Induction: One Step At A Time
Today we will talk about INDUCTION
Induction is the primary way we:
1. Prove theorems
2. Construct and define objects
Let's start with dominoes
Domino Principle: Line up any number of dominos in a row; knock the first one over and they will all fall.
n dominoes numbered 1 to n

$F_k$  $\text{The } k^{th} \text{ domino falls}$

If we set them all up in a row then we know that each one is set up to knock over the next one:

For all $1 = k < n$: 

$$F_k \Rightarrow F_{k+1}$$
n dominoes numbered 1 to n

F_k \ '  The k^{th} domino falls
For all 1 = k < n:
  F_k  \ F_{k+1}

F_1  \ F_2  \ F_3  \ ...  
F_1  \ All Dominoes Fall
Computer Scientists don’t start numbering things at 1, they start at 0.

YOU will spend a career doing this, so GET USED TO IT NOW.
n dominoes numbered 0 to n-1

\[ F_k \quad \text{The } k^{th} \text{ domino falls} \]

For all \( 0 = k < n-1: \)

\[ F_k \quad F_{k+1} \]

\[ F_0 \quad F_1 \quad F_2 \quad \ldots \]

\[ F_0 \quad \text{All Dominoes Fall} \]
Standard Notation/Abbreviation
“for all” is written “\(\forall\)"

Example:

For all \(k > 0\), \(P(k)\)
is equivalent to
\(\forall k > 0, P(k)\)
n dominoes numbered 0 to n-1

\[ F_k \quad \text{The } k^{th} \text{ domino falls} \]

\[ 8k, 0 = k < n-1: \]

\[ F_k \quad F_{k+1} \]

\[ F_0 \quad F_1 \quad F_2 \quad \ldots \quad F_0 \quad \text{All Dominoes Fall} \]
The Natural Numbers

\[ \mathbb{N} = \{0, 1, 2, 3, \ldots\} \]
The Natural Numbers

\[ \mathbb{N} = \{ 0, 1, 2, 3, \ldots \} \]

One domino for each natural number:
The Infinite Domino Principle

$F_k$  \ The $k^{th}$ domino falls

Suppose $F_0$
Suppose for each natural number $k$,

$F_k \rightarrow F_{k+1}$

Then All Dominoes Fall!

$F_0 \rightarrow F_1 \rightarrow F_2 \rightarrow \ldots$
The Infinite Domino Principle

$F_k$ - The $k^{th}$ domino falls

Suppose $F_0$

Suppose for each natural number $k$, $F_k \implies F_{k+1}$

Then All Dominoes Fall!

Proof: If they do not all fall, there must be a least numbered domino $d > 0$ that did not fall. Hence, $F_{d-1}$ and not $F_d$. $F_{d-1} \implies F_d$.

Hence, domino $d$ fell and did not fall. Contradiction.
Mathematical Induction: statements proved instead of dominoes fallen

Infinite sequence of statements: $S_0, S_1, \ldots$

- Infinite sequence of dominoes.
- $F_k$ dominates $k$ falls
- $F_k$ dominates $S_k$ proved

Establish

1) $F_0$
2) $\forall k, F_k \Rightarrow F_{k+1}$

Conclude that $F_k$ is true for all $k$
Inductive Proof / Reasoning
To Prove $\forall k, S_k$

Establish “Base Case”: $S_0$
Establish “Domino Property”: $\forall k, S_k \rightarrow S_{k+1}$

$\forall k, S_k \rightarrow S_{k+1}$

Assume hypothetically that $S_k$ for any particular $k$;

Conclude that $S_{k+1}$
Inductive Proof / Reasoning
To Prove $\forall k, S_k$

Establish "Base Case": $S_0$
Establish "Domino Property": $\forall k, S_k \implies S_{k+1}$

$\forall k, S_k \implies S_{k+1}$

"Induction Hypothesis" $S_k$

Use I.H. to show $S_{k+1}$
Inductive Proof / Reasoning
To Prove $\forall k \geq b, S_k$

Establish “Base Case”: $S_b$
Establish “Domino Property”: $\forall k \geq b, S_k \rightarrow S_{k+1}$

Assume $k \geq b$
Assume “Inductive Hypothesis”: $S_k$
Prove that $S_{k+1}$ follows
Theorem?

The sum of the first $n$ odd numbers is $n^2$. 
Theorem?
The sum of the first $n$ odd numbers is $n^2$.

CHECK IT OUT ON SMALL VALUES:

1 = 1
1+3 = 4
1+3+5 = 9
1+3+5+7 = 16
Theorem: The sum of the first $n$ odd numbers is $n^2$.

The $k^{th}$ odd number is expressed by the formula $(2k - 1)$, when $k > 0$. 
$S_n \equiv$

“The sum of the first $n$ odd numbers is $n^2$.”

