

15-830 – Electric Power Systems 3: Power Flow and Markets

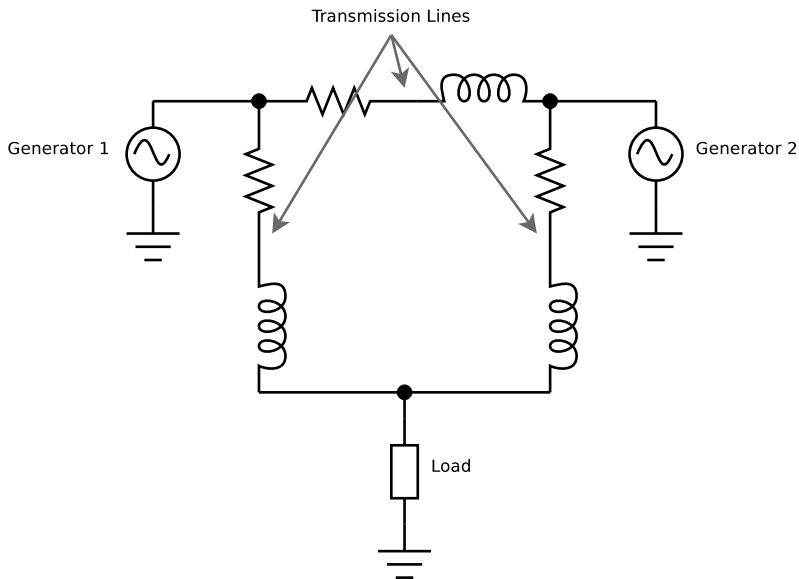
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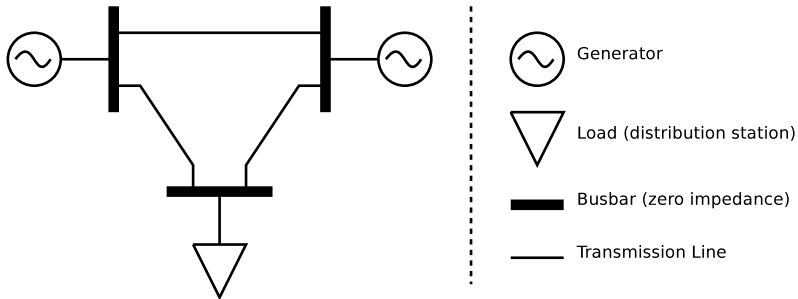
Power Flow in AC Networks

- Basic question: how is electricity supplied in the grid?
 - If these generators produce this much power, how much current will flow through particular transmission lines?
- A fundamentally harder problem than linear circuit analysis (focus on power makes it a non-linear problem)

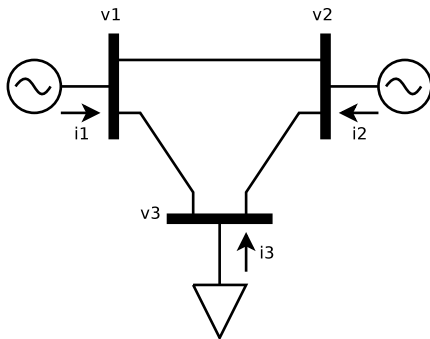
- Simple example setup (showing one of three phases)



- Simpler drawing of the above in standard format



- Imagine we knew bus *voltages* (and impedance of load/lines)



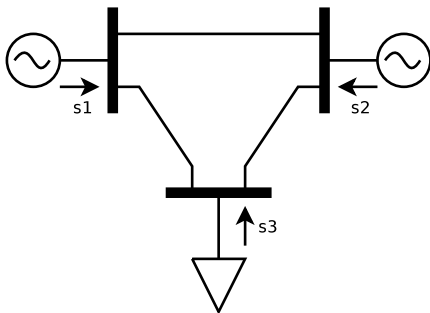
- Then we could compute all bus currents via linear system

$$v = Zi \Leftrightarrow i = Yv \quad v, i \in \mathbb{R}^3, \quad Z, Y \in \mathbb{R}^{3 \times 3}$$

- Could also compute current along each line by Ohm's law

$$i_{12} = \frac{v_1 - v_2}{r_{12} + jx_{12}}$$

- Power flow is similar problem conceptually, except that instead of voltages, we may just know (complex) power injections at each node



- Problem #1: We now know *product* of voltage and current $s_k = v_k i_k^*$, but how can we find each voltage/current individually?

