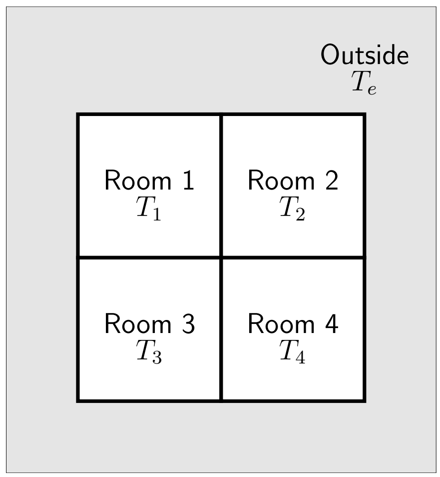
15-884/484 – Problem Set #4

1. **Controllability and Observability**[**20pt**]

   In this problem you'll explore the controllability and observability of the linear system that we discussed in class, representing heating and cooling a room. For the configuration in the slides:

   recall that the dynamical system is given by

$$x_{t+1} = x_t + \Delta t \begin{bmatrix} -4k & k & k & 0 \\ k & -4k & 0 & k \\ k & 0 & -4k & k \\ 0 & k & k & -4k \end{bmatrix} x_t + \Delta t \begin{bmatrix} d & 0 \\ 0 & d \\ 0 & 0 \\ 0 & 0 \end{bmatrix} u_t + \Delta t \begin{bmatrix} 2kT_e \\ 2kT_e \\ 2kT_e \\ 2kT_e \end{bmatrix}$$

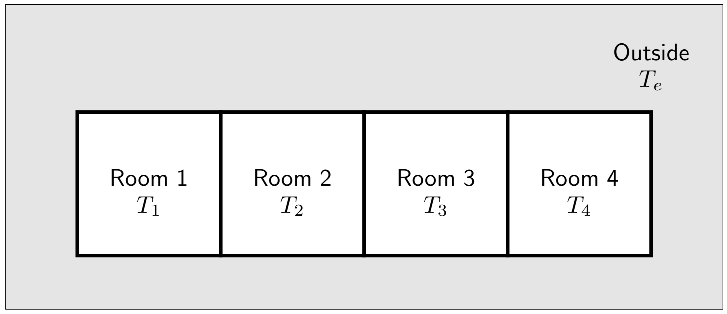
   where k and d are physical constants and Δt is the integration time step.

   (a) Using the constants $k = 0.2$, $d = 0.05$, and $\Delta t = 0.05$, determine if the system is controllable. You should do this using MATLAB and the `rank` command. Hint: remember, you only need to for the controllability matrix for powers A^k up to $n-1$ where n is the dimension of A.

   (b) Find the dynamical systems and determine controllability for the cases where (i) there is only one heater/cooler in Room 1 and (ii) there are heaters/coolers in rooms 1 and 4. Intuitively describe (in a sentence or two) why the controllability is or is not different from the case in part (a).

(c) Write the observation matrix C (for $y_t = Cx_t$) corresponding to the case where there are thermometers in rooms 1 and 2. Is the system observable in this case?

(d) Write the dynamical system for the following configuration of rooms:



where there is a heater/cooler in rooms 1 and 3, and thermometers in rooms 2 and 4. Is the system controllable and/or observable in this case?

(e) Do the choice of model parameters k, d and Δt affect the controllability or observability of the system (assuming they are all non-zero)? That is, is there a situation where a system is controllable for some choice of k, d, Δt, but not for others? If so, give an example of two systems where they do affect the controllability/observability of the system. If not, prove that this is the case.

