On Unifying Deep Generative Models:
Supplementary Materials

1 Adversarial Domain Adaptation (ADA)

ADA aims to transfer prediction knowledge learned from a source domain with labeled data to a target domain without labels, by learning domain-invariant features. Let $D_\phi(x) = q_\phi(y|x)$ be the domain discriminator. The conventional formulation of ADA is as following:

$$
\begin{align*}
\max_{\phi} \mathcal{L}_\phi &= \mathbb{E}_{x = \mathcal{G}_\theta(z), z \sim p(z)} [\log D_\phi(x)] + \mathbb{E}_{x = \mathcal{G}_\theta(z), z \sim p(z)} [\log(1 - D_\phi(x))], \\
\max_{\theta} \mathcal{L}_\theta &= \mathbb{E}_{x = \mathcal{G}_\theta(z), z \sim p(z)} [\log(1 - D_\phi(x))] + \mathbb{E}_{x = \mathcal{G}_\theta(z), z \sim p(z)} [\log D_\phi(x)].
\end{align*}
$$

(1)

Further add the supervision objective of predicting label $t(z)$ of data $z$ in the source domain, with a classifier $f_\omega(t|x)$ parameterized with $\pi$:

$$
\max_{\omega, \theta} \mathcal{L}_{\omega, \theta} = \mathbb{E}_{z \sim p(z)} [\log f_\omega(t(z)|\mathcal{G}_\theta(z))].
$$

(2)

We then obtain the conventional formulation of adversarial domain adaptation used or similar in $[3, 5, 6, 2]$.

2 Proof of Lemma 1

Proof.

$$
\begin{align*}
\mathbb{E}_{p(y)p(y)} [\log q^*(y|x)] &= \\
&= -\mathbb{E}_{p(y)} [\text{KL}(p_\theta(x|y)||q^*(x|y)) - \text{KL}(p_\theta(x|y)||p_{\theta_0}(x))] ,
\end{align*}
$$

(3)

where

$$
\begin{align*}
\mathbb{E}_{p(y)} [\text{KL}(p_\theta(x|y)||p_{\theta_0}(x))] &= p(y = 0) \cdot \text{KL} \left( p_\theta(x|y = 0) || \frac{p_{\theta_0}(x|y = 0) + p_{\theta_0}(x|y = 1)}{2} \right) \\
&+ p(y = 1) \cdot \text{KL} \left( p_\theta(x|y = 1) || \frac{p_{\theta_0}(x|y = 0) + p_{\theta_0}(x|y = 1)}{2} \right) .
\end{align*}
$$

(4)

Note that $p_\theta(x|y = 0) = p_{\theta_0}$, and $p_\theta(x|y = 1) = p_{\text{data}}(x)$. Let $p_M = \frac{p_{\theta_0} + p_{\text{data}}}{2}$. Eq. (4) can be simplified as:

$$
\mathbb{E}_{p(y)} [\text{KL}(p_\theta(x|y)||p_{\theta_0}(x))] = \frac{1}{2} \text{KL} \left( p_{\theta_0} || p_{M_{\theta_0}} \right) + \frac{1}{2} \text{KL} \left( p_{\text{data}} || p_{M_{\theta_0}} \right) .
$$

(5)
We stick to the notational convention in the paper that parameter \( \eta \) is associated with the distribution over \( \mathbf{z} \) of InfoGAN in the graphical model, as shown in Figure 1. To see this, we directly write down the objectives represented by the graphical model in the right panel, and show how one can translate a graphical model representation to the mathematical formulations. Readers can do similarly on graphical models of GANs, InfoGANs, VAEs, and many other relevant variants and write down the respective objectives conveniently.

We stick to the notational convention in the paper that parameter \( \theta \) is associated with the distribution over \( \mathbf{x} \), parameter \( \eta \) with the distribution over \( \mathbf{z} \), and parameter \( \phi \) with the distribution over \( \mathbf{y} \). Besides, we use \( p \) to denote the distributions over \( \mathbf{x} \), and \( q \) the distributions over \( \mathbf{z} \) and \( \mathbf{y} \).

