15-884/484 – Electric Power Systems 3: Power Flow and Markets

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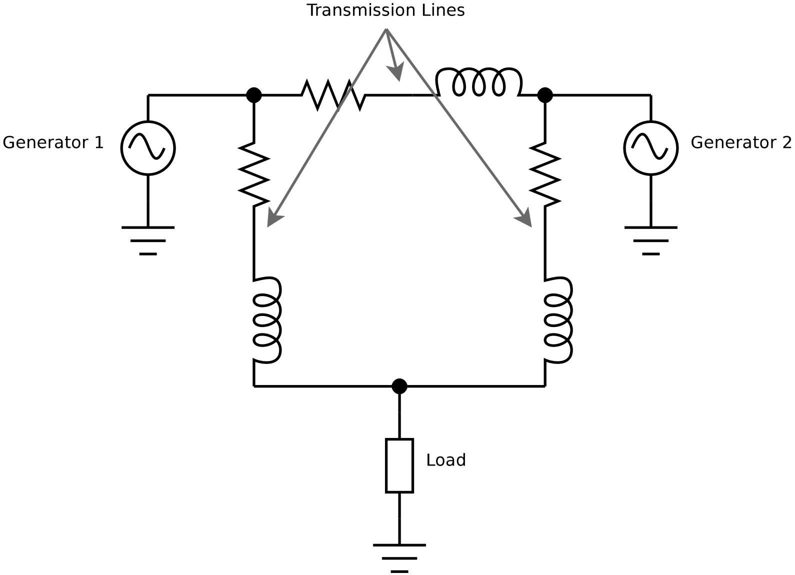
Power Flow in AC Networks

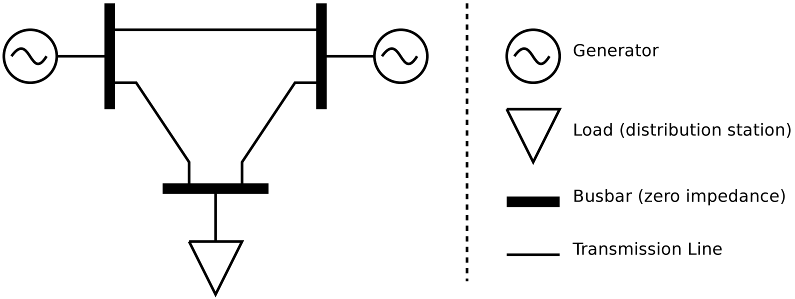
- Basic question: how is electricity supplied in the grid?
  - If these generators produce this much power, how much current will flow through particular transmission lines?

- A fundamentally harder problem than linear circuit analysis (focus on power makes it a non-linear problem)

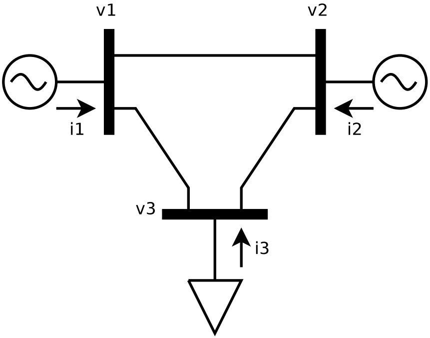
- Simple example setup (showing one of three phases)



- Simpler drawing of the above in standard format



- Imagine we knew bus *voltages* (and impedance of load/lines)



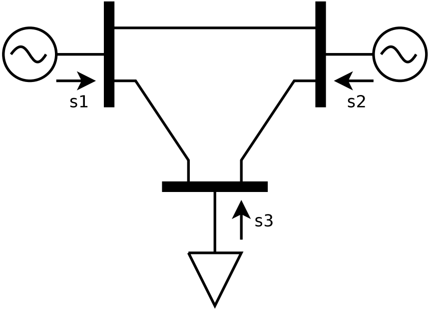
- Then we could compute all bus currents via linear system

$$v = Zi \Leftrightarrow i = Yv \;\; v, i \in \mathbb{R}^3, \; Z, Y \in \mathbb{R}^{3\times 3}$$

- Could also compute current along each line by Ohm's law

$$i_{12} = \frac{v_1 - v_2}{r_{12} + jx_{12}}$$

- Power flow is similar problem conceptually, except that instead of voltages, we may just know (complex) power injections at each node



- Probem #1: We now know *product* of voltage and current $s_k = v_k \bar{i}_k$, but how can we find each voltage/current individually?