Equivalently, $S_n$ is the statement that:

$$\sum_{1 \leq k \leq n} (2k-1) = 1 + 3 + 5 + (2k-1) + \ldots + (2n-1) = n^2$$
$S_n \equiv \text{"The sum of the first } n \text{ odd numbers is } n^2."$

$1 + 3 + 5 + (2k-1) + \ldots + (2n-1) = n^2$

Trying to establish that: $8n_1 \leq 1$ $S_n$

Base case: $S_1$ is true

The sum of the first 1 odd numbers is 1.
$S_n \equiv \text{"The sum of the first } n \text{ odd numbers is } n^2.\"$

"$1 + 3 + 5 + (2k-1) + \ldots +(2n-1)= n^2"$

Trying to establish that: $8n+1 \leq S_n$

Assume "Induction Hypothesis": $S_k$
(for any particular $k \geq 1$)

$1+3+5+\ldots+(2k-1) = k^2$
\[ S_n \equiv \text{"The sum of the first } n \text{ odd numbers is } n^2." \]
\[ 1 + 3 + 5 + (2k-1) + \ldots + (2n-1) = n^2 \]

Trying to establish that: \( 8n, 1 \leq S_n \)

Assume “Induction Hypothesis”: \( S_k \) (for any particular \( k \geq 1 \))

\[
1 + 3 + 5 + \ldots + (2k-1) = k^2
\]

Add \((2k+1)\) to both sides.

\[
1 + 3 + 5 + \ldots + (2k-1) + (2k+1) = k^2 + (2k+1)
\]

Sum of first \( k + 1 \) odd numbers = \((k+1)^2\)

CONCLUDE: \( S_{k+1} \)
\( S_n \equiv \text{“The sum of the first } n \text{ odd numbers is } n^2.\) 
\[ 1 + 3 + 5 + (2k-1) + \ldots + (2n-1) = n^2 \]

Trying to establish that: \( 8n \leq 1 \cdot S_n \)

Established base case: \( S_1 \)

Established domino property: \( 8k \leq 1 \cdot S_k \implies S_{k+1} \)

By induction of } n, \text{ we conclude that:} 
\[ 8n \leq 1 \cdot S_n \]
THEOREM:

The sum of the first $n$ odd numbers is $n^2$. 
Theorem?
The sum of the first \( n \) numbers is \( \frac{1}{2}n(n+1) \).
Theorem? The sum of the first $n$ numbers is $\frac{1}{2}n(n+1)$.

Try it out on small numbers!

1 $\quad = \quad 1 \quad = \quad \frac{1}{2} \quad 1(1+1)$.

1+2 $\quad = \quad 3 \quad = \quad \frac{1}{2} \quad 2(2+1)$.

1+2+3 $\quad = \quad 6 \quad = \quad \frac{1}{2} \quad 3(3+1)$.

1+2+3+4 $\quad = \quad 10 \quad = \quad \frac{1}{2} \quad 4(4+1)$. 
Theorem? The sum of the first n numbers is \( \frac{1}{2}n(n+1) \).

\[
\begin{align*}
0 &= \frac{1}{2} \cdot 0(0+1) \\
1 &= 1 = \frac{1}{2} \cdot 1(1+1) \\
1+2 &= 3 = \frac{1}{2} \cdot 2(2+1) \\
1+2+3 &= 6 = \frac{1}{2} \cdot 3(3+1) \\
1+2+3+4 &= 10 = \frac{1}{2} \cdot 4(4+1).
\end{align*}
\]
Notation:
\[ \Delta_0 = 0 \]
\[ \Delta_n = 1 + 2 + 3 + \ldots + n-1 + n \]

Let \( S_n \)

“\( \Delta_n = \frac{n(n+1)}{2} \)”
Use induction to prove $\forall k \geq 0, S_k$

**Establish “Base Case”:** $S_0, \Delta_0$ = The sum of the first 0 numbers = 0. Setting $n=0$ the formula gives $0(0+1)/2 = 0$.

**Establish “Domino Property”:** $\forall k \geq 0, S_k$) $S_{k+1}$

“Inductive Hypothesis” $S_k$: $\Delta_k = k(k+1)/2$

$\Delta_{k+1} = \Delta_k + (k+1)$

$= k(k+1)/2 + (k+1)$ [Using I.H.]

$= (k+1)(k+2)/2$ [which proves $S_{k+1}$]
THEOREM:

The sum of the first $n$ numbers is $\frac{1}{2}n(n+1)$.
A natural number $n > 1$ is prime if it has no divisors besides 1 and itself.

N.B. 1 is not considered prime.
Easy theorem:
Every natural number $> 1$ can be factored into primes.