2. **Control via least squares [25 pts]**

In class we discussed how optimal control in linear dynamical systems with quadratic costs can be solved using the LQR algorithm. Here we'll show that these problems (for a finite horizon) can also be solved as least squares problems, though this is typically less efficient that the LQR solution. Consider the linear dynamical system

$$x_{t+1} = Ax_t + Bu_t$$

for $x_t \in \mathbb{R}^n$, $u_t \in \mathbb{R}^m$ (where we assume that $x_1 = x_1^{(0)}$ is known) and the quadratic cost function

$$C(x_t, u_t) = \|Qx_t\|_2^2 + \|Ru_t\|_2^2$$

(a) Consider the vectors formed by concatenating the control and state vectors

$$\bar{x} \in \mathbb{R}^{Tn} \equiv \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_T \end{bmatrix}, \quad \bar{u} \in \mathbb{R}^{Tm} \equiv \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_T \end{bmatrix}.$$

Show that the sequences of state vectors $\bar{x}$ can be represented as a linear function of $\bar{u}$, i.e., find $F \in \mathbb{R}^{Tn \times Tm}$ and $g \in \mathbb{R}^{Tn}$ such that

$$\bar{x} = F\bar{u} + g$$

(b) Using the above relationship, show that we can solve the optimal control problem

$$\begin{aligned} \underset{u_{1:T}}{\text{minimize}} \quad & \sum_{t=1}^{T} \left(\|Qx_t\|_2^2 + \|Ru_t\|_2^2 \right) \\ \text{subject to} \quad & x_{t+1} = Ax_t + Bu_t, \;\; t = 1, \ldots, T-1 \end{aligned}$$

can be solved using a least squares formulation – i.e., specify some H and c (defined in terms of F and g from the previous problem) such that

$$\underset{\bar{u}}{\text{minimize}} \, \|H\bar{u} - c\| \Longrightarrow \bar{u}^\star = (H^T H)^{-1} H^T c$$

gives the optimal solution to the optimization problem above.

(c) Can the same procedure be applied to optimal control for the time-varying affine system

$$x_{t+1} = A_t x_t + B_t u_t + a_t$$

where $a_t \in \mathbb{R}^n$ (assuming the same cost function as above)? If so, write the equivalent expressions for this setting (a sketch in fine, you don't need complete detail); if not, elaborate upon why.

(d) We can express non-linear dynamics much the same way we did with least squares, by using a linear function of state features. Assume dynamics that are non-linear in state, but linear in control

$$x_{t+1} = \tilde{A}\phi(x_t) + Bu_t$$

where $\phi : \mathbb{R}^n \to \mathbb{R}^k$ is a feature function. Can the same principles as above be applied to find the optimal control for this system? If so, write the equivalent expressions for this setting (a sketch is fine, you don't need complete detail); if not, elaborate upon why.

3. **Control of multi-generator system [30 pts]** In this problem, we will look at controlling power generators in a grid using a variety of different control techniques.

(a) Using the data files from PS3, load the IEEE 14 bus power network, and form the DC power flow approximation using the following code:

```
[Y,s,v,slack,pv,pq,mva] = load_cdf('ieee14cdf.txt');
B0 = 1./imag(1./Y);
B0(B0 == Inf) = 0;
B0(1:n+1:end) = -sum(B0 - diag(diag(B0)));
G = [slack;pv];
L = pq;
p_des = real(s);
```

Now, construct the continuous time dynamical system for the 5 generators using the states and controls

$$x \in \mathbb{R}^{10} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_5 \\ \omega_1 - \omega^{\text{ref}} \\ \omega_2 - \omega^{\text{ref}} \\ \vdots \\ \omega_5 - \omega^{\text{ref}} \end{bmatrix}, \quad u \in \mathbb{R}^5 = \begin{bmatrix} p_1^{\text{mech}} \\ p_2^{\text{mech}} \\ \vdots \\ p_5^{\text{mech}} \end{bmatrix}$$

(in the slides we showed how to do this with the states just being ω_i, so you'll need to make minor changes). In other words, write the A, B, and a matrices such that

$$\dot{x} = Ax + Bu + a$$

where these matrices will be a function of: B_0 above (the DC approximate succeptance matrix for the power system); the moment of interias of the generators, represented here as a vector $H \in \mathbb{R}^5 = (5.15, 6.54, 6.54, 5.06, 5.06)$; and the powers at the load nodes in the network, $p_L \in \mathbb{R}^9$ (which we assume to be fixed).

