Who to Blame: Ryan. That is, tell him if you find errors, and ask him for clarifications.

Rules: As usual, please cite all sources that you may use.
For the first two questions, you need not prove your answer. A reference suffices.

1. True or false? For all primes \( p \) and integers \( e \geq 1 \), \( \mathbb{Z}_p^* \) has a generator.

2. True or false? For all odd primes \( p \) and integers \( e \geq 1 \), \( \mathbb{Z}_p^* \) has a generator.

3. Give a polynomial time algorithm that takes a number \( N \) as input, and outputs “yes” if and only if \( N = p^e \) for some prime \( p \) and positive integer \( e \).

4. Let \( n \) be a positive integer. Recall that the prime density function, \( \pi(n) \), is defined as the number of prime numbers less than or equal to \( n \). The Prime Number Theorem states that \( \lim_{n \to \infty} \frac{\pi(n)}{n/\log n} = 1 \). Here, you will prove a weaker statement: \( \pi(n) = \Omega(n/\log n) \). Sophisticated number theory is not required to prove this.

   (a) Let \( p \) be a prime in the following. Show that \( p \) divides \( n! \), at least \( \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor \) times.
   
   Hint: First, count the numbers between 1 and \( n \) that can be divided by \( p \) at least once. Then count the number that can be divided by \( p \) at least twice, and so forth.

   (b) Define \( r(p) \) as the natural number such that \( p^{r(p)} \leq 2n < p^{r(p)+1} \).
   
   Prove that \( p \) does not divide \( \binom{2n}{n} \) more than \( r(p) \) times.
   
   Conclude that
   
   \[
   2^n \leq \binom{2n}{n} \leq \prod_{\text{prime } p \leq 2n} p^{r(p)} \leq (2n)^{\pi(2n)}.
   \]

   (c) Prove that \( \pi(n) \geq \frac{n}{2 \log_2 n} \).