial Formula

\[(1+X)^0 = 1\]
\[(1+X)^1 = 1 + X\]
\[(1+X)^2 = 1 + 2X + X^2\]
\[(1+X)^3 = 1 + 3X + 3X^2 + X^3\]
\[(1+X)^4 = 1 + 4X + 6X^2 + 4X^3 + X^4\]

Pascal’s Triangle:

The \(k^{th}\) row are the coefficients of \((1+X)^k\)

\[(1+X)^0 = 1\]
\[(1+X)^1 = 1 + X\]
\[(1+X)^2 = 1 + 2X + X^2\]
\[(1+X)^3 = 1 + 3X + 3X^2 + X^3\]
\[(1+X)^4 = 1 + 4X + 6X^2 + 4X^3 + X^4\]

\[\text{Inductive definition of } k\text{th entry of } n\text{th row:}\]
\[Pascal(n,0) = Pascal(n,n) = 1;\]
\[Pascal(n,k) = Pascal(n-1,k-1) + Pascal(n,k)\]

\[k^{th} \text{ Row Of Pascal’s Triangle:}\]

\[(1+X)^0 = 1\]
\[(1+X)^1 = 1 + X\]
\[(1+X)^2 = 1 + 2X + X^2\]
\[(1+X)^3 = 1 + 3X + 3X^2 + X^3\]
\[(1+X)^4 = 1 + 4X + 6X^2 + 4X^3 + X^4\]

“Pascal’s Triangle”

Al-Karaji, Baghdad 953-1029

Chu Shin-Chieh 1303

The Precious Mirror of the Four Elements

. . . Known in Europe by 1529

Blaise Pascal 1654

Pascal’s Triangle

\[\begin{array}{ccccccc}
1 & & & & & & \\
1 & 1 & & & & & \\
1 & 2 & 1 & & & & \\
1 & 3 & 3 & 1 & & & \\
1 & 4 & 6 & 4 & 1 & & \\
1 & 5 & 10 & 10 & 5 & 1 & \\
1 & 6 & 15 & 20 & 15 & 6 & 1 \\
\end{array}\]

“It is extraordinary how fertile in properties the triangle is. Everyone can try his hand.”
Summing The Rows

\[ 2^n = \sum_{k=0}^{n} \binom{n}{k} \]

\[
\begin{align*}
1 &= 1 \\
1 + 1 &= 2 \\
1 + 2 + 1 &= 4 \\
1 + 3 + 3 + 1 &= 8 \\
1 + 4 + 6 + 4 + 1 &= 16 \\
1 + 5 + 10 + 10 + 5 + 1 &= 32 \\
1 + 6 + 15 + 20 + 15 + 6 + 1 &= 64
\end{align*}
\]

Summing on 1st Avenue

\[
\sum_{i=0}^{n} \binom{n}{i} \left( \binom{n+1}{2} \right) = \sum_{i=0}^{n} \binom{n+1}{2} \binom{n}{i}
\]

Summing on kth Avenue

\[
\sum_{i=0}^{n} \binom{i}{k} \binom{n+1}{i+1} = \sum_{i=0}^{n} \binom{1}{i} \binom{n+1}{i+1}
\]
Al-Karaji Squares

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 2 + 2*1 & 1 & 4 \\
1 & 3 + 2*3 & 1 & 9 \\
1 & 4 + 2*6 & 4 & 1 \\
1 & 5 + 2*10 & 10 & 5 \\
1 & 6 + 2*15 & 20 & 15 \\
\end{array}
\]

All these properties can be proved inductively and algebraically. We will give combinatorial proofs using the Manhattan block walking representation of binomial coefficients.

How many shortest routes from A to B?

Manhattan

There are \( \binom{j+k}{k} \) shortest routes from (0,0) to (j,k).