(b) In matrix form, write a PD controller that (attempts to) drive the system to the state $x = 0$ (note that this won't actually be achievable, so there will be some steady state error). That is, write a matrix $K \in \mathbb{R}^{5\times 10}$ such that PD control is given by

$$u = Kx.$$

The choice of K_p and K_d gains is up to you.

Using this PD controller, simulate the system using MATLAB, starting at the initial point $x = 0$. Convert this system to a discrete time representation, using $\Delta t = 0.01$, and in MATLAB simulate the system for 10 seconds. Plot the resulting graphs of the θ and $\omega - \omega^{\text{ref}}$ variables.

(c) **(15-884 only)** Simulate the continuous time system with PD control directly using the MATLAB function **ode45** in the following manner:

```
[t,X] = ode45(dynamics, [0 10], zeros(10,1))
```

where `dynamics` is a function that takes x as input, computes the control, and outputs the time derivative $\dot{x}$ (`dynamics` technically takes two arguments, the first is the time t, but since our dynamics are time invariant this has no affect here; see the help on `ode45` for details).

i) Plot the resulting sequences of θ and $\omega - \omega^{\text{ref}}$, comparing to the discrete time approximation from the previous problem.

ii) Create a controller using LQR, and plot the dynamics of this controller in continuous time; for continuous time systems, you will use the command `lqr(A,B,Q,R)` to create the controller K, where A and B are the dynamics above, and Q and R

are cost functions of your choosing (you can choose whatever Q or R you want, but be sure to mention what values you chose in your plot).

(d) **(15-884 only)** Finally, note that in all the examples above, the power produced by generator converges to some set amount Kx_T, for the final state x_T. What if we instead want to ensure that each generator is producing a specific amount of power, given for example by the `p_des` variable in the code in part (a). i) Design an LQR feedback controller that will control the system to reach this particular power. Your control law should be of the form

$$u = K(x - x^\star) + u^\star$$

for properly chosen $x^\star$ and $u^\star$, which you will compute using the model of the system and the desired power for the generators. (Hint: since the state contains the power angles θ, you'll want to use DC power flow to determine a "desired" set of power angles corresponding to the desired power at the generators). Implement this method in MATLAB and plot the resulting state sequences.

ii) The method above is somewhat roundabout, in that in order to achieve a certain power we need to compute the resulting power angles. Instead, it is possible to directly control the generators to match a desired power, through integral control. In particular, let

$$e(t) = \int_0^t (u(t) - p^{\text{des}})dt$$

and apply the additional control $-K_i e(t)$ at each time. [Watch out, this part is tricky: how do you compute an integral like that within a continuous time dynamical system? Think about adding additional states to the system that correspond to these integrator terms for each u_i variable]. Implement this controller and plot the resulting state sequences.

4. **(Challenge problem) Controlling a grid [25pts]** *Important: this question is **optional**, and any points you get here will count as extra credit on this problem set.*

This problem brings together several of the elements you have studied this far in class to solve the problem of scheduling generation and storage to minimize carbon emissions in a small smart grid setting. The basis for this problem is the IEEE 30 bus test case, which you saw in Problem Set 3, but instead of having six total generators producing a fixed amount of power (the slack bus and five PV buses), there are two generators, two energy storage facilities, and two wind farms. In addition, instead of just computing the solution for a single instant in time (given fixed demands), you'll need to use control methods to schedule generation and storage over time, taking into account varying load and different constraints of the generation.

