Inductive Reasoning
Lecture 2 (August 28, 2008)

1. prove theorems
2. construct and define objects
**Dominoes**

**Domino Principle:**
Line up any number of dominos in a row; knock the first one over and they will all fall.

---

**n dominoes numbered 1 to n**

\[ F_k \equiv \text{The } k^{\text{th}} \text{ domino falls} \]

For all \( 1 \leq k < n \):
\[ F_k \Rightarrow F_{k+1} \]

\[ F_1 \Rightarrow F_2 \Rightarrow F_3 \Rightarrow \ldots \]

\[ F_1 \Rightarrow \text{All Dominos Fall} \]

---

**n dominoes numbered 0 to n-1**

\[ F_k \equiv \text{The } k^{\text{th}} \text{ domino falls} \]

For all \( 0 \leq k < n-1 \):
\[ F_k \Rightarrow F_{k+1} \]

\[ F_0 \Rightarrow F_1 \Rightarrow F_2 \Rightarrow \ldots \]

\[ F_0 \Rightarrow \text{All Dominos Fall} \]
The Natural Numbers

\[ \mathbb{N} = \{0, 1, 2, 3, \ldots\} \]

One domino for each natural number:

Plato: The Domino Principle works for an infinite row of dominoes

Aristotle: Never seen an infinite number of anything, much less dominoes.

Theorem: An infinite row of dominoes, one domino for each natural number. Knock over the first domino and they all will fall

Proof:

Suppose they don’t all fall. Let \( k > 0 \) be the lowest numbered domino that remains standing. Domino \( k-1 \geq 0 \) did fall, but \( k-1 \) will knock over domino \( k \). Thus, domino \( k \) must fall and remain standing. Contradiction.
Mathematical Induction

Statements proved instead of dominoes fallen

Infinite sequence of dominoes
Infinite sequence of statements: \( S_0, S_1, \ldots \)

\( F_k = \text{“domino } k \text{ fell”} \)

\( F_k = \text{“} S_k \text{ proved”} \)

Establish:
1. \( F_0 \)
2. For all \( k \), \( F_k \Rightarrow F_{k+1} \)
Conclude that \( F_k \) is true for all \( k \)

Inductive Proofs

To Prove \( \forall k \in \mathbb{N}, S_k \)

1. Establish “Base Case”: \( S_0 \)
2. Establish that \( \forall k, S_k \Rightarrow S_{k+1} \)

To prove \( \forall k, S_k \Rightarrow S_{k+1} \)
Assume hypothetically that \( S_k \) for any particular \( k \);
Conclude that \( S_{k+1} \)

Theorem?

The sum of the first \( n \) odd numbers is \( n^2 \)

Check on small values:
\[
\begin{align*}
1 & = 1^2 \\
1+3 & = 4 = 2^2 \\
1+3+5 & = 9 = 3^2
\end{align*}
\]

Theorem?

The sum of the first \( n \) odd numbers is \( n^2 \)

Check on small values:
\[
\begin{align*}
1 & = 1 \\
1+3 & = 4 \\
1+3+5 & = 9 \\
1+3+5+7 & = 16
\end{align*}
\]
Theorem
The sum of the first n odd numbers is $n^2$

The $k^{th}$ odd number is $(2k - 1)$, when $k > 0$

$S_n$ is the statement that:
\[1 + 3 + 5 + (2k-1) + ... + (2n-1) = n^2\]

Establishing that $\forall n \geq 1 \ S_n$

$S_n = "1 + 3 + 5 + (2k-1) + ... + (2n-1) = n^2"$

Base Case: $S_1$

Domino Property:
Assume "Induction Hypothesis": $S_k$
That means:
\[1 + 3 + 5 + ... + (2k-1) = k^2\]
\[1 + 3 + 5 + ... + (2k-1) + (2k+1) = k^2 + (2k+1)\]
Sum of first $k+1$ odd numbers = $(k+1)^2$

Establishing that $\forall n \geq 1 \ S_n$

$S_n = "1 + 3 + 5 + (2k-1) + ... + (2n-1) = n^2"

Base case: $n=1$, $S_1 = 1 = 1^2 = n^2$ ✓

$\forall k \geq 1 \ S_k \Rightarrow S_{k+1}$

Induction Hypothesis (IH): $S_k = "1 + 3 + 5 + ... + (2k-1) = k^2"$

Induction Step:
\[1 + 3 + 5 + ... + (2k-1) + (2k+1) = k^2 + (2k+1)\]
\[= (k+1)^2\] (by IH.)