- Solution: Using the fact that $i = Yv$, we can write power as

$$\begin{aligned} s &= \mathrm{diag}(v)\bar{i} \\ &= \mathrm{diag}(v)\bar{Y}\bar{v} \end{aligned}$$

  where $\bar{A}$ denotes elementwise conjugations

- So we can determine voltage from power, but we need to solve a *non-linear* equation to do so
  - But we've already seen how to do this with Newton's method

- Problem #2: While it's reasonable to assume we can know (complex) power for loads, generators can't directly control reactive power (can control voltage magnitude)

- Solution: For loads, assume real and reactive power are known (PQ load); for generators, we assume real power and voltage magnitude are known (PV generators)

- Problem #3: Can't know the exact amount of real power we need until we've finished the analysis (losses due to line resistance, but not known until currents are known)

- Solution: Treat one generator as a "slack" generator that generates enough extra power to overcome line losses

- Outline of the final power flow problem:
  - Variables ($i = 1, \dots, n$):

$$\text{Power: } s_i = p_i + jq_i,$$
$$\text{Voltage: } v_i = \hat{v}_i \angle \theta_i \equiv \hat{v}_i (\cos\theta_i + j \sin\theta_i)$$

  - Problem data:

Bus Type	**Known Variables**	**Unknown Variables**
Slack Generator	$\hat{v}_i, \theta_i$	p_i, q_i
PV Generator	$\hat{v}_i, p_i$	θ_i, q_i
PQ Load	p_i, q_i	$\hat{v}_i, \theta_i$

  - Problem:

$$\text{Find: } s, v \in \mathbb{C}^n, \text{ such that } s = \mathrm{diag}(v) \bar{Y} \bar{v}$$

$2n$ equations, $2n$ unknowns: can apply Newton's method to solve

Aside: Computing the Admittance Matrix

- For simplified model of transmission lines, admittance matrix Y is very easy to compute

- Let

$$\begin{aligned} r_{ij} &= \text{Resistance of trasmission line between buses } i \text{ and } j \\ &\quad (\infty \text{ if } i \text{ and } j \text{ not connected}) \\ x_{ij} &= \text{Reactance between buses } i \text{ and } j \end{aligned}$$

- Then

$$Y_{ij} = \begin{cases} -\frac{1}{r_{ij}+jx_{ij}} & i \neq j \text{ (0 if } i, j \text{ not connected)} \\ \sum_{k \neq i} \frac{1}{r_{ik}+jx_{ik}} & i = j \end{cases}$$

Mathematical Details of Power Flow

- Common to write out complex equation explicitly

$$p_i + jq_i = v_i \sum_{k=1}^{n} \bar{Y}_{ik} \bar{v}_k$$

$$= \hat{v}_i(\cos\theta_i + j\sin\theta_i) \sum_{k=1}^{n} \hat{v}_k (G_{ik} - jB_{ik})(\cos\theta_k - j\sin\theta_k)$$

where $Y \equiv G + jB$

- Multiplying terms out leads to canonical power flow equations

$$p_i = \hat{v}_i \sum_{k=1}^{n} \hat{v}_k (G_{ik}\cos(\theta_i - \theta_k) + B_{ik}\sin(\theta_i - \theta_k))$$

$$q_i = \hat{v}_i \sum_{k=1}^{n} \hat{v}_k (G_{ik}\sin(\theta_i - \theta_k) - B_{ik}\cos(\theta_i - \theta_k))$$