- Solution: Using the fact that $i = Yv$, we can write power as

$$\begin{aligned} s &= \text{diag}(v)i^* \\ &= \text{diag}(v)Y^*v^* \end{aligned}$$

where A^* denotes elementwise conjugations (though it doesn't matter too much here, since Y is symmetric)

- So we can determine voltage from power, but we need to solve a *non-linear* equation to do so
 - But we've already seen how to do this with Newton's method

- Problem #2: While it's reasonable to assume we can know (complex) power for loads, generators can't directly control reactive power (can control voltage magnitude)
- Solution: For loads, assume real and reactive power are known (PQ load); for generators, we assume real power and voltage magnitude are known (PV generators)
- Problem #3: Can't know the exact amount of real power we need until we've finished the analysis (losses due to line resistance, but not known until currents are known)
- Solution: Treat one generator as a "slack" generator that generates enough extra power to overcome line losses

- Outline of the final power flow problem:

- Variables ($i = 1, \dots, n$):

$$\text{Power: } s_i = p_i + jq_i,$$

$$\text{Voltage: } v_i = \hat{v}_i \angle \theta_i \equiv \hat{v}_i (\cos \theta_i + j \sin \theta_i)$$

- Problem data:

Bus Type	Known Variables	Unknown Variables
Slack Generator	\hat{v}_i, θ_i	p_i, q_i
PV Generator	\hat{v}_i, p_i	θ_i, q_i
PQ Load	p_i, q_i	\hat{v}_i, θ_i

- Problem:

$$\text{Find: } s, v \in \mathbb{C}^n, \text{ such that } s = \text{diag}(v)Y^*v^*$$

$2n$ equations, $2n$ unknowns: can apply Newton's method to solve

Aside: Computing the Admittance Matrix

- For simplified model of transmission lines, admittance matrix Y is very easy to compute
- Let

r_{ij} = Resistance of transmission line between buses i and j
(∞ if i and j not connected)

x_{ij} = Reactance between buses i and j

- Then

$$Y_{ij} = \begin{cases} -\frac{1}{r_{ij} + jx_{ij}} & i \neq j \text{ (0 if } i, j \text{ not connected)} \\ \sum_{k \neq i} \frac{1}{r_{ik} + jx_{ik}} & i = j \end{cases}$$

Mathematical Details of Power Flow

- Common to write out complex equation explicitly

$$\begin{aligned} p_i + jq_i &= v_i \sum_{k=1}^n Y_{ik}^* v_k^* \\ &= \hat{v}_i (\cos \theta_i + j \sin \theta_i) \sum_{k=1}^n \hat{v}_k (G_{ik} - jB_{ik}) (\cos \theta_k - j \sin \theta_k) \end{aligned}$$

where $Y \equiv G + jB$

- Multiplying terms out leads to canonical power flow equations

$$\begin{aligned} p_i &= \hat{v}_i \sum_{k=1}^n \hat{v}_k (G_{ik} \cos(\theta_i - \theta_k) + B_{ik} \sin(\theta_i - \theta_k)) \\ q_i &= \hat{v}_i \sum_{k=1}^n \hat{v}_k (G_{ik} \sin(\theta_i - \theta_k) - B_{ik} \cos(\theta_i - \theta_k)) \end{aligned}$$

