Unary and Binary
How to play the 9 stone game?

9 stones, numbered 1-9
Two players alternate moves.
Each move a player gets to take a new stone
Any subset of 3 stones adding to 15, wins.
For enlightenment, let’s look to ancient China in the days of Emperor Yu.

A tortoise emerged from the river Lo...
Magic Square: Brought to humanity on the back of a tortoise from the river Lo in the days of Emperor Yu
**Magic Square**: Any 3 in a vertical, horizontal, or diagonal line add up to 15.

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>9</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>7</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>6</td>
</tr>
</tbody>
</table>
Conversely, any 3 that add to 15 must be on a line.

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>9</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>7</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>6</td>
</tr>
</tbody>
</table>
TIC-TAC-TOE on a Magic Square
Represents The Nine Stone Game

Alternately choose squares from 1-9.
Get 3 in a row to win.

<table>
<thead>
<tr>
<th>4</th>
<th>9</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>5</td>
<td>7</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>6</td>
</tr>
</tbody>
</table>
BIG IDEA!

Don’t stick with the representation in which you encounter problems, always seek the more useful one!
This IDEA takes practice, practice, practice to understand and use.
Your Ancient Heritage

Let’s take a historical view on abstract representations
Mathematical Prehistory
30,000 BC

Paleolithic peoples in Europe record unary numbers on bones

1 represented by 1 mark
2 represented by 2 marks
3 represented by 3 marks

. . .
Prehistoric Unary

1

2

3

4
Hang on a minute!

Isn’t unary too literal as a representation?

Does it deserve to be an “abstract” representation?
It’s important to respect each representation, no matter how primitive

Unary is a perfect example
Consider the problem of finding a formula for the sum of the first $n$ numbers.

You already used induction to verify that the answer is $\frac{1}{2}n(n+1)$. 
\[ 1 + 2 + 3 + \ldots + n-1 + n = S \]
\[ n + n-1 + n-2 + \ldots + 2 + 1 = S \]
\[ n+1 + n+1 + n+1 + \ldots + n+1 + n+1 = 2S \]
\[ n(n+1) = 2S \]

\[ S = \frac{n(n+1)}{2} \]
There are $n(n+1)$ dots in the grid!

$$S = \frac{n(n+1)}{2}$$

$n(n+1) = 2S$
Very convincing! Unary brings out the geometry of the problem and makes each step look natural.

By the way, my name is Bonzo. And you are?
Odette

Yes, Bonzo. Let’s take it even further...
$$\Delta_n = 1 + 2 + 3 + \ldots + n-1 + n$$

$$= n(n+1)/2$$
\[ \square_n = n^2 \]
\[ = \Delta_n + \Delta_{n-1} \]
Breaking a square up in a new way
Breaking a square up in a new way
1 + 3

Breaking a square up in a new way
Breaking a square up in a new way

\[1 + 3 + 5\]
Breaking a square up in a new way

1 + 3 + 5 + 7

Breaking a square up in a new way
Breaking a square up in a new way

1 + 3 + 5 + 7 + 9

Breaking a square up in a new way
$1 + 3 + 5 + 7 + 9 = 5^2$

Breaking a square up in a new way
The sum of the first $n$ odd numbers is $n^2$.

Here is an alternative dot proof of the same sum....
$n^{th}$ Square Number

$n$ = $\Delta n + \Delta_{n-1}$

= $n^2$
$\square_n = \Delta_n + \Delta_{n-1}$

$= n^2$
n^{th} Square Number

\[ \square_n = \Delta_n + \Delta_{n-1} \]
$n^{th}$ Square Number

$n = \Delta_n + \Delta_{n-1}$

= Sum of first $n$ odd numbers
Same proof in high school notation

\[ \Delta n + \Delta_{n-1} = \]

\[ 1 + 2 + 3 + 4 \ldots \]
\[ + \quad 1 + 2 + 3 + 4 + 5 \ldots \]
\[ \underline{\quad 1 + 3 + 5 + 7 + 9 \ldots} \]

Sum of the first \( n \) odd numbers
Check the next one out...
Area of $\Delta_n \times \Delta_n$ square = $(\Delta_n)^2$
Area of $\Delta_n \times \Delta_n$ square $= (\Delta_n)^2$
Area of $\Delta_n \times \Delta_n$ square $= (\Delta_n)^2$
Area of $\Delta_n \times \Delta_n$ square $= (\Delta_n)^2$
Area of $\Delta_n \times \Delta_n$ square = $(\Delta_n)^2$
Area of $\Delta_n \times \Delta_n$ square\n
$= (\Delta_n)^2$

