**INTRODUCTION AND OBJECTIVE**

- **Higher-order TV denoising**
  - Underlying signal and data
  - Laplacian smoothing, large d
  - Laplacian smoothing, small d

  TV denoising

  Graph trend filtering

  Kronecker trend filtering

- **Optimal rates (d-dim grids, Higher-order TV denoising)**

  - **Nonparametric Regression on Graphs (d-dim grids)**
    - $y_i \sim N(\theta_i, \sigma^2), \text{ i.i.d., for } i = 1, \ldots, n$.
    - $y$ is observed on every vertex of a graph.
    - Estimate $\theta_0$ using noisy observation $y$.

  - **Optimal rates (d-dim grids, kth order TV)**
    - $d = 1 \quad n^{-2(k+2)/(k+3)}$ (Trend Filter)
    - $d > 1 \quad C_n/n$ (TV-denoising)

- **Questions of interest**
  1. What is the discrete analog of kth order TV on grids ($d > 1$)?
  2. Theoretically quantifying the denoising performance
    - How fast does MSE converge to 0 as we get more pixels?
  3. Information-theoretic limit
    - How fast does it get for any method?

**KRONESKER TF AND GRAPH TF**

- **Kronecker Trend Filtering (KTF)**
  - Penalty is the sum of univariate penalties along rows and columns
  - $\|\Delta_{k}^{(k+1)}\|_1 = \sum_{j=1}^{N} \|D_{id}^{(k+1)}\|_1 + \sum_{i=1}^{N} \|D_{id}^{(k+1)}\|_1$

**Graph Trend Filtering (GTF)** (Wang et al., 2014):

- $\Delta_{k}^{(1)} = \Delta_{k}^{(2)} = \Delta_{k}^{(3)} = \Delta_{k}^{(4)} = D, L, DL, L^2, \ldots$

  - Where $L = D^T D$ is the Laplacian of the grid. For $k = 1$,
    - GTF: $|\theta_1 - 2\theta_0 + \theta_2| + |\theta_3 - 2\theta_0 + \theta_4|$
    - KTF: $|\theta_1 - 2\theta_0 + \theta_2| + |\theta_3 - 2\theta_0 + \theta_4|$

- $\text{null}(\Delta_{k}^{(k+1)})$: $p \otimes q$ where $p, q$ polynomials of degree $\leq k$
- $\text{null}(\Delta_{k}^{(k+1)})$ is 1: constant function.

**FUNCTION CLASSES/SMOOTHNESS**

- **GT class**
  - $T_1(\theta_0) = \{\theta \mid |\Delta_{k}^{(1)}|_{\theta_0} \leq \theta_{k} \}$

- **KTF class**
  - $T_2(\theta_0) = \{\theta \mid |\Delta_{k}^{(2)}|_{\theta_0} \leq \theta_{k} \}$

- **Holder class**
  - $T_3\beta(\theta_0) = \{\theta \mid |\Delta_{k}^{(3)}|_{\theta_0} \leq \theta_{k} \}$

**REFERENCES**