A Take-Away Game

Two Players: I and II
A move consists of removing one, two, or three chips from the pile
Players alternate moves, with Player I starting
The player that removes the last chip wins

Which player would you rather be?

Try Small Examples!

If there were 1, 2, or 3 only, player who moves next wins

If there are 4 chips left, player who moves next must leave 1, 2 or 3 chips left, and his opponent will be able to win

If there are 5, 6 or 7 chips left, the player who moves next can win by moving to the position with four chips left
0, 4, 8, 12, 16, … are target positions; if a player moves to that position, they can win the game.

Therefore, with 21 chips, Player I can win!

Combinatorial Games
There are two players
There is a finite set of possible positions
The rules of the game specify for both players and each position which moves to other positions are legal moves.
The players alternate moving
The game ends in a finite number of moves (no draws!)

Normal Versus Misère
Normal Play Rule: The last player to move wins
Misère Play Rule: The last player to move loses
A Terminal Position is one where neither player can move anymore

What is Omitted
No random moves  
(This rules out games like poker)
No hidden moves  
(This rules out games like battleship)
No draws in a finite number of moves  
(This rules out tic-tac-toe)
P-Positions and N-Positions

P-Position: Positions that are winning for the Previous player (the player who just moved)

N-Position: Positions that are winning for the Next player (the player who is about to move)

0, 4, 8, 12, 16, … are P-positions; if a player moves to that position, they can win the game.

21 chips is an N-position

What’s a P-Position?

“Positions that are winning for the Previous player (the player who just moved)”

That means:

For any move that N makes

There exists a move for P such that

For any move that N makes

There exists a move for P such that

:

There exists a move for P such that

There are no possible moves for N

P-positions and N-positions can be defined recursively by the following:

1. All terminal positions are P-positions
2. From every N-position, there is at least one move to a P-position
3. From every P-position, every move is to an N-position
Chomp!

Two-player game, where each move consists of taking a square and removing it and all squares to the right and above.

Player who takes position (1,1) loses

Show That This is a P-position

N-Positions!

Show That This is an N-position

P-position!

Let’s Play! I’m player 1
No matter what you do, I can mirror it!

Mirroring is an extremely important strategy in combinatorial games!

Theorem: Player I can win in any square starting position of Chomp

Proof:

The winning strategy for player I is to chomp on (2,2), leaving only an “L” shaped position

Then, for any move that Player II takes, Player I can simply mirror it on the flip side of the “L”

Theorem: Player I can win in any rectangular starting position

Proof:

Look at this first move:

If this is a P-position, then player 1 wins
Otherwise, there exists a P-position that can be obtained from this position
And player 1 could have just taken that move originally
The Game of Nim

Two players take turns moving

Each move consists of selecting one of the piles and removing chips from it (you can take as many as you want, but you have to at least take one)

In one move, you cannot remove chips from more than one pile

Winner is the last player to remove chips

We use \((x,y,z)\) to denote this position

\((0,0,0)\) is a P-position

Analyzing Simple Positions

One-Pile Nim

What happens in positions of the form \((x,0,0)\)??

The first player can just take the entire pile, so \((x,0,0)\) is an N-position

Two-Pile Nim

P-positions are those for which the two piles have an equal number of chips

If it is the opponent’s turn to move from such a position, he must change to a position in which the two piles have different number of chips

From a position with an unequal number of chips, you can easily go to one with an equal number of chips (perhaps the terminal position)
Nim-Sum
The nim-sum of two non-negative integers is their addition without carry in base 2
We will use $\oplus$ to denote the nim-sum

$2 \oplus 3 = (10)_2 \oplus (11)_2 = (01)_2 = 1$
$5 \oplus 3 = (101)_2 \oplus (011)_2 = (110)_2 = 6$
$7 \oplus 4 = (111)_2 \oplus (100)_2 = (011)_2 = 3$

$\oplus$ is associative: $(a \oplus b) \oplus c = a \oplus (b \oplus c)$
$\oplus$ is commutative: $a \oplus b = b \oplus a$

For any non-negative integer $x$,

$x \oplus x = 0$

Cancellation Property Holds
If $x \oplus y = x \oplus z$
Then $x \oplus x \oplus y = x \oplus x \oplus z$
So $y = z$

Bouton’s Theorem: A position $(x,y,z)$ in Nim is a P-position if and only if $x \oplus y \oplus z = 0$

Proof:
Let $Z$ denote the set of Nim positions with nim-sum zero
Let $NZ$ denote the set of Nim positions with non-zero nim-sum
We prove the theorem by proving that $Z$ and $NZ$ satisfy the three conditions of P-positions and N-positions
(1) All terminal positions are in Z
   The only terminal position is (0,0,0)
(2) From each position in NZ, there is a move to a position in Z

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Look at leftmost column with an odd # of 1s
Rig any of the numbers with a 1 in that column so that everything adds up to zero

(3) Every move from a position in Z is to a position in NZ
If (x,y,z) is in Z, and x is changed to x′ < x, then we cannot have
   \[ x \oplus y \oplus z = 0 = x' \oplus y \oplus z \]
Because then x = x′

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Combinatorial Games
- P-positions versus N-positions
- When there are no draws, every position is either P or N

Nim
- Definitions of the game
- Nim-sum
- Bouton’s Theorem

Here’s What You Need to Know...