- Let $z \in \mathbb{R}^{2n}$ be a vector of all the unknown variables

$$z = \begin{bmatrix} \left.\begin{matrix} p_0 \\ q_0 \end{matrix}\right\} & \text{slack generator} \\ \left.\begin{matrix} q_1 \\ \theta_1 \\ \vdots \\ q_{n_{PV}+1} \\ \theta_{n_{PV}+1} \end{matrix}\right\} & \text{PV generators} \\ \left.\begin{matrix} \hat{v}_{n_{PV}+2} \\ \theta_{n_{PV}+2} \\ \vdots \\ \hat{v}_n \\ \theta_n \end{matrix}\right\} & \text{PQ Loads} \end{bmatrix}$$

- Let $g : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a function that contains all the power flow equations

$$g(z) = \begin{bmatrix} p_1 - \hat{v}_1 \sum_{k=1}^n \hat{v}_k (G_{1k} \cos(\theta_1 - \theta_k) + B_{1k} \sin(\theta_1 - \theta_k)) \\ \vdots \\ q_1 - \hat{v}_1 \sum_{k=1}^n \hat{v}_k (G_{1k} \sin(\theta_1 - \theta_k) - B_{1k} \cos(\theta_1 - \theta_k)) \\ \vdots \end{bmatrix}$$

- Want to find z such that $g(z) = 0$

- Newton's method, repeat:

$$z \leftarrow z - (D_z g(z))^{-1} g(z)$$

  - Computing the Jacobian terms can get a bit cumbersome

- A couple Jacobian terms

$$\frac{\partial g_{p,i}}{\partial \theta_i} = \hat{v}_i \sum_{k \neq i} \hat{v}_k (G_{ik} \sin(\theta_i - \theta_k) - B_{ik} \cos(\theta_i - \theta_k))$$

$$\frac{\partial g_{q,i}}{\partial \theta_i} = -2\hat{v}_i B_{ii} - \sum_{k \neq i} (G_{ik} \sin(\theta_i - \theta_k) - B_{ik} \cos(\theta_i - \theta_k))$$

- Need 8 of these total, can compute Jacobian element by element

- The typical formulation of Newton's method for power flow you'll find in most textbooks

- Jacobians are easier to compute if we consider all variables

$$z = \begin{bmatrix} \theta \\ \hat{v} \\ p \\ q \end{bmatrix}$$

and

$$g(z) = \begin{bmatrix} \mathrm{Re}\{\mathrm{diag}(v)\bar{Y}\bar{v} - s\} \\ \mathrm{Im}\{\mathrm{diag}(v)\bar{Y}\bar{v} - s\} \end{bmatrix}.$$

Then

$$J = D_z g(z) = \begin{bmatrix} \mathrm{Re}\{J_1\} & \mathrm{Re}\{J_2\} & -I & 0 \\ \mathrm{Im}\{J_1\} & \mathrm{Im}\{J_2\} & 0 & -I \end{bmatrix}$$

$$J_1 = j\,\mathrm{diag}(v)(\mathrm{diag}(\bar{Y}\bar{v}) - \bar{Y}\,\mathrm{diag}(\bar{v}))$$

$$J_2 = \mathrm{diag}(v)\bar{Y}\,\mathrm{diag}(e^{-j\theta}) + \mathrm{diag}(e^{j\theta})\,\mathrm{diag}(\bar{Y}\bar{v})$$

(J_1, J_2 have the same sparsity pattern as Y)

- Let $\mathcal{I}$ be the indices of unknown variables in z ($|\mathcal{I}| = 2n$)

- Then Newton's method is just

$$z_{\mathcal{I}} \leftarrow z_{\mathcal{I}} - J_{:,\mathcal{I}}^{-1} g$$

- Some notes:
  - Newton's method is not guaranteed to converge in all cases (even if solution exists)

