15-830 – Control 1: Introduction

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Control Problems

- Thus far in the course, we have focused on allocation problems (e.g., optimal power flow) that are static in time
- Many tasks in sustainability have large *temporal* components
- We'll use the term "control tasks" to talk broadly about settings where we take actions over time in a dynamical system to minimize some total cost

- Control tasks are ubiquitous in energy/sustainability problems
  - Heating/cooling of buildings in response to occupancy
  - Storing electrical power in response to renewable resources and electricity pricing
  - Maximum power point tracking for renewable energy systems
  - Scheduling generation over time in the power grid

Some definitions

- **State:** $x \in \mathbb{R}^n$
  - Captures those elements of system that are relevant to its evolution but not controlled by agent
- **Control input:** $u \in \mathbb{R}^m$
  - Captures elements that are directly specified by the agent
- **Policy/controller:** $\pi : \mathbb{R}^n \to \mathbb{R}^m$
  - $u = \pi(x)$ prescribes input u as a function of state x

Dynamical Systems

- A dynamical system describes how state evolves over time, in response to control inputs
- Typical setting: *first-order* dynamics (next state or time derivative depends only on current state and control)
  - Discrete-time: $x_{t+1} = f(x_t, u_t)$
  - Continuous-time: $\frac{dx}{dt} \equiv \dot{x} = f(x, u)$
  - Can (approximately) transform between these two via Euler integration

$$x_{t+1} = x_t + (\Delta t) f(x_t, u_t)$$

- Lots of extensions and special cases of dynamical systems (we will cover these in more detail in the next lecture)
  - *Partially observable systems*: can't observe state x_t directly, just some observation $y_t \in \mathbb{R}^p$

$$y_t = h(x_t)$$

  - *Stochastic systems*: some uncertainty or noise in the dynamics

$$x_{t+1} = f(x_t, u_t) + \epsilon_t$$

  - *Markov Decision Processes*: a setting with general dynamics and stochasticity, typically applied to discrete state / discrete action settings

Control as Optimization

- Basis of *optimal control* is the notion of a cost function

$$C : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$$

  – $C(x, u)$ captures "badness" of being in state x and executing action u.

- Goal is to pick actions that minimize the sum of costs over some horizon T (also called cost-to-go, value function)

$$J(u_{1:T}) = \sum_{t=1}^{T} C(x_t, u_t) \quad \text{subject to} \quad x_{t+1} = f(x_t, u_t)$$

$$J(u_{1:T}) = \int_0^T C(x, u) dt \quad \text{subject to} \quad \dot{x} = f(x, t)$$

- Optimization formulation of (deterministic) optimal control

$$\begin{array}{ll} \underset{x_{1:T}, u_{1:T}}{\text{minimize}} & \displaystyle\sum_{t=1}^{T} C(x_t, u_t) \\ \text{subject to} & x_{t+1} = f(x_t, u_t), \;\; t = 1, \ldots, T-1 \\ & g(x_{1:T}, u_{1:T}) \leq 0, \;\; \text{(arbitrary inequality constraints)} \\ & h(x_{1:T}, u_{1:T}) = 0, \;\; \text{(arbitrary equality constraints)} \end{array}$$

- A very general way of formulating optimal control, but solving the optimization problem is still hard, except for special cases of systems

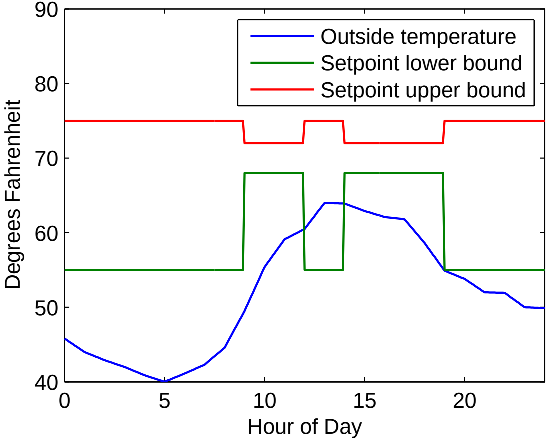
- Today, we'll focus on a specific case that can be solved efficiently: affine system + convex cost and constraints

$$
\begin{aligned}
\underset{x_{1:T},u_{1:T}}{\text{minimize}} \quad & \sum_{t=1}^{T} C(x_t, u_t) \\
\text{subject to} \quad & x_{t+1} = Ax_t + Bu_t + a_t, \quad t = 1, \ldots, T-1 \\
& g(x_{1:T}, u_{1:T}) \leq 0 \\
& h(x_{1:T}, u_{1:T}) = 0
\end{aligned}
$$

- Can again be solved using our off-the-shelf solvers (e.g. YALMIP)

Example 1: Building Heating

- Maintain building temperature within setpoints:



- Heat transfer dynamics (temperature T)

$$\rho c_p V \frac{dT}{dt} = -UA(T - T^{(\mathrm{external})})$$

- Define state space and control

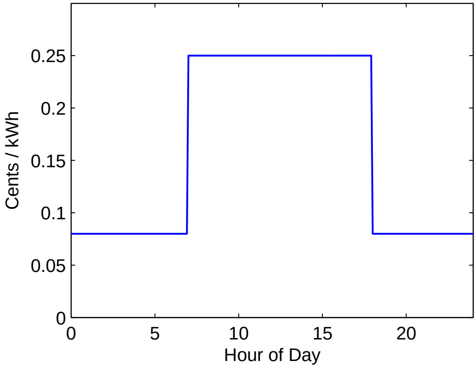
$$\begin{aligned} x_t &= \text{ internal temperature at time } t \\ u_t &= \text{ additional injected energy (i.e., heating) at time } t \end{aligned}$$