$= (\Delta_{n-1})^2 + n\Delta_{n-1} + n\Delta_n$

$= (\Delta_{n-1})^2 + n(\Delta_{n-1} + \Delta_n)$

$= (\Delta_{n-1})^2 + n(\square_n)$

$= (\Delta_{n-1})^2 + n^3$
\((\Delta_n)^2 = n^3 + (\Delta_{n-1})^2\)

\[= n^3 + (n-1)^3 + (\Delta_{n-2})^2\]

\[= n^3 + (n-1)^3 + (n-2)^3 + (\Delta_{n-3})^2\]

\[= n^3 + (n-1)^3 + (n-2)^3 + \ldots + 1^3\]
\((\Delta_n)^2 = 1^3 + 2^3 + 3^3 + \ldots + n^3 = \left[ \frac{n(n+1)}{2} \right]^2\)
Can you find a formula for the sum of the first $n$ squares?

Babylonians needed this sum to compute the number of blocks in their pyramids.
Ancients grappled with abstraction in representation

Let’s look back to the dawn of symbols...
Sumerians [modern Iraq]
Sumerians [modern Iraq]

8000 BC  Sumerian tokens use multiple symbols to represent numbers

3100 BC  Develop Cuneiform writing

2000 BC  Sumerian tablet demonstrates base 10 notation (no zero), solving linear equations, simple quadratic equations

Biblical timing: Abraham born in the Sumerian city of Ur in 2000 BC
Babylonians Absorb Sumerians

1900 BC  Sumerian/Babylonian Tablet:  
Sum of first n numbers  
Sum of first n squares  
“Pythagorean Theorem”  
“Pythagorean Triples”  
some bivariate equations

1600 BC  Babylonian Tablet:  
Take square roots  
Solve system of n linear equations
Egyptians

6000 BC  Multiple symbols for numbers
3300 BC  Developed Hieroglyphics
1850 BC  Moscow Papyrus: Volume of truncated pyramid
1650 BC  Rhind Papyrus [Ahmes/Ahmose]: Binary Multiplication/Division
          Sum of 1 to n
          Square roots
          Linear equations

Biblical timing: Joseph Governor is of Egypt
Moscow Papyrus
Harrappans
[Indus Valley Culture] Pakistan/India

3500 BC  Perhaps the first writing system?!

2000 BC  Had a uniform decimal system of weights and measures
<table>
<thead>
<tr>
<th>Year</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1200 BC</td>
<td>Independent writing system (Surprisingly late)</td>
</tr>
<tr>
<td>1200 BC</td>
<td>I Ching [Book of changes]: Binary system developed to do numerology</td>
</tr>
</tbody>
</table>
Rhind Papyrus
Scribe Ahmes was Martin Gardener of his day!
Rhind Papyrus
Scribe Ahmes was Martin Gardener of his day!
A man has 7 houses,  
Each house contains 7 cats,  
Each cat has killed 7 mice,  
Each mouse had eaten 7 ears of spelt,  
Each ear had 7 grains on it.  
What is the total of all of these?

**Sum of powers of 7**
We’ll use this fundamental sum again and again:

The Geometric Series
A Frequently Arising Calculation

\[(X-1) \left( 1 + X^1 + X^2 + X^3 + \ldots + X^{n-2} + X^{n-1} \right) \]

\[= \frac{X^1 + X^2 + X^3 + \ldots + X^{n-2} + X^{n-1} + X^n}{-1} - 1 \]

\[= X^n - 1 \]

\[1 + X^1 + X^2 + X^3 + \ldots + X^{n-2} + X^{n-1} = \frac{X^n - 1}{X - 1} \]

(when \(x \neq 1\))
Geometric Series for $X=2$

$$1 + 2^1 + 2^2 + 2^3 + \ldots + 2^{n-1} = \frac{2^n - 1}{2 - 1}$$

$$1 + X^1 + X^2 + X^3 + \ldots + X^{n-2} + X^{n-1} = \frac{X^n - 1}{X - 1}$$

(when $x \neq 1$)

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^{n-1}} = \frac{\left(\frac{1}{2}\right)^n - 1}{\frac{1}{2} - 1}$$
Numbers and their properties can be represented as strings of symbols.
Strings Of Symbols

We take the idea of symbol and sequence of symbols as primitive.