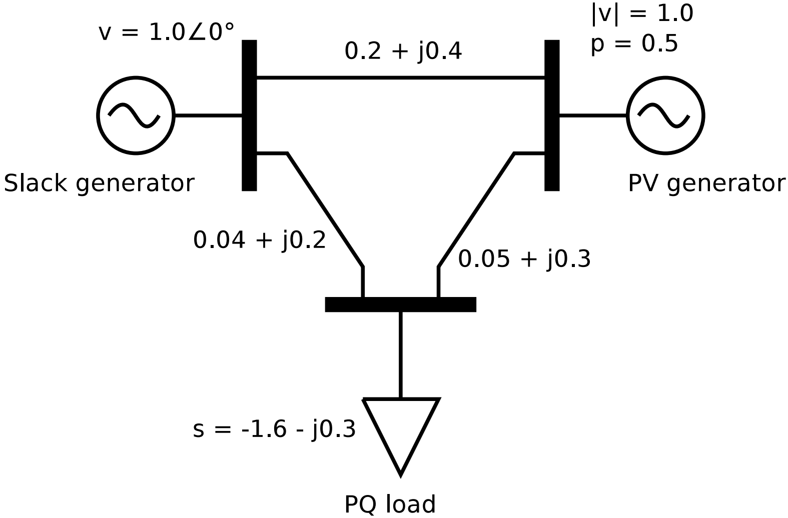
  - But, works very well for many "typical" power flow problems

- Typically, not much relevance to absolute voltage magnitudes; what matters is *relative* magnitudes

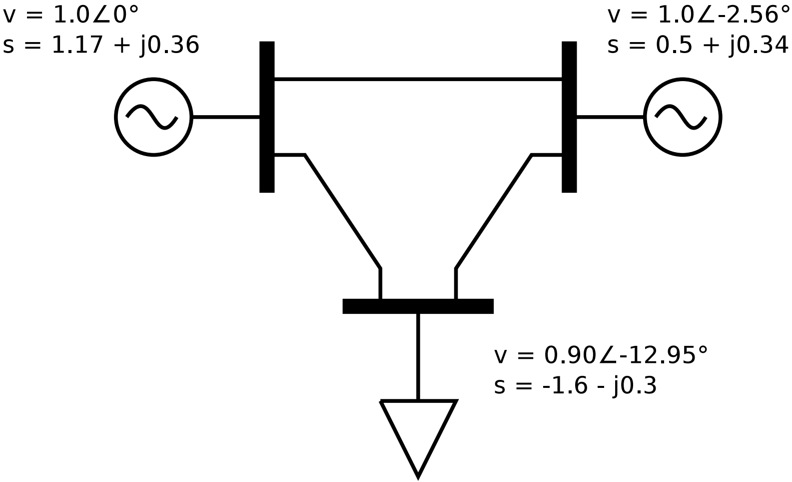
- Therefore, common to report voltages in terms of per-unit (p.u.) units, scalled by nominal voltage of the system

- Also, power, admittance, etc are scalled (typical parameter of power flow problems is “base MVA” that says how to scale power)

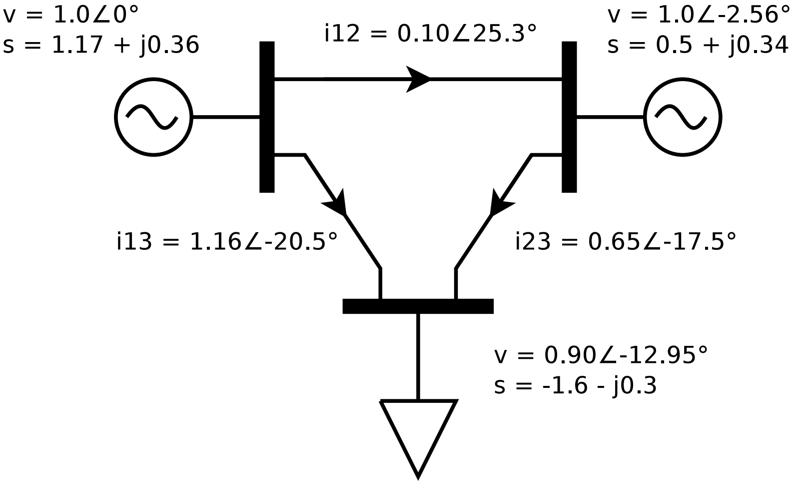
- A simple three-bus example:

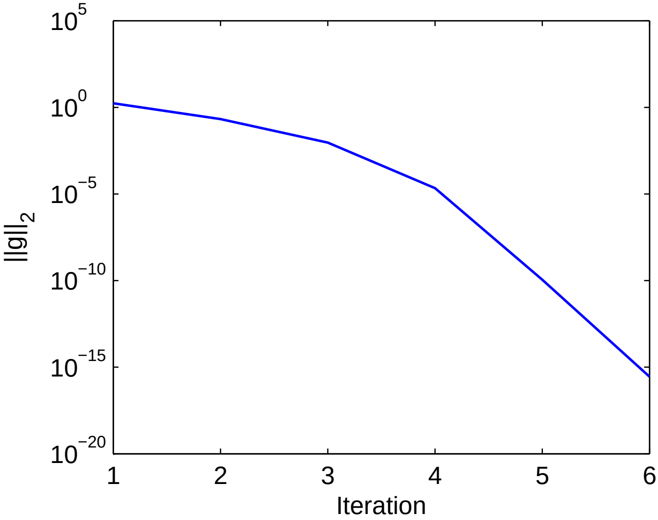


- Power flow solution:

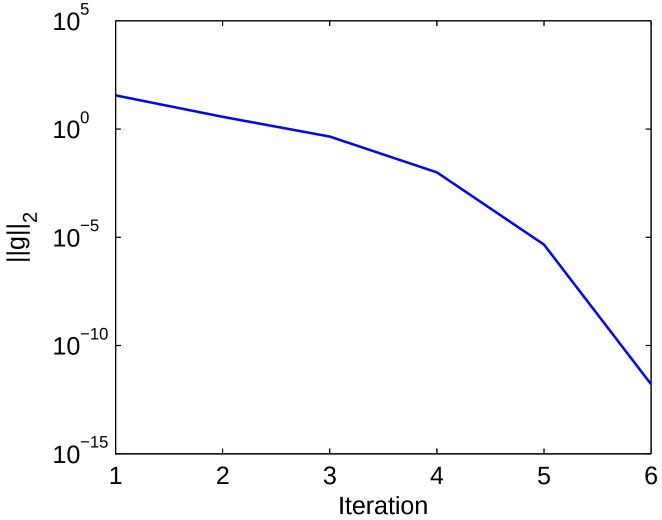


- Power flow solution w/ resulting currents:





Progress of Newton's method on 3 bus example



Progress of Newton's method on IEEE 300 bus example

- Some notes:
  - Many extensions to Newton's method are possible, help with convergence or speed things up (less of an issue with current computation)
  - E.g. Fast Decoupled Power Flow (FDPF): ignore off-diagonal blocks in Jacobian (typically smaller in value)
  - Solution to power flow also not guranteed to exist (can be physically unrealizable)

DC Power Flow

- A linear approximation to power flow
- *Not* power flow computation for DC network (this is still a non-linear problem)
- Simplifying assumptions:
  - All line resistances are zero (reactances are non-zero)
  - All voltage magnitudes are equal (to 1 p.u.)
  - Voltage angles are small, so that

$$\sin(\theta_i - \theta_k) \approx \theta_i - \theta_k, \quad \cos(\theta_i - \theta_k) \approx 1$$

- Under these assumptions

  $p = -B\theta$ (imaginary part of admittance matrix $Y = G + jB$)
  $q = 0$ (all reactive power is equal to zero)

  (you will derive these in problem set)

- Given solution to θ's, can again compute current flowing through each line

$$i_{12} = \frac{v_1 - v_2}{jx_{12}} = \frac{\theta_1 - \theta_2}{x_{12}} = B_{12}(\theta_1 - \theta_2)$$

- We are back to a set of *linear* equations, easier for solving and optimization

Optimal Power Flow

- In power flow problem, real powers at all generators and loads (except slack) are *given* as input

- In some situations (e.g. power markets, demand response, etc), we might want to *optimize* the power at each generator (or loads, if we have demand response capabilites)

- The general term for power flow where we are optimizing certain variables (which are typically fixed in power flow) is *optimal power flow* (OPF)