You can load the data for all these problems using the command

```
load ps4_data.mat
```

This will load several variables into MATLAB. For reference, all the loaded variables are listed below, but we'll also describe them in more detail in the relevant questions

- `gen = [1 2]` — bus indices of the controllable generators
- `storage = [5 8]` — bus indices of the storage facilities
- `wind = [11 13]` — bus indices of the wind farms
- `pq` — bus indices for all the PQ loads
- `winds` — matrix of time series data giving power production for the two wind farms over every hour in a year
- `loads` — matrix of time series data giving electrical demand for each of the PQ loads for every hour in a year (following convention, these values are negative to indicate consumed power)
- `Belec` — DC power flow approximation susceptance matrix; this matrix relates power and voltage angles via the DC power flow approximation $p = B\theta$. Note that for convenience we include the negative sign (that you saw in the derivation of power flow for the previous assignment) into B itself, so that you don't need to include it explicitly.
- `G,h` — line constraints; these specify constraints on the amount of power than can flow over different branches in the power network. The line constraints are given in the form $G\theta \leq h$.

The grid has the following elements[1]

- There are two generators in the system, at buses 1 and 2. Bus 2 represents an open-cycle gas turbine (essentially a combustion system that directly burns the gas to move an electrical turbine); the advantage of such systems is that they can step up generation very quickly (in this setting, we assume it can change it's power generation arbitrarily quickly), but produce more carbon emissions than other approaches, due to their decreased efficiency. Bus 1 represents a combined-cycled gas turbine; these systems combine an open-cycle turbine with a steam generator powered by the exhaust heat of the turbine. The combined systems are about 60% more efficient that open-cycle systems, but because they rely on steam power (which has to be boiled before it can produce power), they have to ramp power up and down more slowly than the open-cycle generators.
- There are two energy storage facilities, a pumped-hydro facility at bus 5 and a battery storage facility at bus 8. As with generators, there are trade-offs involved: pumped hydro can store significantly more energy, but has lower efficiency than batteries; in this example, we assume both types of storage are roughly comparable in terms of their MW capacity.

[1] Note that these precise generation and storage quantities do not necessarily represent realistic generators in terms of the absolute numbers, but they do capture the overall properties and relative strengths and weaknesses of the different types of generation and storage.

- There are two wind farms in our system, at nodes 11 and 13. These wind farms generate power according to however the wind blows, and cannot be controlled to generate more power (though they can be turned off to stop generated power if needed). The power generated by these systems is given by the `winds` variables, discussed in the previous problem set.
- Finally, every node that is not one of the above is a PQ load, with power consumption given by the `loads` variable. Several of these nodes consume no power, which correspond to internal junction nodes in the network; although they consume no power, they are treated at PQ nodes because we know the real and reactive power consumption at the nodes (namely zero).

(a) We will define the state of the system as

$$x_t \in \mathbb{R}^4 = \begin{bmatrix} \text{Power from generator 1} \\ \text{Power from generator 2} \\ \text{Energy stored in pumped storage} \\ \text{Energy stored in battery} \end{bmatrix}$$

and the control input

$$u_t \in \mathbb{R}^6 = \begin{bmatrix} \text{Change in generator 1 power} \\ \text{Change in generator 2 power} \\ \text{Power used to charge pumped storage} \\ \text{Power used to charge battery} \\ \text{Power released by pumped storage} \\ \text{Power released by battery} \end{bmatrix}$$

Write the dynamics of the system in the form of a linear system

$$x_{t+1} = Ax_t + Bu_t$$

(i.e., explicitly write down A and B), capturing the fact that the pumped storage and battery storage are 80% and 90% efficiency respectively — i.e., if we put in 1 unit of power, this will only result in 0.8 or 0.9 units of energy stored (when we take power *out* of the storage, we assume that 100% of the energy is delivered).

(b) Write a set of constraints on the state and control variables, in the form

$$\underline{u} \le u_t \le \overline{u}, \quad \underline{x} \le x_t \le \overline{x}$$

that captures the constraints that

- The power produced by the generators, as well as the charge of the batteries, can never be negative.
- The pumped storage has a maximum energy capacity of 10 unit-hours, while the battery has a maximum capacity of 0.3 unit-hours.
- The combined-cycle generator can change its generator by at most 0.1 units per hour (we assume the open-cycle turbine can change as much as desired between hours).