Let \( \Sigma \) be any fixed finite set of symbols. \( \Sigma \) is called an alphabet, or a set of symbols.

Examples:

\[ \Sigma = \{0,1,2,3,4\} \]
\[ \Sigma = \{a,b,c,d,\ldots,z\} \]
\[ \Sigma = \text{all typewriter symbols} \]
\[ \Sigma = \{\alpha, \beta, \gamma, d, \ldots, z\} \]
Strings Over the Alphabet $\Sigma$

A **string** is a sequence of symbols from $\Sigma$

Let $s$ and $t$ be strings

Then $st$ denotes the **concatenation** of $s$ and $t$

i.e., the string obtained by the string $s$
followed by the string $t$

Now define $\Sigma^+$ by these inductive rules:

$x \in \Sigma \Rightarrow x \in \Sigma^+$

$s,t \in \Sigma^+ \Rightarrow st \in \Sigma^+$
The Set $\Sigma^*$

Define $\varepsilon$ to be the empty string
i.e., $X\varepsilon Y = XY$ for all strings $X$ and $Y$
$\varepsilon$ is also called the string of length 0

Define $\Sigma^* = \Sigma^+ \cup \{\varepsilon\}$
\[ \Sigma^+ \] is set of all finite strings that we can make using (at least one) letters from \( \Sigma \).

\[ \Sigma^* \] is the set of all finite strings that we can make using letters from \( \Sigma \), including the empty string.
Let $\text{DIGITS} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ be a symbol alphabet.

Any string in $\text{DIGITS}^+$ will be called a decimal number.
Let $\text{BITS} = \{0,1\}$ be a symbol alphabet. Any string in $\text{BITS}^+$ will be called a binary number.
Let ROCK = {•} be a symbol alphabet.

Any string in ROCK⁺ will be called a unary number.
Let $\text{BASE-}X = \{0,1,2,\ldots, X-1\}$ be a symbol alphabet.

Any string in $\text{BASE-}X^+$ will be called a base-$X$ number.
Each of these sets of sequences can represent numbers.

We just need to specify the correct map between sets of sequences and numbers.
Inductively defined function

\[ f : \text{ROCK}^+ \rightarrow \mathbb{N} \]

\[ f(\bullet) = 1 \]
\[ f(\bullet X) = f(X) + 1 \]
Inductively defined function

\[ f : \text{BITS}^+ \rightarrow \mathbb{N} \]

\[ f(0) = 0; \quad f(1) = 1 \]

If \(|W| > 1\) then 
\[ W = Xb \]
(b is a bit)
\[ f(Xb) = 2f(X) + b \]
Non-inductive representation of $f$:

$$f(a_{n-1} \ a_{n-2} \ ... \ a_0) = a_{n-1}2^{n-1} + a_{n-2}2^{n-2} + ... + a_02^0$$
Two identical maps from sequences to numbers:

\[ f(0) = 0; \quad f(1) = 1 \]
\[ f(Xb) = 2f(X) + b \]

and

\[ f(a_{n-1} a_{n-2} \ldots a_0) = a_{n-1}2^{n-1} + a_{n-2}2^{n-2} + \ldots + a_02^0 \]
The symbol $a_0$ is called the Least Significant Bit or the Parity Bit

\[
a_0 = 0
\]
iff

\[
f(a_{n-1}a_{n-2}...a_0) = a_{n-1}2^{n-1} + a_{n-2}2^{n-2} + ... + a_02^0
\]
is an even number
Theorem: Each natural has a binary representation

**Base Case:** 0 and 1 do

**Induction hypothesis:** Suppose all natural numbers less than \( n \) have a binary representation

**Induction Step:** Note that \( n = 2m+b \) for some \( m < n \), with \( b = 0 \) or \( 1 \)

Represent \( n \) as the left-shifted sequence for \( m \) concatenated with the symbol for \( b \)
No Leading Zero Binary (NLZB)

A binary string that is either 0 or 1, or has length > 1, and does not have a leading zero.