- General formulation

$$\begin{array}{lll} \text{minimize}_z & f(z) & \text{(some objective function)} \\ \text{subject to} & h_e(z) = 0 & \text{(equality constraints)} \\ & h_i(z) \leq 0 & \text{(inequality constraints)} \\ & s = \text{diag}(v)\bar{Y}\bar{v} & \text{(power flow constraints)} \end{array}$$

where z is defined as before

$$z = \begin{bmatrix} \theta \\ \hat{v} \\ p \\ q \end{bmatrix}$$

- Note that we must pre-specify *fewer* variables than in the power flow problem, or there won't be anything to optimize over

- In principle, just a (non-convex) optimization problem, could solve with YALMIP
  - In practice, YALMIP has a lot of difficulties with the non-convex equality constraint $s = \mathrm{diag}(v)\bar{Y}\bar{v}$.

- If f, h_i and h_e are such that the OPF problem would be convex without the power flow constraint, can solve with *iterative optimization*

- Basic idea: linearize the power flow constraint at each iteration, and solve resulting convex problem

Iterative Optimization Procedure

- Begin with some initial guess z_0
- Repeat:
  - Compute Jacobian

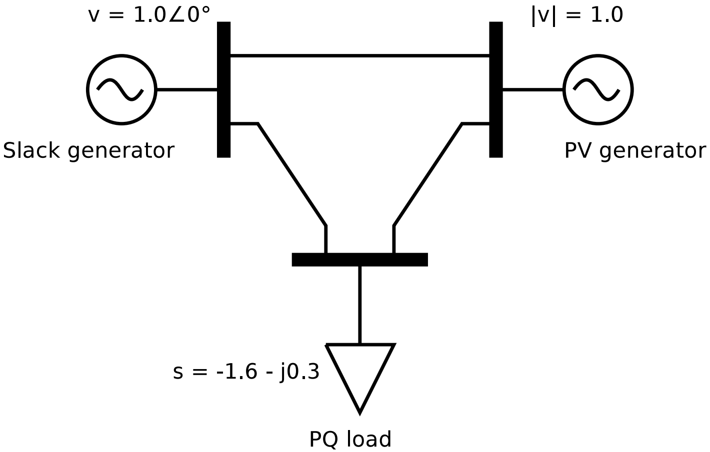
$$J \leftarrow D_z g(z_0)$$

  - Solve linearized optimization problem

$$\begin{aligned} \underset{z}{\text{minimize}} \quad & f(z) \\ \text{subject to} \quad & h_e(z) = 0 \\ & h_i(z) \leq 0 \\ & J(z - z_0) = -g(z_0) \end{aligned}$$

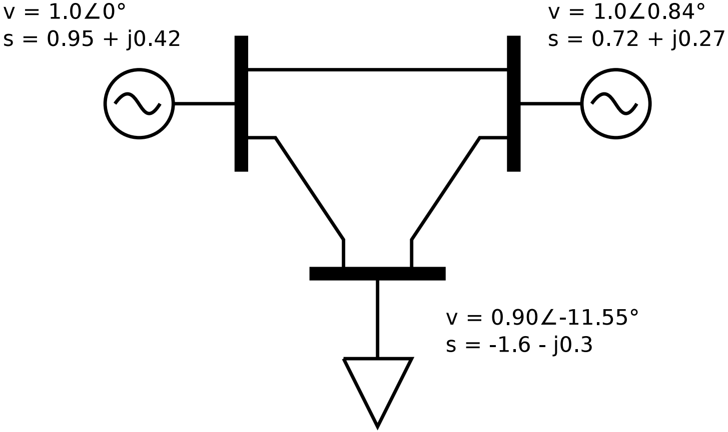
  - Set $z_0 \leftarrow z^\star$ (solution to optimization problem)