N.B.:
It is much more subtle to argue for the existence of a unique prime factorization.
Easy theorem:
Every natural number $>1$ can be factored into primes.
$S_n = \text{“}n \text{ can be factored into primes} \text{“}$

$S_2$ is true because 2 is prime.
Every natural number \( \geq 1 \) can be factored into primes. Base case: 2

Assume 2, 3, \ldots, k-1 all can be factored into primes. Show k can be factored into primes.
Assume 2, 3, ......, k-1 all can be factored into primes. Show k can be factored into primes.

If k is prime, we are done. If not, k = ab where 1 < a, b < k, hence a and b can be factored into primes. Thus, k is the product of the factors of a and the factors of b.
This illustrates a technical point about using and defining mathematical induction.
To Prove $\forall k, S_k$

Establish "Base Case": $S_0$

Establish that $\forall k, S_k$ ) $S_{k+1}$

Let $k$ be any natural number.

Induction Hypothesis:

Assume $\forall j<k, S_j$

Derive $S_k$
"Strong" Induction
To Prove \( \forall k, S_k \)

Establish "Base Case": \( S_0 \)

Establish that \( \forall k, S_k \) \( S_{k+1} \)

Let \( k \) be any natural number.

Assume \( \forall j < k, S_j \)

Prove \( S_k \)
Least Counter-Example Induction to Prove $\forall k, S_k$

Establish “Base Case”: $S_0$
Establish that $\forall k, S_k \Rightarrow S_{k+1}$

Assume that $S_k$ is the least counter-example.

Derive the existence of a smaller counter-example
All numbers > 1 has a prime factorization.

Let $n$ be the least counter-example. $n$ must not be prime - so $n = ab$. If both $a$ and $b$ had prime factorizations, then $n$ would. Thus either $a$ or $b$ is a smaller counter-example.
Inductive reasoning is the high level idea:

“Standard” Induction

“Least Counter-example”

“All Previous” Induction

all just different packaging.
"All Previous" Induction Can Be Repackaged As Standard Induction

Establish "Base Case": $S_0$

Establish that $\forall k, S_k \rightarrow S_{k+1}$
Let $k$ be any natural number.
Assume $\forall j < k, S_j$
Prove $S_k$

Define $T_i = \forall j \cdot i, S_j$

Establish "Base Case": $T_0$

Establish that $\forall k, T_k \rightarrow T_{k+1}$
Let $k$ be any natural number.
Assume $T_{k-1}$
Prove $T_k$
Induction is also how we can define and construct our world.
So many things, from buildings to computers, are built up stage by stage, module by module, each depending on the previous stages.
Well, almost always
Inductive Definition Of Functions

Stage 0, Initial Condition, or Base Case:
Declare the value of the function on some subset of the domain.

Inductive Rules
Define new values of the function in terms of previously defined values of the function.

F(x) is defined if and only if it is implied by finite iteration of the rules.
Inductive Definition Of Functions

Stage 0, Initial Condition, or Base Case:
Declare the value of the function on some subset of the domain.

Inductive Rules
Define new values of the function in terms of previously defined values of the function.

If there is an x such that F(x) has more than one value – then the whole inductive definition is said to be inconsistent.
Inductive Definition
Recurrence Relation for $F(X)$

**Initial Condition, or Base Case:**
$F(0) = 1$

**Inductive Rule**
For $n > 0$, $F(n) = F(n-1) + F(n-1)$

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Inductive Definition
Recurrence Relation for $F(X) = 2^X$

**Initial Condition, or Base Case:**
$F(0) = 1$

**Inductive Rule**
For $n>0$, $F(n) = F(n-1) + F(n-1)$

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**Recurrence Relation**

**Initial Condition, or Base Case:**
\[ F(1) = 1 \]

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For \( n > 1 \), \( F(n) = F(n/2) + F(n/2) \)

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**Inductive Definition**

Recurrence Relation

\[ F(X) = X \text{ for } X \text{ a whole power of 2.} \]

**Initial Condition, or Base Case:**

\[ F(1) = 1 \]

**Inductive Rule**

For \( n > 1 \), \( F(n) = F(n/2) + F(n/2) \)

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Base Case: $8 \times 2 \mathbb{N}$ \( P(X,0) = X \)

Inductive Rule: $8 \times y \mathbb{N}$, \( y > 0 \), \( P(x,y) = P(x,y-1) + 1 \)

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$8 \times y 2 \mathbb{N}, y > 0, P(x, y) = P(x, y-1) + 1$

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Inductive Rule:
\(8 \times, y \mathbb{N}, y > 0, P(x,y) = P(x,y-1) + 1\)

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Inductive Rule:
$8 \times y, y \in 2 \mathbb{N}$, $y > 0$, $P(x, y) = P(x, y-1) + 1$