Theorem
The sum of the first n odd numbers is $n^2$
Inductive Proofs

To Prove $\forall k \in \mathbb{N}, S_k$

1. Establish “Base Case”: $S_0$
2. Establish that $\forall k, S_k \Rightarrow S_{k+1}$

To prove $\forall k, S_k \Rightarrow S_{k+1}$

Assume hypothetically that $S_k$ for any particular $k$;

Conclude that $S_{k+1}$

Primes:
A natural number $n > 1$ is a prime if it has no divisors besides 1 and itself
Note: 1 is not considered prime

Theorem?
Every natural number $n > 1$ can be factored into primes

$S_n =$ “$n$ can be factored into primes”

Base case:
2 is prime $\Rightarrow$ $S_2$ is true

How do we use the fact: $S_{k-1} \Rightarrow S_k$

$S_{k-1} =$ “$k$-1 can be factored into primes”

to prove that:

$S_k = $ “$k$ can be factored into primes”

This shows a technical point about mathematical induction
A different approach:
Assume 2, 3, ..., k-1 all can be factored into primes.
Then show that k can be factored into primes:

Either k is prime

\[ k = p \cdot q \]

or factored into primes.

All Previous Induction
To Prove \( \forall k, S_k \)

Establish Base Case: \( S_0 \)

Establish Domino Effect:
Assume \( \forall j < k, S_j \)
use that to derive \( S_k \)

“All Previous” Induction
Repackaged As Standard Induction

Define \( T_i = \forall j \leq i, S_j \)

Establish Base Case: \( T_0 \)

Establish that
\( \forall k, T_k \implies T_{k+1} \)

Let k be any number
Assume \( \forall j < k, S_j \)
Prove \( S_k \)

Prove \( T_k \)
Regular Induction
All-previous Induction
And there are more ways to do inductive proofs

Method of Infinite Descent
Pierre de Fermat
Show that for any counter-example you can find a smaller one
Hence, if a counter-example exists there would be an infinite sequence of smaller and smaller counter examples

Theorem:
Every natural number > 1 can be factored into primes

Let n be a counter-example
Hence n is not prime, so n = ab
If both a and b had prime factorizations, then n would too
Thus a or b is a smaller counter-example
Regular Induction

All-previous Induction

Infinite Descent

And one more way of packaging induction...

Invariant (n):

1. Not varying; constant.
2. Mathematics. Unaffected by a designated operation, as a transformation of coordinates.

Invariant (n):

   A rule, such as the ordering of an ordered list, that applies throughout the life of a data structure or procedure. Each change to the data structure maintains the correctness of the invariant

Invariant Induction

Suppose we have a time varying world state: $W_0, W_1, W_2, \ldots$

Each state change is assumed to come from a list of permissible operations. We seek to prove that statement $S$ is true of all future worlds

Argue that $S$ is true of the initial world

Show that if $S$ is true of some world – then $S$ remains true after one permissible operation is performed
Odd/Even Handshaking Theorem

At any party at any point in time define a person’s parity as ODD/EVEN according to the number of hands they have shaken.

Statement: The number of people of odd parity must be even.

Statement: The number of people of odd parity must be even

Initial case: Zero hands have been shaken at the start of a party, so zero people have odd parity.

Invariant Argument:

If 2 people of the same parity shake, they both change and hence the odd parity count changes by 2 – and remains even.

If 2 people of different parities shake, then they both swap parities and the odd parity count is unchanged.

Inductive reasoning is the high level idea

“Standard” Induction
“All Previous” Induction
“Least Counter-example”
“Invariants”
all just different packaging

One more useful tip…
Here’s another problem

Let $A_m = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^m$  $A_4 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$
Prove that all entries of $A_m$ are at most $m \geq 1$

Base case: $A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$
all entries $\leq 1$

IH. $A_m = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $a, b, c, d \leq m$

$A_{m+1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b + c \\ c & d \end{pmatrix} \checkmark$

So, is it false?

$A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  $A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  $A_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Prove a stronger statement!

Claim: $A_m = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$

Base case: $m = 1$  $A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ \checkmark

IH. $A_m = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$

$A_{m+1} = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2m \\ 0 & 1 \end{pmatrix}$ \checkmark

Corollary: All entries of $A_m$ are at most $m$.

Often, to prove a statement inductively you may have to prove a stronger statement first!
Using induction to define mathematical objects

Induction is also how we can define and construct our world

So many things, from buildings to computers, are built up stage by stage, module by module, each depending on the previous stages

Inductive Definition

Example

Initial Condition, or Base Case:
F(0) = 1

Inductive Rule:
For n > 0, F(n) = F(n-1) + F(n-1)

Leonardo Fibonacci

In 1202, Fibonacci proposed a problem about the growth of rabbit populations

<table>
<thead>
<tr>
<th>n</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>F(n)</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>16</td>
<td>32</td>
<td>64</td>
<td>128</td>
</tr>
</tbody>
</table>
**Rabbit Reproduction**

A rabbit lives forever

The population starts as a single newborn pair

Every month, each productive pair begets a new pair which will become productive after 2 months old

\[ F_n = \text{# of rabbit pairs at the beginning of the } n^{\text{th}} \text{ month} \]

<table>
<thead>
<tr>
<th>month</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>rabbits</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td>13</td>
</tr>
</tbody>
</table>

**Fibonacci Numbers**

Stage 0, Initial Condition, or Base Case:

\[ \text{Fib}(1) = 1; \text{Fib}(2) = 1 \]

Inductive Rule:

For \( n > 3 \), \[ \text{Fib}(n) = \text{Fib}(n-1) + \text{Fib}(n-2) \]

**Example**

If you define a function inductively, it is usually easy to prove it’s properties using induction!