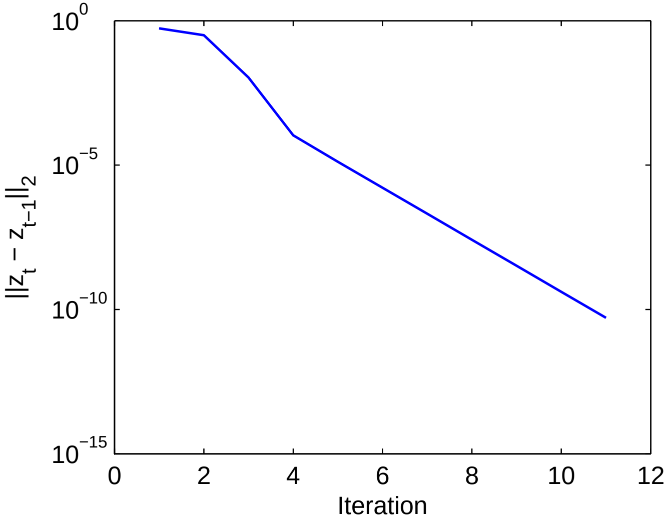
- Three-bus example with power costs



$$f(z) = p_1^2 + 2p_1 + 2p_2^2 + p_2$$

- OPF Solution



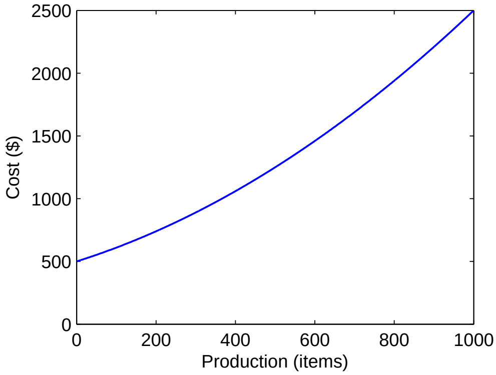


Convergence of solution in iterative optimization procedure

Power Markets

- How do we actually coordinate and price generation with consumption?
- Previously, done by single monopoly, could observe all loads and optimize generation accordingly.
- With competition (even just in generation), need a way to coordinate market instantaneously
  - Unlike traditional markets, can't wait for market to coverge through bidding system

- Theory of the firm



- Given market price π, determine how much to make by optimization problem

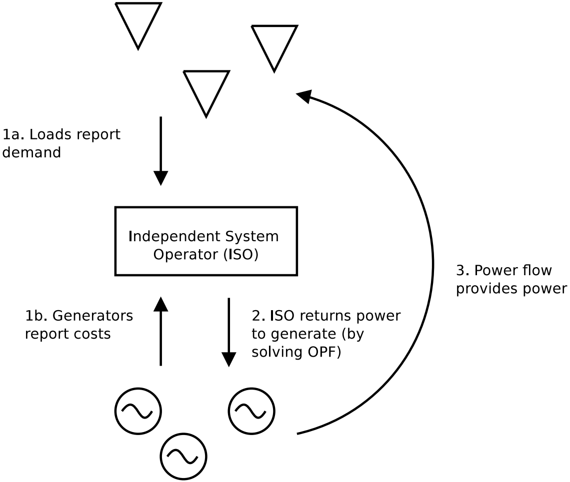
$$\underset{y}{\text{maximize}} \; \{\pi y - C(y)\} \implies \pi = \frac{dC}{dy}$$

- $\frac{dC}{dy}$ is known as marginal price, cost to make one more item
  - By above relationship, a producer will produce items until the marginal price equals the market price

- In traditional market, market price is set by interaction of supply and demand

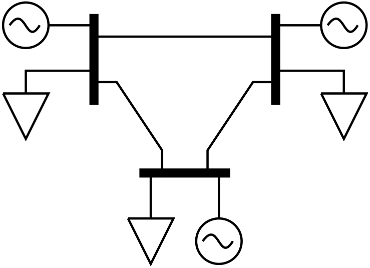
- Many idiosyncracies to power markets
  - Need to reponsd instantaneously to demand (no time for market to “settle” via a manual bidding system)
  - Can’t (yet) store significant portions of energy
  - Demand is typically viewed as fixed, producers (generators) must in total produce enough generation to match
  - Distribution of product (i.e., transmission of power) must obey the laws of physics

- Operation of centralized power market



Optimal Power Flow for Power Markets

- For generality, we'll usually consider case of power generators and loads at the different nodes