- Let $z \in \mathbb{R}^{2n}$ be a vector of all the unknown variables

$$z = \left[\begin{array}{c} p_0 \\ q_0 \\ q_1 \\ \theta_1 \\ \vdots \\ q_{n_{PV}+1} \\ \theta_{n_{PV}+1} \\ \hat{v}_{n_{PV}+2} \\ \theta_{n_{PV}+2} \\ \vdots \\ \hat{v}_n \\ \theta_n \end{array} \right] \left. \begin{array}{l} \} \text{slack generator} \\ \\ \\ \} \text{PV generators} \\ \\ \\ \} \text{PQ Loads} \end{array} \right.$$

- Let $g : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ be a function that contains all the power flow equations

$$g(z) = \begin{bmatrix} p_1 - \hat{v}_1 \sum_{k=1}^n \hat{v}_k (G_{1k} \cos(\theta_1 - \theta_k) + B_{1k} \sin(\theta_1 - \theta_k)) \\ \vdots \\ q_1 - \hat{v}_1 \sum_{k=1}^n \hat{v}_k (G_{1k} \sin(\theta_1 - \theta_k) - B_{1k} \cos(\theta_1 - \theta_k)) \\ \vdots \end{bmatrix}$$

- Want to find z such that $g(z) = 0$
- Newton's method, repeat:

$$z \leftarrow z - (D_z g(z))^{-1} g(z)$$

- Computing the Jacobian terms can get a bit cumbersome

- A couple Jacobian terms

$$\frac{\partial g_{p,i}}{\partial \theta_i} = \hat{v}_i \sum_{k \neq i} \hat{v}_k (G_{ik} \sin(\theta_i - \theta_k) - B_{ik} \cos(\theta_i - \theta_k))$$

$$\frac{\partial g_{q,i}}{\partial \theta_i} = -2\hat{v}_i B_{ii} - \sum_{k \neq i} (G_{ik} \sin(\theta_i - \theta_k) - B_{ik} \cos(\theta_i - \theta_k))$$

- Need 8 of these total, can compute Jacobian element by element
- The typical formulation of Newton's method for power flow you'll find in most textbooks

- Jacobians are easier to compute if we consider all variables

$$z = \begin{bmatrix} \theta \\ \hat{v} \\ p \\ q \end{bmatrix}$$

and

$$g(z) = \begin{bmatrix} \operatorname{Re}\{\operatorname{diag}(v)(Yv)^* - s\} \\ \operatorname{Im}\{\operatorname{diag}(v)(Yv)^* - s\} \end{bmatrix}.$$

Then

$$J = D_z g(z) = \begin{bmatrix} \operatorname{Re}\{J_1\} & \operatorname{Re}\{J_2\} & -I & 0 \\ \operatorname{Im}\{J_1\} & \operatorname{Im}\{J_2\} & 0 & -I \end{bmatrix}$$

where (A^* denotes elementwise conjugation)

$$J_1 = j \operatorname{diag}(v)(\operatorname{diag}(Yv) - Y \operatorname{diag}(v))^*$$

$$J_2 = \operatorname{diag}(v)(Y \operatorname{diag}(e^{j\theta}))^* + \operatorname{diag}(e^{j\theta}) \operatorname{diag}(Yv)^*$$

(J_1, J_2 have the same sparsity pattern as Y)

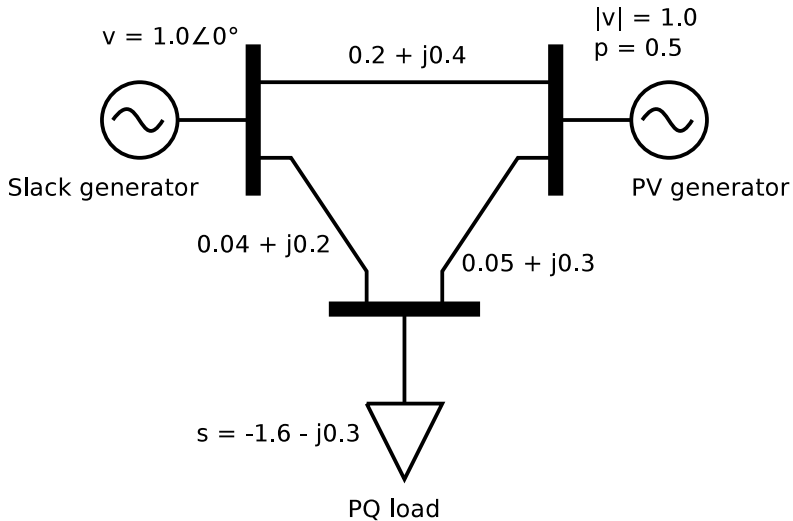
- Let \mathcal{I} be the indices of unknown variables in z ($|\mathcal{I}| = 2n$)
- Then Newton's method is just