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Definition of $P$:

$$8 \times 2^\mathbb{N} P(X,0) = X$$

$$8x, y \in 2^\mathbb{N}, y>0, P(x,y) = P(x,y-1) + 1$$

Any inductive definition with a finite number of base cases, can be translated into a program. The program simply calculates from the base cases on up.
Definition of $P$:

$8 \times 2 \{0, 1, 2, 3\}$  $P(X, 0) = X$

$8 \times y \mathbb{N}, y > 0$, $P(x, y) = P(x, y-1) + 1$

What would be the bottom up implementation of $P$?
For \( k = 0 \) to 3
\[ P(k,0) = k \]
For \( j = 1 \) to 7
For \( k = 0 \) to 3
\[ P(k,j) = P(k,j-1) + 1 \]

Bottom-Up Program for \( P \)

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Suppose we wanted to know $P(2,3)$ in particular, but we had not yet done any calculation.
**Base Case:** $8 \times 2 \mathbb{N}$ \( P(X, 0) = X \)

**Inductive Rule:**

\( 8 \times y \mathbb{N}, y > 0, P(x, y) = P(x, y-1) + 1 \)

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**Base Case:** \(8 \times 2^N\) \(P(X,0) = X\)

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\(8 \times y \in \mathbb{N}, y > 0, P(x,y) = P(x,y-1) + 1\)

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\( 8 \times y 2^{\mathbb{N}}, y > 0, P(x,y) = P(x,y-1) + 1 \)

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Procedure $P(x,y)$:  
If $y=0$ return $x$  
Otherwise return $P(x,y-1)+1$;

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Inductive Definition:
\[ 8 \times 2^N \ P(X,0) = X \]
\[ 8 \times Y \in 2^N, \ Y > 0, \ P(X,Y) = P(X,Y-1) + 1 \]

Bottom-Up, Iterative Program:
For k = 0 to 3
  P(k,0) = k
For j = 1 to 7
  For k = 0 to 3
    P(k,j) = P(k,j-1) + 1

Top-Down, Recursive Program:
Procedure P(X,Y):
  If Y = 0 return X
  Otherwise return P(X,Y-1)+1
Leonardo Fibonacci

In 1202, Fibonacci proposed a problem about the growth of rabbit populations.
Leonardo Fibonacci

In 1202, Fibonacci proposed a problem about the growth of rabbit populations.
Leonardo Fibonacci

In 1202, Fibonacci proposed a problem about the growth of rabbit populations.
The rabbit reproduction model

• A rabbit lives forever
• The population starts as a single newborn pair
• Every month, each productive pair begets a new pair which will become productive after 2 months old

\[ F_n = \# \text{ of rabbit pairs at the beginning of the } n^{th} \text{ month} \]

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13853211 rabbits
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The rabbit reproduction model

- A rabbit lives forever
- The population starts as a single newborn pair
- Every month, each productive pair begets a new pair which will become productive after 2 months old

\[ F_n = \text{# of rabbit pairs at the beginning of the } n\text{th month} \]

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Inductive Definition or Recurrence Relation for the Fibonacci Numbers

Stage 0, Initial Condition, or Base Case:
Fib(1) = 1; Fib (2) = 1

Inductive Rule
For n>3, Fib(n) = Fib(n-1) + Fib(n-2)

<table>
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<tr>
<th>n</th>
<th>0</th>
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<th>2</th>
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<td>Fib(n)</td>
<td>%</td>
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Inductive Definition or Recurrence Relation for the Fibonacci Numbers

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For \( n > 1 \), \( Fib(n) = Fib(n-1) + Fib(n-2) \)

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Top-Down, Recursive Program:
Procedure Fib(k)
    If k=0 return 0
    If k=1 return 1
    Otherwise return Fib(k-1)+Fib(k-2);

Inductive Definition:
Fib(0)=0, Fib(1)=1, k>1, Fib(k)=Fib(k-1)+Fib(k-2)

Bottom-Up, Iterative Program:
Fib(0) = 0; Fib(1) =1;
Input x;
For k= 2 to x do Fib(x)=Fib(x-1)+Fib(x-2);
Return Fib(k);
What is a closed form formula for $\text{Fib}(n)$? 

**Stage 0, Initial Condition, or Base Case:**
$\text{Fib}(0) = 0$; $\text{Fib}(1) = 1$

**Inductive Rule**
For $n > 1$, $\text{Fib}(n) = \text{Fib}(n-1) + \text{Fib}(n-2)$

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Leonhard Euler (1765)
J. P. M. Binet (1843)
August de Moivre (1730)

\[ F_{3n} = \frac{\phi^{5+1}}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right) ^ n + \frac{\phi^{-5-1}}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right) ^ n \]
Study Bee

Inductive Proof
  Standard Form
  All Previous Form
  Least-Counter Example Form
  Invariant Form

Inductive Definition
  Bottom-Up Programming
  Top-Down Programming
  Recurrence Relations
  Solving a Recurrence