Surviving rates of trees and outerplanar graphs for the firefighter problem

Leizhen Cai* Yongxi Cheng† Elad Verbin‡ Yuan Zhou‡

Abstract

The firefighter problem is a discrete-time game on graphs introduced by Hartnell in an attempt to model the spread of fire, diseases, computer viruses and suchlike in a macro-control level. To measure the defence ability of a graph as a whole, Cai and Wang defined the *surviving rate* of a graph G for the firefighter problem to be the average percentage of vertices that can be saved when a fire starts randomly at one vertex of G.

In this paper, we prove that the surviving rate of every n-vertex outerplanar graph is at least $1 - \Theta(\frac{\log n}{n})$, which is asymptotically tight. We also show that the greedy strategy of Hartnell and Li for trees saves at least $1 - \Theta(\frac{\log n}{n})$ percentage of vertices on average for an n-vertex tree.

Key words: firefighter problem, surviving rate, tree, outerplanar graph

1 Introduction

The study of a discrete-time game, the *firefighter problem*, was initiated by Hartnell [7] in 1995 in an attempt to model the spread of fire, diseases, computer viruses and suchlike in a macro-control level. At time 0, a fire breaks out at a vertex v of a graph $G = (V, E)$. At each subsequent time, a firefighter *protects* one vertex not yet on fire, and the fire then spreads from all burning vertices to all their unprotected neighbors. The process ends when the fire can no longer spread. Once a vertex is burning or protected, it remains so during the whole process, and in the end all vertices that are not burning are *saved*. A general objective of the firefighter is to save as many vertices as possible.

Finbow et al. [4] showed that it is NP-hard for the firefighter to save the maximum number of vertices, even for trees of maximum degree three, but polynomial-time solvable

*Department of Computer Science and Engineering, The Chinese University of Hong Kong, Shatin, Hong Kong SAR, China. E-mail address: lcai@cse.cuhk.edu.hk.

†Department of Computing Science, University of Alberta, Edmonton, Alberta T6G 2E8, Canada. E-mail address: chengyx@gmail.com.

‡The Institute for Theoretical Computer Science, Tsinghua University, Beijing 100084, China. E-mail address: {elad.verbin, timzhouyuan}@gmail.com.

for graphs of maximum degree three with the fire starting at a vertex of degree at most two. MacGillivray and Wang [9] gave a 0-1 integer programming formulation of the problem for trees and solved the problem in polynomial time for some subclasses of trees, and recently Cai et al. [1] considered several algorithmic issues concerning firefighting on trees. Develin and Hartke [3], Fogarty [6], and Wang and Moeller [11] considered a variation that allows more than one firefighters, and studied the number of firefighters required to contain the fire for d-dimensional grids. Recently, Ng and Raff [10] investigated a generalization of the firefighter problem on two dimensional infinite grids when the number of firefighters available per time step is not a constant but periodic. Other variations of the problem have also been considered in the literature; for instance, to put out the fire as quickly as possible, or to save a given subset of vertices. We refer the reader to a recent survey by Finbow and MacGillivray [5].