1  Is in NLZB
000001101001  Is NOT in NLZB
0  Is in NLZB
01  Is NOT in NLZB
10000001  Is in NLZB
Theorem: Each natural has a unique NLZB representation

**Base Case:** 0 and 1 do

**Induction hypothesis:** Suppose all natural numbers less than n have a unique NLZB representation

**Induction Step:** Suppose n = 2m+b has 2 NLZB representations. Their parity bit b must be identical. Hence, m also has two distinct NLZB representations, which contradicts the induction hypothesis. So n must have a unique representation.
Inductive definition is great for showing UNIQUE representation:
\[ f(Xb) = 2f(X) + b \]

Let \( n \) be the smallest number reprinted by two different binary sequences. They must have the same parity bit, thus we can make a smaller number that has distinct representations.
Each natural number has a unique representation as a (No Leading Zeros) Binary number!
BASE X Representation

\[ S = a_{n-1} a_{n-2} \ldots a_1 a_0 \] represents the number:

\[ a_{n-1} X^{n-1} + a_{n-2} X^{n-2} + \ldots + a_0 X^0 \]

---

**Base 2 [Binary Notation]**
101 represents:
\[ 1 (2^2) + 0 (2^1) + 1 (2^0) \]
= \[ \text{□□□□□□} \]

---

**Base 7**
015 represents:
\[ 0 (7^2) + 1 (7^1) + 5 (7^0) \]
= \[ \text{□□□□□□□□□□□□} \]
Bases In Different Cultures

- Sumerian-Babylonian: 10, 60, 360
- Egyptians: 3, 7, 10, 60
- Maya: 20
- Africans: 5, 10
- French: 10, 20
- English: 10, 12, 20
BASE 10 Representation

\[ S = (a_{n-1} a_{n-2} \ldots a_1 a_0)_{10} \] represents the number:

\[ a_{n-1} \times 10^{n-1} + a_{n-2} \times 10^{n-2} + \ldots + a_0 \times 10^0 \]

Largest number representable in base-10 with \( n \) “digits”

\[ = (999999\ldots9)_x \] [with \( n \) 9’s]
\[ = 9 \times (10^{n-1} + 10^{n-2} + \ldots + 10^0) \]
\[ = (10^n - 1) \]
BASE X Representation

\[ S = (a_{n-1} a_{n-2} \ldots a_1 a_0)_X \] represents the number:

\[ a_{n-1} X^{n-1} + a_{n-2} X^{n-2} + \ldots + a_0 X^0 \]

Largest number representable in base-X with n “digits”

\[ = (X-1 \ X-1 \ X-1 \ X-1 \ X-1 \ X-1 \ldots \ X-1)_X \]
\[ = (X-1)(X^{n-1} + X^{n-2} + \ldots + X^0) \]
\[ = (X^n - 1) \]
Fundamental Theorem For Binary

Each of the numbers from 0 to $2^n-1$ is uniquely represented by an $n$-bit number in binary

$k$ uses $\lfloor \log_2 k \rfloor + 1$ digits in base 2
Fundamental Theorem For Base-X

Each of the numbers from 0 to $X^{n-1}$ is uniquely represented by an $n$-“digit” number in base $X$

$k$ uses $\lceil \log_X k \rceil + 1$ digits in base $X$
n has length $n$ in unary, but has length $\lceil \log_2 n \rceil + 1$ in binary

Unary is exponentially longer than binary
Other Representations: Egyptian Base 3

Conventional Base 3:
Each digit can be 0, 1, or 2

Here is a strange new one:

**Egyptian Base 3 uses** -1, 0, 1

Example: $1 -1 -1 = 9 - 3 - 1 = 5$

*We can prove a unique representation theorem*
How could this be Egyptian?

Historically, negative numbers first appear in the writings of the Indian mathematician Brahmagupta (628 AD)
One weight for each power of 3
Left = “negative”. Right = “positive”
Unary and Binary
Triangular Numbers
Dot proofs

Geometric sum
\[(1+x+x^2 + ... + x^{n-1}) = \frac{x^n - 1}{x-1}\]

Base-X representations
unique binary representations
proof for no-leading zero binary

\[k \text{ uses } \left\lfloor \log_2 k \right\rfloor + 1 = \left\lceil \log_2 (k+1) \right\rceil \text{ digits in base 2}\]

Largest length \(n\) number in base \(X\)