$$z_{\mathcal{I}} \leftarrow z_{\mathcal{I}} - J_{:, \mathcal{I}}^{-1} g$$

- Some notes:
 - Newton's method is not guaranteed to converge in all cases (even if solution exists)
 - But, works very well for many “typical” power flow problems

- Typically, not much relevance to absolute voltage magnitudes; what matters is *relative* magnitudes
- Therefore, common to report voltages in terms of per-unit (p.u.) units, scaled by nominal voltage of the system
- Also, power, admittance, etc are scaled (typical parameter of power flow problems is “base MVA” that says how to scale power)

- A simple three-bus example:



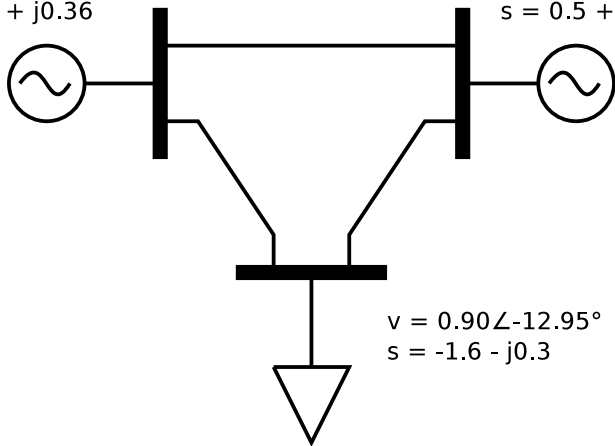
- Power flow solution:

$$v = 1.0 \angle 0^\circ$$

$$s = 1.17 + j0.36$$

$$v = 1.0 \angle -2.56^\circ$$

$$s = 0.5 + j0.34$$



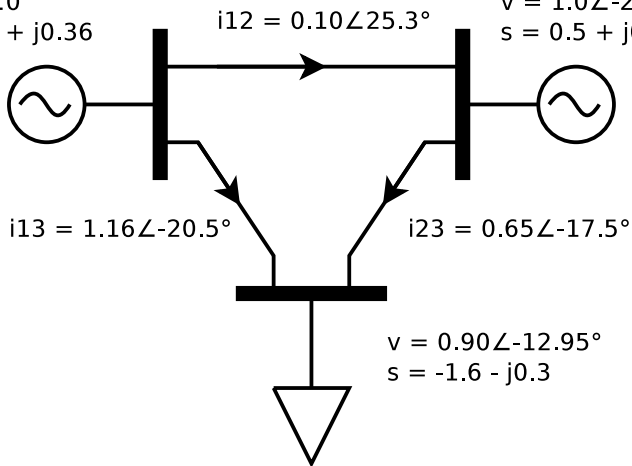
- Power flow solution w/ resulting currents:

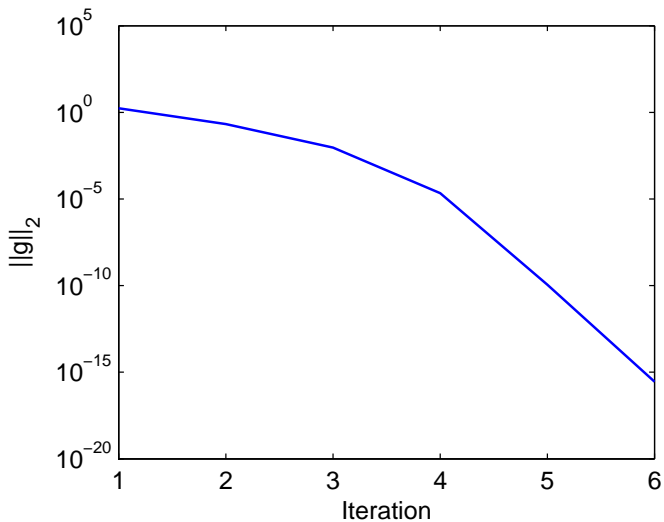
$$v = 1.0 \angle 0^\circ$$

$$s = 1.17 + j0.36$$

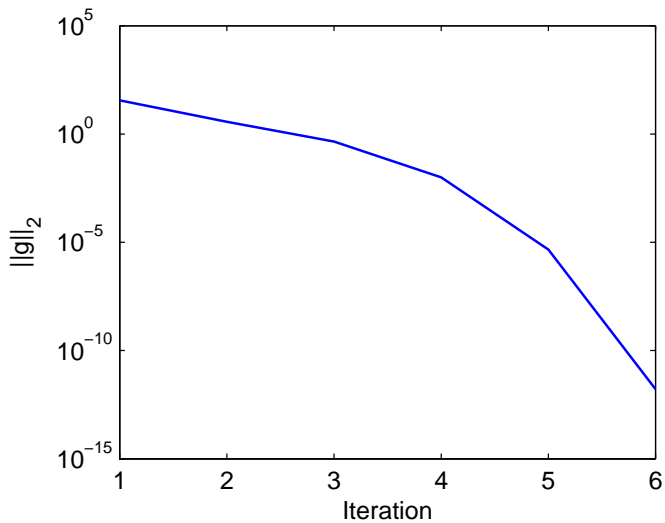
$$v = 1.0 \angle -2.56^\circ$$

$$s = 0.5 + j0.34$$





Progress of Newton's method on 3 bus example



Progress of Newton's method on IEEE 300 bus example

- Some notes:
 - Many extensions to Newton's method are possible, help with convergence or speed things up (less of an issue with current computation)
 - E.g. Fast Decoupled Power Flow (FDPF): ignore off-diagonal blocks in Jacobian (typically smaller in value)
 - Solution to power flow also not guaranteed to exist (can be physically unrealizable)

DC Power Flow

- A linear approximation to power flow
- *Not* power flow computation for DC network (this is still a non-linear problem)
- Simplifying assumptions:
 - All line resistances are zero (reactances are non-zero)
 - All voltage magnitudes are equal (to 1 p.u.)
 - Voltage angles are small, so that

$$\sin(\theta_i - \theta_k) \approx \theta_i - \theta_k, \quad \cos(\theta_i - \theta_k) \approx 1$$

- Under these assumptions

$$p = -B\theta \quad (\text{imaginary part of admittance matrix } Y = G + jB)$$

$$q = 0 \quad (\text{all reactive power is equal to zero})$$

(you will derive these in problem set)

- Given solution to θ 's, can again compute current flowing through each line

$$i_{12} = \frac{v_1 - v_2}{jx_{12}} = \frac{\theta_1 - \theta_2}{x_{12}} = B_{12}(\theta_1 - \theta_2)$$

- We are back to a set of *linear* equations, easier for solving and optimization

Optimal Power Flow

- In power flow problem, real powers at all generators and loads (except slack) are *given* as input
- In some situations (e.g. power markets, demand response, etc), we might want to *optimize* the power at each generator (or loads, if we have demand response capabilities)
- The general term for power flow where we are optimizing certain variables (which are typically fixed in power flow) is *optimal power flow* (OPF)