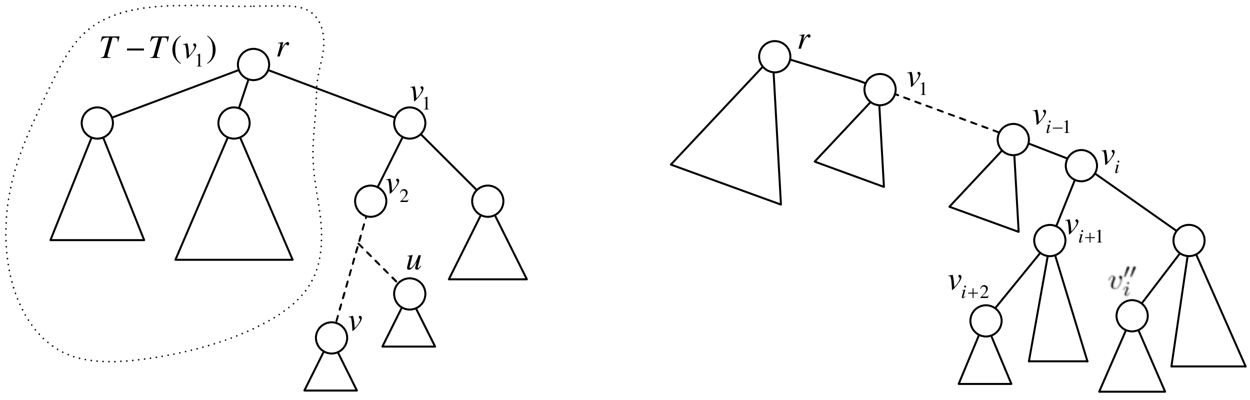
For a vertex v in G, let $sn(v)$ denote the maximum number of vertices in G the firefighter can save when the fire breaks out at v. To mensure the defence ability of an n-vertex graph G as a whole, Cai and Wang [2] defined the *surviving rate* $\rho(G)$ of G to be the average percentage of vertices that can be saved when the fire starts randomly at one vertex of the graph, i.e., $\rho(G) = \frac{\sum_{v \in V} sn(v)}{n^2}$. They showed that $\rho(T) \geq 1 - \sqrt{\frac{2}{n}}$ for every tree T, $\rho(G) > 1/6$ for every outerplanar graph G, and $\rho(H) > 0.3$ for every Halin graph H with at least 5 vertices. They also conjectured that $\rho(T) \geq 1 - \Theta(\frac{\log n}{n})$ for every tree T and asked whether $\rho(G)$ tends to 1 for any n-vertex outerplanar graph when n tends to infinity.

In this paper, we study surviving rates of trees and outerplanar graphs. We show in Section 2 that the greedy strategy of Hartnell and Li [8] for trees on average saves $1-\Theta(\frac{\log n}{n})$ percentage of vertices, and in Section 3 we obtain the asymptotically tight bound $1-\Theta(\frac{\log n}{n})$ for the surviving rate of outerplanar graphs. Our results confirm a conjecture of Cai and Wang [2] and also answer two of their questions in affirmative. In the paper, log is of base 2, and for a graph G we use $|G|$ to denote the number of vertices of G. An *outerplanar graph* is a graph that can be embedded on the plane without crossing edges and with all vertices on the boundary of the exterior face.

2 Firefighting on trees

For the firefighter problem on trees, the following greedy method of Hartnell and Li [8] achieves approximation ratio 1/2 for the number of saved vertices: the firefighter always protects a vertex that cuts off the maximum number of non-burning vertices from the fire. In this section, we prove that their greedy method on average saves $1 - \Theta(\frac{\log n}{n})$ percentage of vertices, thus achieves an approximation ratio of $1 - \Theta(\frac{\log n}{n})$ for the surviving rates of trees. This also settles the conjecture of Cai and Wang [2] that the surviving rate of a tree is at least $1 - \Theta(\frac{\log n}{n})$. On the other hand, we also construct a tree whose surviving rate is at most $1 - \Theta(\frac{\log n}{n})$.

Let T be a tree. The greedy method of Hartnell and Li produces a strategy for the firefighter, which will be called an *HL-strategy* for T. Note that an HL-strategy for T is not



(a) (b)

Figure 1. (a): All fire-sources of v are on the (r, v)-path; (b): $|T(v_i)| \geq 2|T(v_{i+2})|$.

unique, since according to the greedy method at one time step there may be more than one vertex that the firefighter can choose to protect. A vertex u is a *fire-source* for a vertex v if when the fire starts at u, the greedy method of Hartnell and Li cannot always save v, i.e., there is an HL-strategy that will not save v.

Theorem 2.1 *For a tree T, the greedy method of Hartnell and Li saves on average at least $1 - \Theta(\frac{\log n}{n})$ percent of vertices when the fire starts randomly at one vertex of T.*

Proof: To prove the theorem, we need only prove that no vertex $v \in T$ has more than $3 + 2\log_2 n$ fire-sources. Let r be a fire-source for v that is farthest away from v in T, and $P = v_0, v_1, \ldots, v_k$ the (r, v)-path in T, where $v_0 = r$ and $v_k = v$. Regard T as a rooted tree with root r, and denote the subtree rooted at vertex x by $T(x)$.