From the graphical model, the inference process (dashed-line arrows) involves implicit distribution \( q_\eta (\mathbf{z}|\mathbf{y}) \) (where \( \mathbf{x} \) is encapsulated). As in the formulations of GANs (Eq.4 in the paper) and VAEs
With the above components, we maximize the log likelihood of the generative distributions. Again, the only difference between the objectives of

\[ q_\eta(z | y) = \begin{cases} q_\eta(z | y = 0) & y = 0 \\ q(z) & y = 1, \end{cases} \tag{9} \]

where, as \( z \) is the hidden code, \( q(z) \) is the prior distribution over \( z \) and the space of \( x \) is degenerated. Here \( q_\eta(z | y = 0) \) is the implicit distribution such that

\[ z \sim q_\eta(z | y = 0) \iff z = E_\eta(x), \ x \sim p_{data}(x), \tag{10} \]

where \( E_\eta(x) \) is a deterministic transformation parameterized with \( \eta \) that maps data \( x \) to code \( z \). Note that as \( x \) is a visible variable, the pre-fixed distribution of \( x \) is the empirical data distribution.

On the other hand, the generative process (solid-line arrows) involves \( p_\theta(x|z,y)q_\phi^{(r)}(y|z) \) (here \( q^{(r)} \) means we will swap between \( q^{(r)} \) and \( q \)). As the space of \( x \) is degenerated given \( y = 1 \), thus \( p_\theta(x|z,y) \) is fixed without parameters to learn, and \( \theta \) is only associated to \( y = 0 \).

With the above components, we maximize the log likelihood of the generative distributions \( \log p_\theta(x|z,y)q_\phi^{(r)}(y|z) \) conditioning on the variable \( z \) inferred by \( q_\eta(z | y) \). Adding the prior distributions, the objectives are then written as

\[
\max_\phi L_\phi = \mathbb{E}_{q_\eta(z|y)p(y)} \left[ \log p_\theta(x|z,y)q_\phi(y|z) \right] \\
\max_{\theta,\eta} L_{\theta,\eta} = \mathbb{E}_{q_\eta(z|y)p(y)} \left[ \log p_\theta(x|z,y)q_\phi^{(r)}(y|z) \right]. \tag{11} \]

Again, the only difference between the objectives of \( \phi \) and \( \{\theta, \eta\} \) is swapping between \( q_\phi(y|z) \) and its reverse \( q_\phi^{(r)}(y|z) \).

To make it clearer that Eq.(11) is indeed the original AAE proposed in [4], we transform \( L_\phi \) as

\[
\max_\phi L_\phi = \mathbb{E}_{q_\eta(z|y)p(y)} \left[ \log q_\phi(y|z) \right] \\
= \frac{1}{2} \mathbb{E}_{q_\eta(z|y=0)} \left[ \log q_\phi(y = 0|z) \right] + \frac{1}{2} \mathbb{E}_{q_\eta(z|y=1)} \left[ \log q_\phi(y = 1|z) \right] \\
= \frac{1}{2} \mathbb{E}_{z=E_\eta(x), x \sim p_{data}(x)} \left[ \log q_\phi(y = 0|z) \right] + \frac{1}{2} \mathbb{E}_{z \sim q(z)} \left[ \log q_\phi(y = 1|z) \right]. \tag{12} \]

That is, the discriminator with parameters \( \phi \) is trained to maximize the accuracy of distinguishing the hidden code either sampled from the true prior \( p(z) \) or inferred from observed data example \( x \). The objective \( L_{\theta,\eta} \) optimizes \( \theta \) and \( \eta \) to minimize the reconstruction loss of observed data \( x \) and at the same time to generate code \( z \) that fools the discriminator. We thus get the conventional view of the AAE model.