- We'll also explicitly seperate out generator and load powers (if no generator at a node, then we'll just add a constraint that $s^G = 0$)

$$s_i = s_i^G - s_i^L$$

- A typical OPF formulation for power markets

$$\begin{aligned}\underset{z}{\text{minimize}} \quad & C(p^G) \equiv \sum_{i=1}^{n} C_i(p_i^G) \\ \text{subject to} \quad & s^G - s^L = \mathrm{diag}(v)\bar{Y}\bar{v} \\ & -v_i\bar{Y}_{ij}(\bar{v}_i - \bar{v}_j) \leq S_{ij}^{\max}, \quad \forall i \neq j \\ & s^G \leq s^{G\max}\end{aligned}$$

where constraints are 1) power flow constriants, 2) transmission constraints, and 3) generation constraints

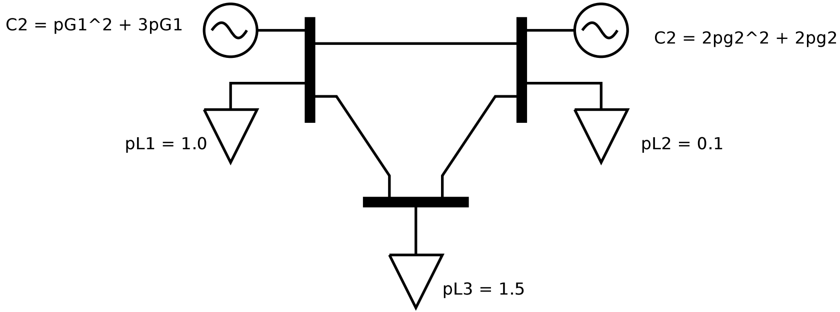
- Can view this as a "black box" that takes as input node loads, and outputs power generation, power flow, prices

$$s^L \to OPF(s^L) \to z^\star, C(z^\star)$$

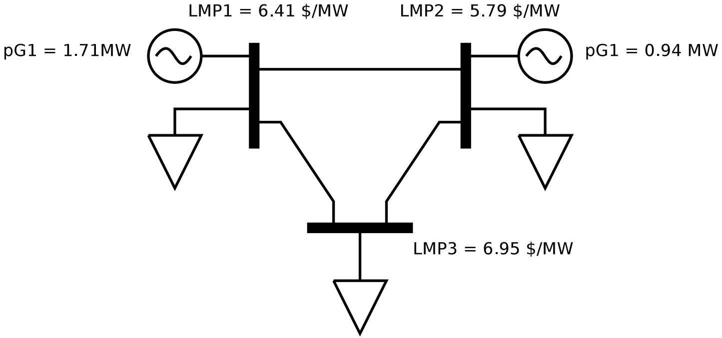
- Locational marginal prices (LMPs)
  - One of the most important elements of power markets is that the marginal price of power can *vary* at different nodes
  - Leads to locational marginal pricing, or nodal pricing

$$\begin{aligned} \mathrm{LMP}_i &= \text{ cost per unit of a little more (real) power at node } i \\ &= \frac{\partial C(z^\star)}{\partial p_i^L} \end{aligned}$$

  - Can compute these numerically (or, optimization methods often generate these types of derivatives automatically via *dual variables*)



Example power market costs and (real) loads



Resulting generation and LMPs

- In above example, LMP was highest at the node without a generator (intuitive, since it had to move power from elsewhere, incur losses)

- LMPs can also be *less* than marginal cost at *any* generator, or even be negative (much less intuitive)
  - Basic issue is that power flow equations + transmission constraints can conspire so that, for example, to obtain 1MW more at node 3 requires we *decrease* generator 2 by 1MW and increase generator 1 by 2MW

  - Given certain prices, this can *lower* the overall cost of power by increasing demand at node 3

- *Many* more elements to power markets
  - Run markets both on real-time and one day ahead
  - How do we handle reserve? Viewed as a "service" to grid and often bought separately by ISO
  - What about reactive power? Losses are much higher, so it's not economical to supply it over longer distances, like with real power
  - Who pays for transmission? How do we handle pricing when transmission lines become congested?