Level n

There are \( \binom{n}{k} \) shortest routes from (0,0) to (n-k,k).
By convention:

- $0! = 1$ (empty product = 1)
- $\binom{n}{k} = 1$ if $k = 0$
- $\binom{n}{k} = 0$ if $k < 0$ or $k > n$

$$\sum_{i=0}^{n} \binom{n}{k} = 2^{n-1}$$

$$\sum_{i=1}^{n} \binom{n}{k} = \binom{2n}{n}$$

Corollary ($k = 1$)

$$\sum_{i=0}^{n} \binom{i+1}{k+1} = \frac{n(n+1)}{2}$$
Application (Al-Karaji):

\[
\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 \\
= (1 \cdot 0 + 1) + (2 \cdot 1 + 2) + (3 \cdot 2 + 3) + \cdots + (n \cdot (n-1) + n) \\
= 1 \cdot 2 + 2 \cdot 3 + \cdots + n(n-1) + \sum_{i=1}^{n} i \\
= 2 \left( \frac{2}{2} + \frac{3}{2} + \frac{4}{2} + \frac{5}{2} + \cdots + \frac{n+1}{2} \right) \\
= 2 \left( \frac{n+1}{3} \right) \cdot \left( \frac{n+1}{2} \right) \\
= \frac{(2n+1)n}{6} + \frac{1}{n} 
\]

Vector Programs

Let’s define a (parallel) programming language called VECTOR that operates on possibly infinite vectors of numbers. Each variable \( V^i \) can be thought of as:

\[
\langle *, *, *, *, *, *, \ldots \rangle \\
0 \ 1 \ 2 \ 3 \ 4 \ 5 \ \ldots . . .
\]

Vector Programs

Let \( k \) stand for a scalar constant.
\( \langle k \rangle \) will stand for the vector \( \langle k, 0, 0, 0, \ldots \rangle \)

\( \langle 0 \rangle = \langle 0, 0, 0, 0, \ldots \rangle \)

\( \langle 1 \rangle = \langle 1, 0, 0, 0, \ldots \rangle \)

\( V^i \ + \ T^i \) means to add the vectors position-wise.

\( \langle 4,2,3,\ldots \rangle + \langle 5,1,1,\ldots \rangle = \langle 9,3,4,\ldots \rangle \)

Vector Programs

RIGHT(\( V^i \)) means to shift every number in \( V^i \) one position to the right and to place a 0 in position 0.

RIGHT( \( \langle 1,2,3,\ldots \rangle \) ) = \( \langle 0,1,2,3,\ldots \rangle \)

Example:

\( V^i = \langle 6 \rangle; \)
\( \langle 13, 2, 42, 6, 0, 0, 0, \ldots \rangle \)

\( V^i = \langle 1 \rangle; \)
Loop n times:
\( V^i = \langle 1,1,0,0,\ldots \rangle \)

\( V^i = V^i + \text{RIGHT}(V^i); \)
\( V^i = \langle 1,2,1,0,\ldots \rangle \)
\( V^i = \langle 1,3,3,1,\ldots \rangle \)

\( V^i = n^{th} \text{ row of Pascal’s triangle.} \)
Programs ----> Polynomials

The vector \( \mathbf{V} = \langle a_0, a_1, a_2, \ldots \rangle \) will be represented by the polynomial:
\[
P_{\mathbf{V}} = \sum_{i=0}^{\infty} a_i X^i
\]

Vector Programs

Example:
\[
\mathbf{V} := \langle 1 \rangle; \quad P_{\mathbf{V}} := 1;
\]
Loop n times:
\[
\mathbf{V} := \mathbf{V} + \text{RIGHT}(\mathbf{V}); \quad P_{\mathbf{V}} := P_{\mathbf{V}} + P_{\mathbf{V}} X;
\]
\( \mathbf{V} \) = \( n \)th row of Pascal’s triangle.

Formal Power Series

The vector \( \mathbf{V} = \langle a_0, a_1, a_2, \ldots \rangle \) will be represented by the formal power series:
\[
P_{\mathbf{V}} = \sum_{i=0}^{\infty} a_i X^i
\]

Example:
\[
\mathbf{V} := \langle 1 \rangle; \quad P_{\mathbf{V}} := 1;
\]
Loop n times:
\[
\mathbf{V} := \mathbf{V} + \text{RIGHT}(\mathbf{V}); \quad P_{\mathbf{V}} := P_{\mathbf{V}} + P_{\mathbf{V}} \times X;
\]
\( \mathbf{V} \) = \( n \)th row of Pascal’s triangle.
Vector Programs

Example:

\[ V' := <1> ; \]

Loop \( n \) times:
\[ V' := V' + \text{RIGHT}(V') ; \]

\[ P_V = (1+X)^n \]

\[ V' = n^{th} \text{ row of Pascal's triangle.} \]