- The pumped storage and battery can consume or produce at most 0.3 units of power at each hour.

Make sure that the constraints you put in place enforce the fact that the storage must lose energy: i.e., putting in 1.0 units of power will increase the pumped storage by 0.8 unit-hours, but you certainly can't take out 0.8 unit hours to get 1 unit of power.

(c) Let $p_t \in \mathbb{R}^{30}$ denote the power injections at each bus at time t. Define the elements p_t^{gen}, p_t^{wind}, and p_t^{storage} in terms of the state and control variables, and the variable $\text{wind}_t \in \mathbb{R}^2$, the wind produced by the wind farms (where we can use all the power generated by the turbines if desired, but could also use less). The power consumed by the 24 PQ loads at time t is assumed to be given by the variables $\text{loads}_t \in \mathbb{R}^{24}$, and cannot be adjusted.

(d) Combining the parts above, write down the model predictive control task as a linear optimization problem over the variables $x_{1:T}$, $u_{1:T}$, $p_{1:T}$, $\theta_{1:T}$. The objective of the optimization problem is to minimize carbon emissions, which are equal to the sum of $(x_t)_1 + 2(x_t)_2$ over all time (the power of the combined-cycled generator plus two times the power of the open-cycle generator, to take into account the fact that it is less efficient). You will want to include all the constraints mentioned above, plus the fact that x_1 equals some constant x^{init} at all indices *except* the second generator (this accounts for the fact that the second generators power can be adjusted arbitrarily, and will need to be adjusted at the first state to ensure the total generation equals the total demand), and also the power flow constraints

$$p_t = B^{\text{elec}}\theta_t, \quad (\theta_t)_1 = 0, \quad G\theta_t \le h. \; \forall t$$

where B^{elec} is the DC power flow susceptance matrix (denoted with the superscript "elec" just to avoid confusing with the "B" matrix from the dynamics) and where G and h enforce line flow constraints (in our case, the fact that no line can transmit than 1 unit of power). You can write the objective and constraints in whatever form is easiest (you don't need to put the problem into standard form).

(e) Implement the above optimization problem as a MATLAB function of the form

```
[f,X,U,P,Theta] = optimize_power(x0, winds, loads, Belec, G, h, ...
                                  gen, storage, wind, pq)
```

where `x0` is the initial state of the system; `winds` and `loads` are respectively a $2 \times T$ and matrix of future wind powers and a $24 \times T$ matrix of future loads (either true or forecasted in both cases); `Belec`, `G`, and `h` are matrices that capture the power flow constraints mentioned above; and `gen`, `storage`, `wind`, and `pq` are the indices of the different bus types. You should infer the length of the horizon T by inspecting the size of these input variables. For the output variables, `f` should be a scalar value indicating the objective function at the solution; `X` should be a $4 \times T$ dimensional matrix of all the states; `U` a $6 \times T$ matrix of all chosen controls; `P` a $30 \times T$ matrix of all power injections; and `Theta` a $30 \times T$ variable of all voltage angles in the DC power flow approximation.

You should use YALMIP to solve this optimization problem, and you'll find the problem much easier to solve if you define all the output variables (other than `f`) directly as matrix variables, i.e.,

```
U = sdpvar(6,T);
```

and write as many constraints as possible in matrix form.

(f) Finally, solve the control problem using the real future values of power consumption and wind, starting at time $t = 4681$ with a time horizon $T = 24$ and an initial state of $x = (0, 0, 0, 0)$, i.e., using the call

```
[f,X,U,P,Theta] = optimize_power(zeros(4,1), winds(:,4681:4740), ...
                                 loads(:,4681:4704), Belec, G, h, ...
                                 gen, storage, wind, pq)
```

Plot the resulting states and controls for the entire time horizon.