- General formulation

$$\begin{array}{ll} \text{minimize}_z & f(z) & \text{(some objective function)} \\ \text{subject to} & h_e(z) = 0 & \text{(equality constraints)} \\ & h_i(z) \leq 0 & \text{(inequality constraints)} \\ & s = \text{diag}(v)(Yv)^* & \text{(power flow constraints)} \end{array}$$

where z is defined as before

$$z = \begin{bmatrix} \theta \\ \hat{v} \\ p \\ q \end{bmatrix}$$

- Note that we must pre-specify *fewer* variables than in the power flow problem, or there won't be anything to optimize over

- In principle, just a (non-convex) optimization problem, could solve with YALMIP
 - In practice, YALMIP has a lot of difficulties with the non-convex equality constraint $s = \text{diag}(v)(Yv)^*$.
- If f , h_i and h_e are such that the OPF problem would be convex without the power flow constraint, can solve with *iterative optimization*
- Basic idea: linearize the power flow constraint at each iteration, and solve resulting convex problem

Iterative Optimization Procedure

- Begin with some initial guess z_0

- Repeat:

- Compute Jacobian

$$J \leftarrow D_z g(z_0)$$

- Solve linearized optimization problem

$$\underset{z}{\text{minimize}} \quad f(z)$$

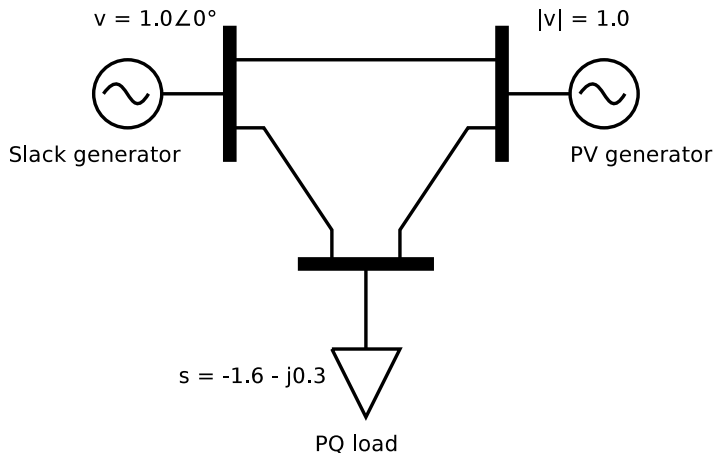
$$\text{subject to} \quad h_e(z) = 0$$

$$h_i(z) \leq 0$$

$$J(z - z_0) = -g(z_0)$$

- Set $z_0 \leftarrow z^*$ (solution to optimization problem)