We first show that all fire-sources for v are on the (r, v)-path P. Suppose that there is a fire-source $u \notin P$ for v. Then u is in $T(v_1)$ as any vertex not in $T(v_1)$ is farther away from v than r. Since r is a fire-source for v, some HL-strategy will not protect v_1 when a fire starts at r. Therefore $|T(v_1)| < n/2$ and hence $|T - T(v_1)| > n/2$, which implies that $|T - T(u)| > n/2$ as $T - T(u)$ contains $T - T(v_1)$. This indicates that, when a fire starts at u, any HL-strategy would have saved the parent of u and hence v, a contradiction to u being a fire-source for v (see Figure 1 (a)).

Next we show that if v_i is a fire-source for v then $|T(v_i)| > 2|T(v_{i+2})|$, where $1 \leq i \leq k-2$. Consider the situation when the fire starts at vertex v_i. Since $|T - T(v_i)| > n/2$ (note that $T - T(v_i)$ contains $T - T(v_1)$), any HL-strategy will protect v_{i-1} at time 1. By the assumption that v_i is a fire-source for v, we see that there is an HL-strategy that does not protect v_{i+2} but another grandchild v_i'' of v_i at time 2. This implies that $|T(v_i'')| \geq |T(v_{i+2})|$ and hence $|T(v_i)| > 2|T(v_{i+2})|$ (see Figure 1 (b)).

Now let $v_{s(0)}, v_{s(1)}, \ldots, v_{s(t)} \in P$ be fire-sources of v ordered from r to v. Then $|T(v_{s(i)})| >$

$2|T(v_{s(i+2)})|$ as $T(v_{s(i+2)})$ is a subtree of $T(v_{s(i)+2})$. Therefore

$$|T(v_{s(1)})| > 2|T(v_{s(3)})| > \cdots > 2^i|T(v_{s(2i+1)})| > \cdots,$$

which implies $t \leq 2 + 2\log_2 n$ as $|T(v_{s(1)})| < n$ and hence the lemma. □

The above theorem answers a question of Cai and Wang [2] in affirmative that the greedy method of Hartnell and Li achieves approximation rate $1 - \Theta(\frac{\log n}{n})$ for the surviving rate of a tree, and also settles the following conjecture of Cai and Wang regarding surviving rates of trees.

Corollary 2.2 *The surviving rate of every tree is at least* $1 - \Theta(\frac{\log n}{n})$.

In fact, the above lower bound for surviving rates of trees is asymptotically best possible, which is established by the following theorem.

Theorem 2.3 *Let T_h be a balanced complete ternary tree (i.e., each non-leaf vertex has three children) of height h and with n vertices. Then* $\rho(T_h) \leq 1 - \Theta(\frac{\log n}{n})$.

Proof: We will prove the following: If the fire starts at a vertex v of height k $(0 \leq k \leq h)$, let T_k denote the subtree with v as its root, then the number of burnt leaves of T_k in the end is at least $\frac{1}{2}(3^k+1)$ under any protecting strategy. Since n, the number of vertices of T_h, is $\sum_{i=0}^{h} 3^i = \frac{3^{h+1}-1}{2} = \Theta(3^h)$, this implies that no matter what protecting strategy is adopted, when the fire starts randomly at one vertex of T_h, the minimum average percentage of vertices that will get burnt in the end is at least

$$\frac{\sum_{k=0}^{h} \frac{1}{2}(3^k+1) \times 3^{h-k}}{n^2} \geq \frac{\sum_{k=0}^{h} \frac{3^h}{2}}{n^2} = \frac{(h+1)3^h}{2n^2} = \Theta(\frac{h}{n}) = \Theta(\frac{\log n}{n}).$$

which implies the theorem.