**Predictability Minimization (PM)** [7] is the early form of adversarial approach which aims at learning code \( z \) from data such that each unit of the code is hard to predict by the accompanying code predictor based on remaining code units. AAE closely resembles PM by seeing the discriminator as a special form of the code predictors.

**CycleGAN** [8] is the model that learns to translate examples of one domain (e.g., images of horse) to another domain (e.g., images of zebra) and vice versa based on unpaired data. Let \( x \) and \( z \) be the variables of the two domains, then the objectives of AAE (Eq.[11]) is precisely the objectives that train the model to translate \( x \) into \( z \). The reversed translation is trained with the objectives of InfoGAN (Eq.9 in the paper), the symmetric counterpart of AAE.

\[^1\text{See section 6 of the paper for the detailed discussion on prior distributions of hidden variables and empirical distribution of visible variables.}\]
4 Proof of Lemme 2

Proof. For the reconstruction term:
\[
E_{p_{θ₀}}(x) \left[ E_{q_η(z|x,y)} q_η^*(y|x) \left[ \log p_θ(x|z,y) \right] \right]
= \frac{1}{2} E_{p_{θ₀}}(x|y=1) \left[ E_{q_η(z|x,y=0), y=0 \sim q_η^*(y|x)} \left[ \log p_θ(x|z,y=0) \right] \right]
+ \frac{1}{2} E_{p_{θ₀}}(x|y=0) \left[ E_{q_η(z|x,y=1), y=1 \sim q_η^*(y|x)} \left[ \log p_θ(x|z,y=1) \right] \right]
= \frac{1}{2} E_{p_{θ₀}}(x) \left[ E_{q_η(z|x)} \left[ \log ρ_θ(x|z) \right] \right] + \text{const.}
\]

where \( y = 0 \sim q_η^*(y|x) \) means \( q_η^*(y|x) \) predicts \( y = 0 \) with probability 1. Note that both \( q_η(z|x, y = 1) \) and \( p_θ(x|z, y = 1) \) are constant distributions without free parameters to learn; \( q_η(z|x, y = 0) = \tilde{q}_η(z|x) \), and \( p_θ(x|z, y = 0) = \tilde{p}_θ(x|z) \).

For the KL prior regularization term:
\[
E_{p_{θ₀}}(x) \left[ \text{KL}(q_η(z|x,y) q_η^*(y|x)||p(z|y)p(y)) \right]
= \frac{1}{2} E_{p_{θ₀}}(x) \left[ \int q_η^*(y|x) \text{KL}(q_η(z|x,y)||p(z|y)) dy + \text{KL}(q_η^*(y|x)||p(y)) \right]
= \frac{1}{2} E_{p_{θ₀}}(x|y=1) \left[ \text{KL}(q_η(z|x,y=0)||p(z|y=0)) + \text{const} \right] + \frac{1}{2} E_{p_{θ₀}}(x|y=1) \left[ \text{const} \right]
= \frac{1}{2} E_{p_{θ₀}}(x) \left[ \text{KL}(\tilde{q}_η(z|x)||\tilde{p}(z)) \right].
\]

Combining Eq.(13) and Eq.(14) we recover the conventional VAE objective in Eq.(7) in the paper. \( \square \)

5 Importance Weighted GANs (IWGAN)

From Eq.(4) in the paper, we can view GANs as maximizing a lower bound of the “marginal log-likelihood”:
\[
\log q(y) = \log \int p_θ(x|y) q_η^*(y|x) p_θ(x|y) dx \\
\geq \int p_θ(x|y) \log \frac{q_η^*(y|x) p_θ(x|y)}{p_θ(x|y)} dx \\
= -\text{KL}(p_θ(x|y)|| q_η^*(x|y)) + \text{const.}
\]