- Leads to discrete time approximation

$$x_{t+1} = x_t + k(T_t^{(\mathrm{external})} - x_t) + bu_t$$

  - A linear dynamical system
  - Physical constants and time step all folded into constants k, b

- Time-based pricing of electricity:



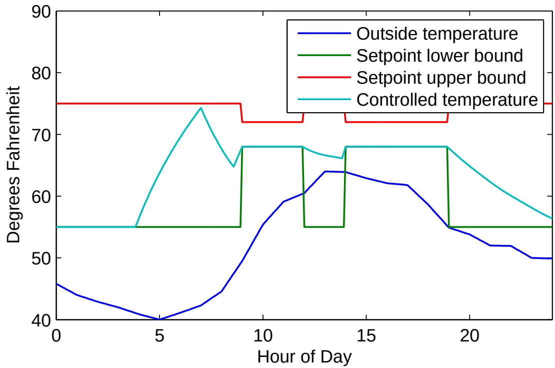
- Task: minimize cost of electricity while maintaining building withing heating bounds

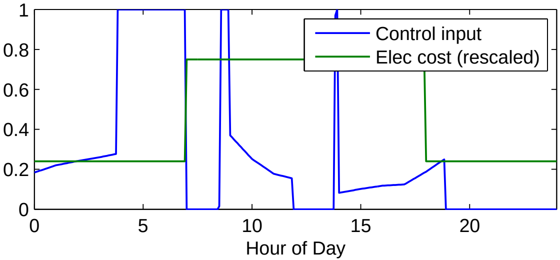
- Formulate this as the optimization problem (LP):

$$
\begin{aligned}
\underset{x,u}{\text{minimize}} \quad & c^T u \\
\text{subject to} \quad & x_{t+1} = x_t + k(T_t^{(\text{external})} - x_t) + bu_t \\
& x_1 = T_1 \\
& T_l \le x \le T_u \\
& 0 \le u \le 1
\end{aligned}
$$

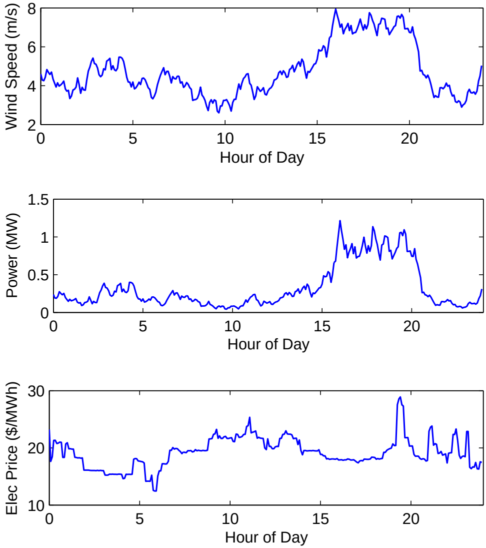
- MATLAB code:

```
x = sdpvar(T,1);
u = sdpvar(T,1);
C = [x(1) == 55];
for i=1:T-1,
  C = [C; x(i+1) == x(i) + k*(Te(i)-x(i)) + b*u(i)];
end
C = [C; x >= Tl; x <= Tu];
C = [C; u >= 0; u <= 1];
solvesdp(C, c'*u);
```


Outside temperature
Setpoint lower bound
Setpoint upper bound
Controlled temperature
Degrees Fahrenheit
90
80
70
60
50
40
0
5
10
15
20
Hour of Day


Control input
Elec cost (rescaled)
1
0.8
0.6
0.4
0.2
0
0
5
10
15
20
Hour of Day

Example 2: Energy Storage for Wind



- Lacking storage, we need to just sell energy generated at time t at the current market price

- If we have an energy storage system attached to the turbine, we can store energy if price is low, sell when high

- State space and control:

$$x_t = \text{ total energy stored in battery at time } t$$
$$u_t = \text{ energy put into or taken out of battery at time } t$$

- System dynamics

$$x_{t+1} = x_t + u_t \;\; (0 \le x_t \le E_{\max})$$

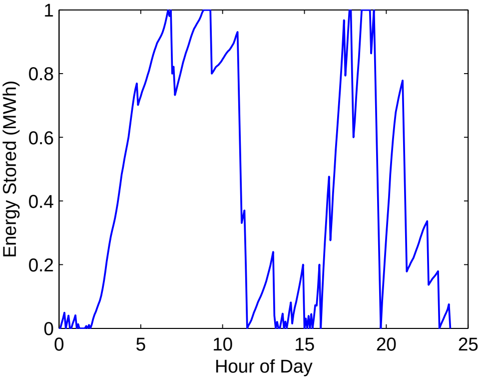
- External variables:

$$e_t = \text{ wind energy generated at time } t$$
$$c_t = \text{ cost of electricity at time } t$$

- Optimization formulation

$$
\begin{aligned}
\underset{x,u}{\text{minimize}} \quad & c^T(e-u) \\
\text{subject to} \quad & x_{t+1} = x_t + u_t, \quad t = 1, \dots, T-1 \\
& 0 \le x \le E_{\max} \\
& -E_{\text{ramp}} \le u \le E_{\text{ramp}} \\
& e - u \ge 0
\end{aligned}
$$

- Solution (energy stored over the day):



In this case, make 17% more money from including storage (but real advantage is when wind penetration high, for avoiding spinning reserve)

A preview of issues to come

- These example were very simple dynamical systems; how do we handle more complex dynamics?
- In our optimization problems we assumed the dynamics were deterministic and future was known; how do we deal with stochasticity and uncertain predictions?
- Exact optimal solutions will often be impossible to obtain, but approximate methods like *Model Predictive Control* often work very well in practice.