In what follows we consider the subtree T_k with root v, and assume that the fire starts at v. Then, within T_k the fire stops to propagate at time k. Thus, we can assume without loss of generality that the number of protected vertices in T_k is at most k. Furthermore, at time i $(0 \leq i \leq k)$, the fire stops to propagate among the vertices in T_k having distance at most i from the root v, therefore we can assume that there are at most i protected vertices in T_k which are within distance i from v. Let a_j denote the number of protected vertices in T_k that have distance j from v, $j = 0, 1, \cdots, k$. Then, $\sum_{j=0}^{i} a_j \leq i$, for $i = 0, 1, \cdots, k$.

If the fire starts at v, for each leaf u of T_k which is saved in the end, there must exist an ancestor w of u such that w is in T_k and w is protected at some time t, where $1 \leq t \leq k$. Therefore, the total number of leaves of T_k that are saved in the end cannot exceed $\sum_{j=0}^{k} a_j \times 3^{k-j}$. Define $b_i = \sum_{j=0}^{i} a_j$, then $b_i \leq i$ for $i = 0, 1, \cdots, k$ and thus

$$\begin{aligned} \sum_{j=0}^{k} a_j \times 3^{k-j} &= \sum_{j=1}^{k} (b_j - b_{j-1}) \times 3^{k-j} = b_k + \sum_{j=1}^{k-1} b_j \times (3^{k-j} - 3^{k-j-1}) \\ &\leq k + \sum_{j=1}^{k-1} j \times (3^{k-j} - 3^{k-j-1}) = \frac{3^k - 1}{2}. \end{aligned}$$

Therefore, the total number of burnt leaves of T_k is at least $3^k - \frac{3^k-1}{2} = \frac{3^k+1}{2}$, which completes the proof of the theorem. □

3 Firefighting on outerplanar graphs

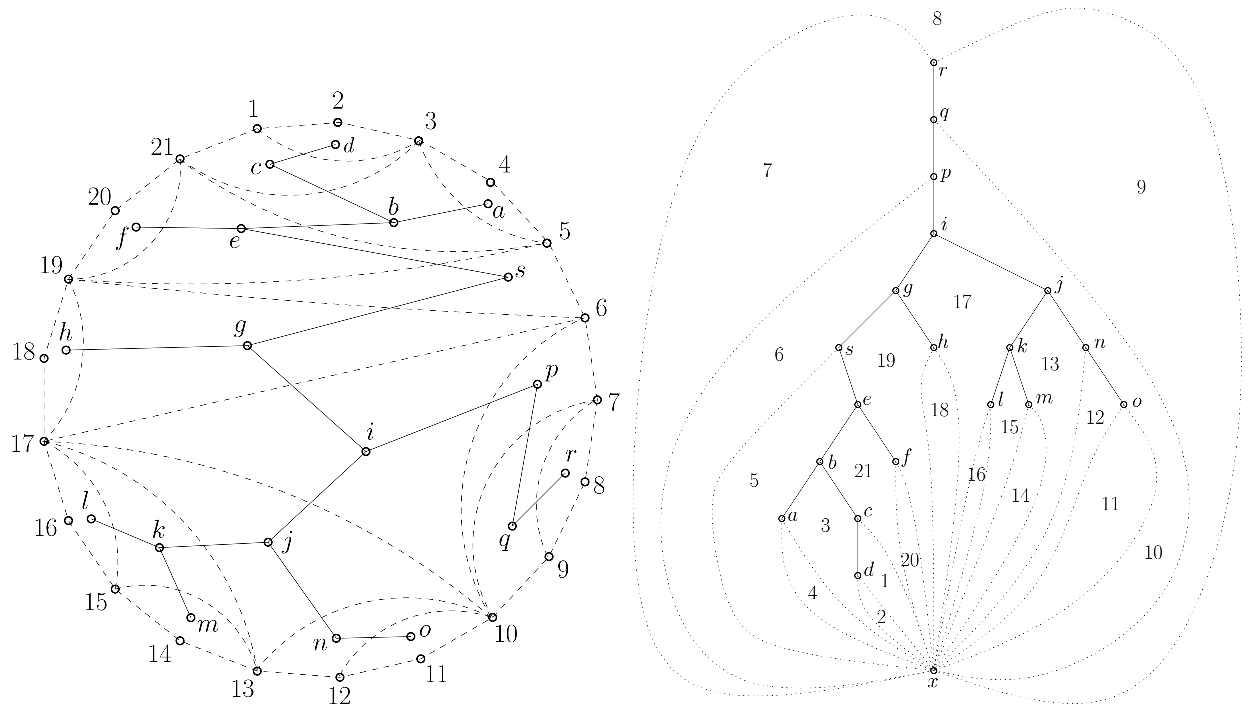
In this section we prove that the surviving rate of outerplanar graphs is also $1-\Theta(\frac{\log n}{n})$, which is asymptotically tight as the same rate is asymptotically tight for trees (Theorem 2.3), and it is not hard to verify that outerplanar graphs form a superset of trees. For this purpose, we need only consider *maximal outerplanar graphs*, i.e., outerplanar graphs where the addition of any edge will destroy the outerplanarity. Let $G = (V, E)$ be a *maximal outerplanar graph*, i.e., a planar embedding of a maximal outerplanar graph with all vertices on the boundary of the exterior face. We will establish our result for G by considering the *dual graph* $G^* = (V^*, E^*)$ of G constructed as follows: place a vertex inside each face of G, and, if two faces have an edge e in common, join their corresponding vertices by an edge e' crossing only e.