- Three-bus example with power costs

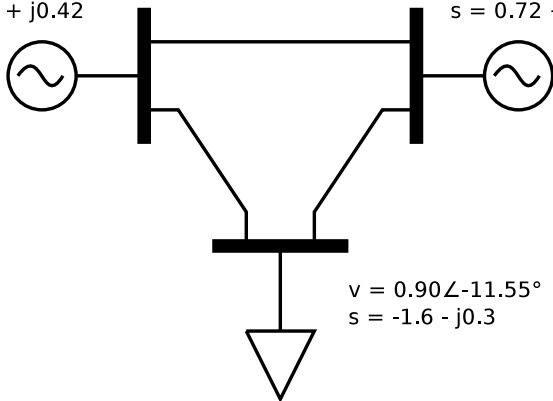


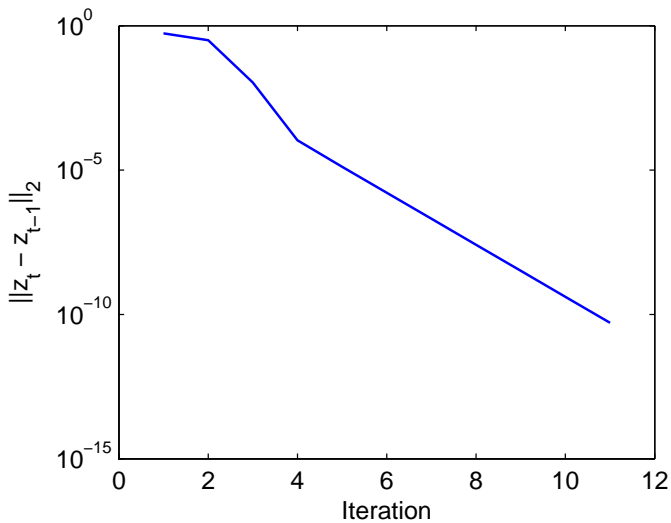
$$f(z) = p_1^2 + 2p_1 + 2p_2^2 + p_2$$

- OPF Solution

$$v = 1.0 \angle 0^\circ$$
$$s = 0.95 + j0.42$$

$$v = 1.0 \angle 0.84^\circ$$
$$s = 0.72 + j0.27$$



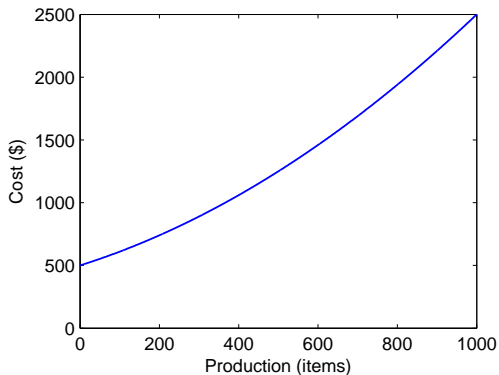


Convergence of solution in iterative optimization procedure

Power Markets

- How do we actually coordinate and price generation with consumption?
- Previously, done by single monopoly, could observe all loads and optimize generation accordingly.
- With competition (even just in generation), need a way to coordinate market instantaneously
 - Unlike traditional markets, can't wait for market to coverge through bidding system

- Theory of the firm



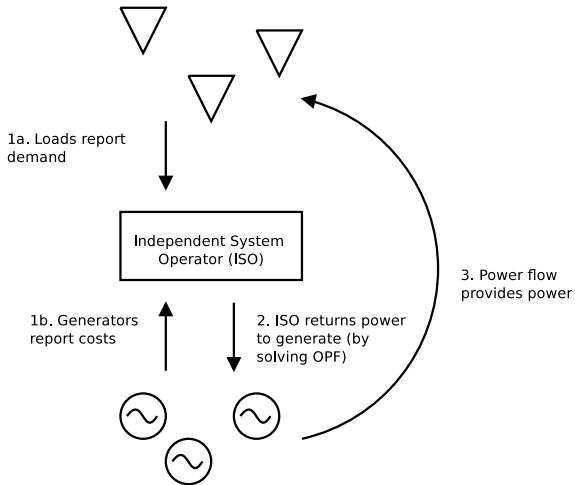
- Given market price π , determine how much to make by optimization problem

$$\underset{y}{\text{maximize}} \{ \pi y - C(y) \} \implies \pi = \frac{dC}{dy}$$

- $\frac{dC}{dy}$ is known as marginal price, cost to make one more item
 - By above relationship, a producer will produce items until the marginal price equals the market price
- In traditional market, market price is set by interaction of supply and demand

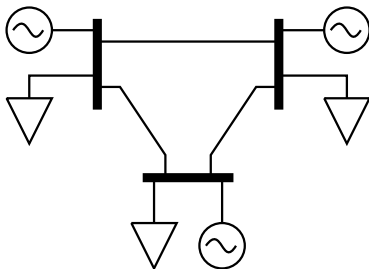
- Many idiosyncracies to power markets
 - Need to repond instantaneously to demand (no time for market to “settle” via a manual bidding system)
 - Can’t (yet) store significant portions of energy
 - Demand is typically viewed as fixed, producers (generators) must in total produce enough generation to match
 - Distribution of product (i.e., transmission of power) must obey the laws of physics

- Operation of centralized power market



Optimal Power Flow for Power Markets

- For generality, we'll usually consider case of power generators and loads at the different nodes