We can apply the same importance weighting method as in IWAE [1] to derive a tighter bound.
\[
\log q(y) = \log \mathbb{E} \left[ \frac{1}{k} \sum_{i=1}^{k} q_η^*(y|x_i) p_θ(x_i|y) \right] \\
\geq \mathbb{E} \left[ \log \frac{1}{k} \sum_{i=1}^{k} q_η^*(y|x_i) p_θ(x_i|y) \right] \\
= \mathbb{E} \left[ \log \frac{1}{k} \sum_{i=1}^{k} w_i \right] := \mathcal{L}_k(y)
\]
where we have denoted \( w_i = \frac{q_η^*(y|x_i) p_θ(x_i|y)}{p_θ(x_i|y)} \). We recover the lower bound of Eq.(15) when setting \( k = 1 \).

To maximize the importance weighted lower bound, we compute the gradient:
\[
\nabla_θ \mathcal{L}_k(y) = \nabla_θ \mathbb{E}_{x_1, \ldots, x_k} \left[ \log \frac{1}{k} \sum_{i=1}^{k} w_i \right] = \mathbb{E}_{x_1, \ldots, x_k} \left[ \nabla_θ \log \frac{1}{k} \sum_{i=1}^{k} w(y, x_i, θ) \right] \\
= \mathbb{E}_{x_1, \ldots, x_k} \left[ \sum_{i=1}^{k} w_i \nabla_θ \log w(y, x_i, θ) \right],
\]

4
where \( \tilde{w}_i = w_i / \sum_{i=1}^{k} w_i \) are the normalized importance weights. We expand the weight at \( \theta = \theta_0 \)

\[
w_i|_{\theta=\theta_0} = \frac{q'(y|x_i)p_{\theta_0}(x_i)}{p_{\theta}(x_i|y)} = \frac{q'(y|x_i)}{\frac{1}{2} p_{\theta}(x_i|y = 0) + \frac{1}{2} p_{\theta}(x_i|y = 1)}|_{\theta=\theta_0}.
\]

The ratio of \( p_{\theta_0}(x_i|y = 0) \) and \( p_{\theta_0}(x_i|y = 1) \) is intractable. Using the Bayes’ rule and approximating with the discriminator distribution, we have

\[
p(x_i|y = 0)p(x_i|y = 1) = p(y = 0|x) p(y = 1|x) \approx q(y = 0|x) q(y = 1|x).
\]

Plug Eq. (19) into the above we have

\[
w_i|_{\theta=\theta_0} \approx \frac{q'(y|x_i)}{q(y|x_i)}.
\]

In Eq. (17), the derivative \( \nabla_\theta \log w_i \) is

\[
\nabla_\theta \log w_i (y, x(z_i, \theta)) = \nabla_\theta \log q'(y|x(z_i, \theta)) + \nabla_\theta \log \frac{p_{\theta_0}(x_i)}{p_{\theta}(x_i|y)}.
\]

Similar to GAN, we omit the second term on the RHS of the equation. Therefore, the resulting update rule of \( p_{\theta}(x|y) \) is

\[
\nabla_\theta \mathcal{L}_k(y) = \mathbb{E}_{x_1, \ldots, x_k} \left[ k \sum_{i=1}^{k} \tilde{w}_i \nabla_\theta \log q'(y|x(z_i, \theta)) \right]
\]

6 Experimental Results of SVAE

Table 1 shows the results.

<table>
<thead>
<tr>
<th></th>
<th>1%</th>
<th>10%</th>
</tr>
</thead>
<tbody>
<tr>
<td>SVAE</td>
<td>0.9412±0.0039</td>
<td>0.9768±0.0009</td>
</tr>
<tr>
<td>AASVAE</td>
<td><strong>0.9425±0.0045</strong></td>
<td><strong>0.9797±0.0010</strong></td>
</tr>
</tbody>
</table>

Table 1: Classification accuracy of semi-supervised VAEs and the adversary activated extension on the MNIST test set, with varying size of real labeled training examples.

References