The firefighting problem on vertices of G can be transformed into that on faces of the dual graph G^*: A fire starts at a face of G^*, and spreads from a burning face f to each unprotected face sharing a common edge with f in one unit of time. In each unit of time, a firefighter can protect one face not yet on fire. See Figure 2 for an example.

Let x denote the vertex in G^* corresponding to the exterior face of G. It is well known that $G^* - x$ is a tree of maximum degree 3 as every face of G (except the exterior face) is a triangle. We turn $G^* - x$ into a rooted binary tree $T = (V_T, E_T)$ by picking up a leaf r as the root (see Figure 2(b) for an example). Note that each edge in $G^* - T$ connects a vertex v of T with vertex x, and for convenience, we also regard vertex x as a child of v. For each vertex $v \in V_T \setminus \{r\}$, all vertices on the (r, v)-path are *ancestors* of v. For each vertex $v \in V_T \setminus \{r\}$, we use $p(v)$ to denote its parent in T, T_v the subtree rooted at v, and $depth(v)$ the depth of v which is the distance from the root to v. We also designate left and right children of each vertex $v \in V_T \setminus \{r\}$ in a natural way: we start with edge $vp(v)$ and turn around v clockwise in a very small circle, the first edge we cross connects v with its *right child*, and the other child of v is its *left child*[1].

We will use T to design our strategy for the dual graph G^*, and we will define a few more terms before we can describe our strategy. A vertex $v \in V_T \setminus \{r\}$ is a *heavy vertex* if T_v contains more than half vertices of $T_{p(v)}$, and a *light vertex* otherwise. Clearly, each vertex has at most one heavy vertex as its child, and every path from the root has at most $\lfloor \log n \rfloor$ light vertices. A path $P(v_0) = v_0, v_1, \ldots, v_t$ is a *heavy path* if each v_i, $1 \le i \le t$, is a heavy vertex (vertex v_0 can be either a heavy or light vertex), and a vertex v_i, $1 \le i \le t-1$, is a *turning vertex* if v_i is the left (right respectively) child of v_{i-1} and v_{i+1} is the right (left) child of v_i.

[1]When designating the left and right child of a vertex $v \in V_T \setminus \{r\}$, we also take the exterior vertex x into account. That is, if v has one (zero respectively) child in T, then there must be one (two) edge(s) in G^* connecting v and x, we also regard x as a child of v. For example in Figure 2(b), e is the right child of s, d is the left child of c, and x is both the left and right child of f.



(a) The maximal outerplanar graph G (dashed edges) (b) The dual G^*. $G^* - x$ forms a rooted binary tree with root r.