Theorem?: \[ F_1 + F_2 + \ldots + F_n = F_{n+2} - 1 \]
Example

Theorem?: $F_1 + F_2 + \ldots + F_n = F_{n+2} - 1$

Example

Theorem?: $F_1 + F_2 + \ldots + F_n = F_{n+2} - 1$

Base cases: $n=1$, $F_1 = F_3 - 1$

$n=2$, $F_1 + F_2 = F_4 - 1$

I.H.: True for all $n < k$.

Induction Step: $F_1 + F_2 + \ldots + F_k$

$= (F_1 + F_2 + \ldots + F_{k-1}) + F_k$

$= (F_{k+1} - 1) + F_k$ (by I.H.)

$= F_{k+2} - 1$ (by defn.)

Another Example

$T(1) = 1$

$T(n) = 4T(n/2) + n$

Notice that $T(n)$ is inductively defined only for positive powers of 2, and undefined on other values

$T(1) = 1$  $T(2) = 6$  $T(4) = 28$  $T(8) = 120$

Guess a closed-form formula for $T(n)$

Guess: $G(n) = 2n^2 - n$

Inductive Proof of Equivalence

Base Case: $G(1) = 1$ and $T(1) = 1$

Induction Hypothesis:

$T(x) = G(x)$ for $x < n$

Hence: $T(n/2) = G(n/2) = 2(n/2)^2 - n/2$

$T(n) = 4 T(n/2) + n$

$= 4 G(n/2) + n$

$= 4 [2(n/2)^2 - n/2] + n$

$= 2n^2 - 2n + n$

$= 2n^2 - n$

$= G(n)$

$G(n) = 2n^2 - n$

$T(1) = 1$

$T(n) = 4T(n/2) + n$
We inductively proved the assertion that $G(n) = T(n)$

Giving a formula for $T$ with no recurrences is called “solving the recurrence for $T$”

**Technique 2**

**Guess Form, Calculate Coefficients**

\[
T(1) = 1, \quad T(n) = 4 \ T(n/2) + n
\]

Guess: $T(n) = an^2 + bn + c$

for some $a, b, c$

Calculate: $T(1) = 1$, so $a + b + c = 1$

\[
T(n) = 4 \ T(n/2) + n
\]

\[
an^2 + bn + c = 4 \ [a(n/2)^2 + b(n/2) + c] + n
\]

\[
= an^2 + 2bn + 4c + n
\]

\[
(b+1)n + 3c = 0
\]

Therefore: $b = -1 \quad c = 0 \quad a = 2$

**The Lindenmayer Game**

Alphabet: \{a, b\}

Start word: a

Productions Rules:

- $Sub(a) = ab$
- $Sub(b) = a$

Next word: $Next(w_1 w_2 \ldots w_n) = Sub(w_1) \ Sub(w_2) \ldots \ Sub(w_n)$

Time 1: a

Time 2: ab

Time 3: aba

Time 4: abaa

Time 5: abaaababa

How long are the strings at time $n$?

Fibonacci($n$)
The Koch Game

Alphabet: \{ F, +, - \}
Start word: F
Productions Rules:
Sub(F) = F+F--F+F
Sub(+) = +
Sub(-) = -
NEXT(w_1 w_2 \ldots w_n) =
Sub(w_1) Sub(w_2) \ldots Sub(w_n)

Time 0: F
Time 1: F+F--F+F
Time 2: F+F--F+F+F+F--F+F+F--F+F+F+F+F--F+F

Visual representation:
- F draw forward one unit
- + turn 60 degree left
- - turn 60 degrees right
**Dragon Game**

Sub(X) = X+YF+
Sub(Y) = -FX-Y

**Hilbert Game**

Sub(L) = +RF-LFL-FLR+
Sub(R) = -LF+RFR+FLR-

Note: Make 90 degree turns instead of 60 degrees

**Peano-Gossamer Curve**
Sierpinski Triangle

Lindenmayer (1968)

Sub(F) = F[-F]F[+F][F]

Interpret the stuff inside brackets as a branch

Inductive Proof
Standard Form
All Previous Form
Least-Counter Example Form
Invariant Form

Strengthening the Inductive Claim

Inductive Definition
Recurrence Relations
Fibonacci Numbers
Guess and Verify

Here’s What You Need to Know…