Claim 1. It is impossible to overcome an opponent’s head start in a foot race.

Proof: Without loss of generality, consider a foot race between Achilles and the Tortoise. We prove that for any $\epsilon$, if the tortoise is given a head start of $\epsilon$, Achilles can never pass the tortoise.

At time $x$, let the tortoise’s position be $p_t$ and let Achilles’ position be $p_a$, with $p_a < p_t$. In order for Achilles to pass the tortoise, it is necessary first for Achilles to travel the distance $p_t - p_a$ which takes time $\delta$. At time $t + \delta$, Denote the new positions of the tortoise and Achilles as $p^2_t$ and $p^2_a$. Note $p^2_a = p_t$, but since the tortoise is moving at some non-zero rate (and since time has passed), $p^2_t > p_t = p^2_a$ and so Achilles has not yet passed the tortoise. But since this argument holds for any headstart $\epsilon$, we may simply substitute in $\epsilon = p^2_t - p^2_a$ and we have reduced the problem of Achilles passing the tortoise to the original one. Therefore, Achilles will not be able to pass the tortoise in the next time step either. By induction, Achilles will never be able to pass the tortoise (since by the time Achilles reaches the tortoise’s last position $p^i_t$, there will always be some epsilon such that $p^{i+1}_t - p^i_t \geq \epsilon$, since the tortoise is always moving at some nonzero rate.) This completes the proof. ■

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1 Special thanks to Zeno of Elea for contributing this proof