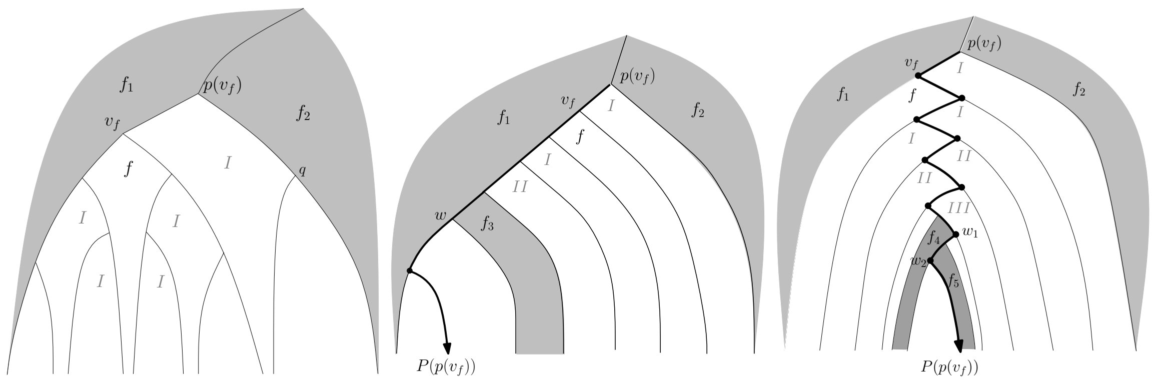
Figure 2. A maximal outerplanar graph G and its dual G^*. For example, a fire starting at vertex 17 in G is equivalent to a fire starting at face 17 in G^*. After one time unit the fire will spread to the face-neighbors 16, 15, 13, 10, 6, 19 and 18 of face 17 in G^*, and the firefighter can protect one of these neighbors (or any face other than 17).

For a subtree T_v of T, we use T_v^x to denote the induced subgraph $G^*[V(T_v) \cup \{x\}]$. For each face f in G^*, its *top vertex*, denoted by v_f, is the vertex on the boundary of f with minimum depth in T. Note that, except root r, each vertex in T is a top vertex of exactly one face. Finally we define two important faces for f: for the two faces enclosing $T^x_{p(v_f)}$ in G^*, the one that contains both v_f and $p(v_f)$ in its boundary is the *critical face* of f and the other one is the *nearly-critical face* of f (see Figure 3(a)). Note that if we protect the critical and near-critical faces of f in sequence, the whole subgraph $T^x_{p(v_f)}$ will be cut off from the rest of T.

We are now ready to describe our strategy for the firefighting in the dual graph G^*. Let f be the face that starts the fire. Our strategy consists of at most four rounds of protection: the first two rounds will contain the fire in $T^x_{p(v_f)}$, and the next two rounds will go along the heavy path $P(p(v_f))$ to save all faces inside a 'heavy subgraph' of $T^x_{p(v_f)}$. The strategy is given by the following 4 rules carried out in order.

Rule 1 We do nothing if $depth(v_f) \leq 1$, otherwise we use Rules 2-4.

Rule 2 For the first two rounds, we protect the critical and nearly-critical faces of f in turn



(a) In the first two rounds, we use Rule 2 to protect critical and nearly-critical faces f_1 and f_2 of f.

(b) In the 3rd round, we use Rule 3 to protect face f_3.

(c) In the 3rd and 4th rounds, we use Rule 4 to protect faces f_4 and f_5.

Figure 3. Our strategy when the fire starts at f. Grey faces are protected. The Roman number in a face indicates the earliest time for the face to catch fire.

to contain the fire inside subgraph $T^x_{p(v_f)}$ (see Figure 3(a)).

Rule 3 If the heavy path $P(p(v_f))$ has at least 6 vertices and no turning vertex among the first 5 vertices, let w be the its 6th vertex. In the 3rd round, we protect the face whose top vertex is $p(w)$ to save all faces inside subgraph T^x_w (see Figure 3(b)). We do nothing in the 4rd round.

Rule 4 If the condition in Rule 3 doesn't hold (thus the strategy in Rule 3 hasn't been carried out), and if the heavy path $P(p(v_f))$ has at least 9 turning vertices, then let w_1 and w_2 be the 8th and 9th turning vertices. Let f_4 and f_5 be the faces whose top vertices are $p(w_1)$ and $p(w_2)$. We protect f_4 and f_5 in the 3rd and 4th rounds to save all faces inside subgraph $T^x_{w_2}$ (see Figure 3(c)).