- We'll also explicitly separate out generator and load powers (if no generator at a node, then we'll just add a constraint that $s^G = 0$)

$$s_i = s_i^G - s_i^L$$

- A typical OPF formulation for power markets

$$\begin{aligned}
 & \underset{z}{\text{minimize}} \quad C(p^G) \equiv \sum_{i=1}^n C_i(p_i^G) \\
 & \text{subject to} \quad s^G - s^L = \text{diag}(v)(Yv)^* \\
 & \quad \quad \quad -v_i Y_{ij}^* (v_i - v_j)^* \leq S_{ij}^{\max}, \quad \forall i \neq j \\
 & \quad \quad \quad s^G \leq s^{G\max}
 \end{aligned}$$

where constraints are 1) power flow constraints, 2) transmission constraints, and 3) generation constraints

- Can view this as a “black box” that takes as input node loads, and outputs power generation, power flow, prices

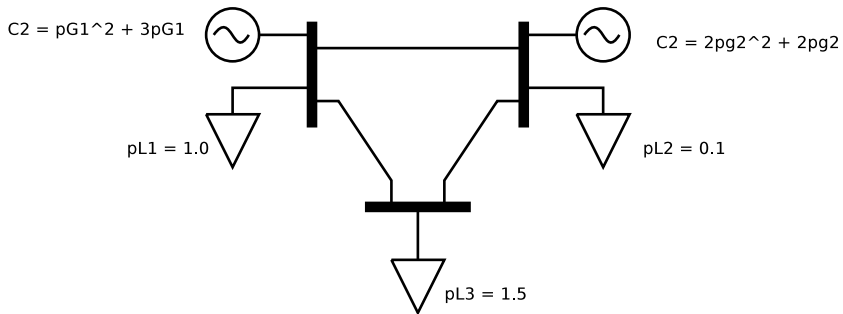
$$s^L \rightarrow OPF(s^L) \rightarrow z^*, C(z^*)$$

- Locational marginal prices (LMPs)

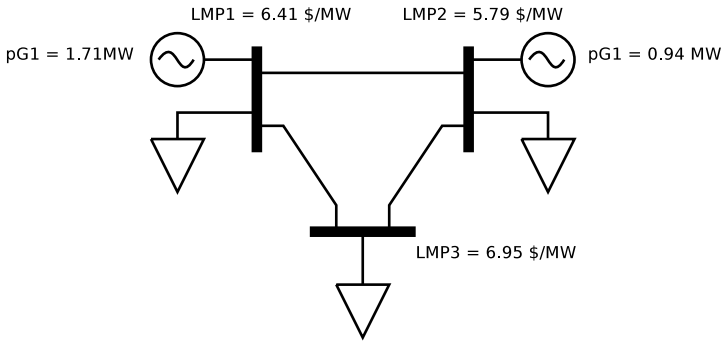
- One of the most important elements of power markets is that the marginal price of power can *vary* at different nodes
- Leads to locational marginal pricing, or nodal pricing

$$\begin{aligned} \text{LMP}_i &= \text{cost per unit of a little more (real) power at node } i \\ &= \frac{\partial C(z^*)}{\partial p_i^L} \end{aligned}$$

- Can compute these numerically (or, optimization methods often generate these types of derivatives automatically via *dual variables*)



Example power market costs and (real) loads



Resulting generation and LMPs

- In above example, LMP was highest at the node without a generator (intuitive, since it had to move power from elsewhere, incur losses)
- LMPs can also be *less* than marginal cost at *any* generator, or even be negative (much less intuitive)
 - Basic issue is that power flow equations + transmission constraints can conspire so that, for example, to obtain 1MW more at node 3 requires we *decrease* generator 2 by 1MW and increase generator 1 by 2MW
 - Given certain prices, this can *lower* the overall cost of power by increasing demand at node 3

- *Many* more elements to power markets
 - Run markets both on real-time and one day ahead
 - How do we handle reserve? Viewed as a “service” to grid and often bought separately by ISO
 - What about reactive power? Losses are much higher, so it’s not economical to supply it over longer distances, like with real power
 - Who pays for transmission? How do we handle pricing when transmission lines become congested?