It is easy to see that the strategy is valid as it does not protect any burning faces. A face f is a *fire-source* for g if when the fire starts at face f, our strategy may not save face g. A vertex u is a *bad-ancestor* of a face g if there is a fire-source f for g with $depth(v_f) > 1$ such that v_f is a child of u. We now show that our strategy ensures that the average number of burnt faces in G^* is $O(\log n)$. For this purpose, we first put an upper bound on the number of bad-ancestors.

Lemma 3.1 *Every face g has at most $O(\log n)$ bad-ancestors.*

Proof: By Rule 2, we know that when the fire starts at a face f with $depth(v_f) > 1$, all faces outside $T^x_{p(v_f)}$ are saved because of the protection of the critical and nearly-critical faces of f in the first two rounds. Therefore any bad-ancestor of g is an ancestor of v_g, and hence is on the (r, v_g)-path P of T.

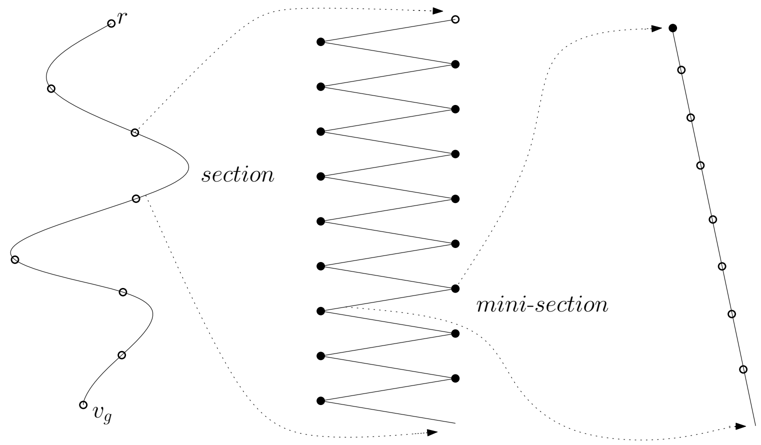


Figure 4. Decomposition of the (r, v_g)-path into sections, and further into mini-sections.

To put an upper bound on the number of bad ancestors of g in P, we first decompose P into sections. Because P contains at most $\lfloor \log n \rfloor$ light vertices, the deletion of these light vertices divides P into at most $\lfloor \log n \rfloor + 1$ sections. We add each light vertex to the section immediately follows it. Let S be an arbitrary such section. Then turning vertices on S further divide S into sections, which we refer to as *mini-sections* (see Figure 4). We add each turning vertex to the end of the mini-section immediately follows it. Suppose $v \in P$ is a bad-ancestor of g. By Rule 4, we see that v can only reside in the last 9 mini-sections for each section. For each mini-section, we see from Rule 3 that v can only be one of the last 4 vertices in the mini-section. It follows that the number of possible bad-ancestors of g in each section is at most 36, implying that the total number of possible bad-ancestors on P is at most $36(\lfloor \log n \rfloor + 1)$, which completes the lemma. □

With Lemma 3.1 at hand, we can now establish the surviving rate of outerplanar graphs.

Theorem 3.2 *The surviving rate of every n-vertex outerplanar graph is* $1 - \Theta(\frac{\log n}{n})$.

Proof: As mentioned earlier, the firefighting problem on vertices of an outerplanar graph G is equivalent to that on faces of its dual graph G^*. From Rules 1 and 2, we see that every fire-source f for a face g satisfies either (a) $depth(v_f) \leq 1$ or (b) $p(v_f)$ is an ancestor of v_g. There are at most 4 faces satisfying (a), and the number of fire-sources satisfying (b) is at most twice the number of bad-ancestors of g as each vertex in T has at most two children.

Since the number of bad-ancestors of g is at most $O(\log n)$ (by Lemma 3.1), the number of fire-sources for g is no more than $4 + 2O(\log n) = O(\log n)$. Therefore our strategy saves at least $n - O(\log n)$ vertices on average, and the theorem easily follows from this and Theorem 2.3. □

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