On the Hardness of Pricing Loss-leaders

Preyas Popat
New York University
popat@cs.nyu.edu

Yi Wu
IBM Almaden
wuyi@us.ibm.com

September 30, 2011

Abstract

Consider the problem of pricing n items under an unlimited supply with m buyers. Each buyer is interested in a bundle of at most k of the items. These buyers are single minded, which means each of them has a budget and they will either buy all the items if the total price is within their budget or they will buy none of the items. The goal is to price each item with profit margin $p_1, p_2, ..., p_n$ so as to maximize the overall profit. When $k = 2$, such a problem is called the GRAPH-VERTEX-PRICING problem. Another special case of the problem is the HIGHWAY-PRICING problem when the items (toll-booths) are arranged linearly on a line and each buyer (as a driver) is interested in paying for a path that consists of consecutive items. The goal again is to price the items (tolls) so as to maximize the total profits.

There is an $O(k)$-approximation algorithm by [BB06] when the price on each item must be above its margin cost; i.e., $p_i > 0$ for every $i \in [n]$. As for the highway problem, a PTAS is shown in [GR11]. We investigate the above problem when the seller is allowed to price some of the items below their margin cost. It is shown in [BB06, BBCH07] that by pricing some of the items below cost, the maximum profit can increase by a factor of $\Omega(\log n)$. These items sold at low prices to stimulate other profitable sales are called "loss leaders". Given the possibility of making more profit, understanding the approximability of pricing loss leaders for GRAPH-VERTEX-PRICING, HIGHWAY-PRICING as well as the general item pricing problem are formulated as open problems in [BB06,BBCH07].

In this paper, we obtain strong hardness of approximation result for the problem of pricing loss leaders. First we show that it is NP-hard to get better than $O(\log \log \log n)$-approximation when $k \geq 3$. This improves a previous super-constant hardness result assuming the Unique Games Conjecture [Wu11]. In addition, we show a super-constant UNIQUE-GAMES hardness for the HIGHWAY-PRICING problem as well as the GRAPH-VERTEX-PRICING problem.

1 Introduction

Imagine a seller (e.g., the owner of a large supermarket) who has many different types of items with an unlimited supply. There are buyers each of which is interested in a bundle (subset) of these items. These buyers all have *single minded valuations* over the bundles (e.g., a desktop and a monitor or ingredients of a recipe); i.e., each buyer has certain budget over the bundles and will only purchase if the total price is below their budget. Suppose the seller has the valuation of all the customers, the algorithmic task is to price all the item with a proper profit margin so as to maximize the overall profit. Intuitively, setting low prices will attract more buyers while setting high prices will have more profits on each individual sales.

There is a flurry of work on understanding the approximability of the above pricing problem as well its special case such as the HIGHWAY-PRICING problem (e.g., see [GHK+05, HK05, BB06, GVLSU06, BK06, DHFS06, ESZ07, BBCH07, KKMS09, ERRS09, AGU10, GS10, GR11, Wu11]). Assuming that there are n different items and m different buyers, the best approximation algorithm is an $O(\log n + \log m)$-approximation given by Guruswami, Hartline, Karlin, Kempe, Kenyon and McSherry [GHK+05]. The best hardness of approximation result is a factor of $(\log n)^\epsilon$ for some $\epsilon > 0$ assuming NP $\not\subseteq$ BPTIME(2^{n^δ}) for some $0 \le \delta \le 1$ given by Demaine, Feige, Hajiaghayi and Salavatipour [DHFS06]. If we further assume that each customer is only interested in at most k of the items (as in many settings, each customer is only interested in constant number of items), Balcan and Blum [BB06] give an algorithm with approximation ratio $O(k)$. In particular, for $k = 2$ when the problem is also called GRAPH-VERTEX-PRICING, their algorithm gives a 4-approximation. On the hardness side, it is known that the even the simple GRAPH-VERTEX-PRICING problem is APX-hard [GHK+05] and UG-hard to get better than 2-approximation by Khandekar, Kimbrel, Makarychev and Sviridenko[KKMS09]. This UGC hardness result even holds when the underlying graph is bipartite, which is tight since [BB06] give a 2-approximation for bipartite graphs. Note that a factor c hardness for $k = l$ also translates to a factor c hardness for all $k > l$ since we allow bundles with *at most* k items.

Another special case of the pricing problem that has received much attention is the highway problem when the items (toll-booths) are arranged linearly on a line and each buyer (as a driver) is interested in paying for a path that consists of consecutive items. The goal again is to price the items (tolls) so as to maximize the total profit. Such a problem is known to be strongly NP-hard by Elbassioni, Raman, Ray and Sitters [ERRS09] and very recently a PTAS is obtained by Grandoni and Rothvoß [GR11].

All of the above results assume the seller always prices each item with a positive profit margin. Much less is known for the problem when the seller is allowed to price some of the items below their margin cost. The motivation for pricing certain item below cost is to stimulate the sales of other more profitable items. Such a pricing strategy is also widely used in practice and these items sold at price below cost are called *"loss leaders"*. For example, a printer company may sell the printer at a low price (as the loss leader) to make more profits from selling the ink cartridges. Consider the following concrete example: there are three items A, B, C and three customers: one values $\{A\}$ at \$10 above the margin cost, one values $\{C\}$ at \$10 above the cost and one values $\{A, B, C\}$ at \$10 above the cost. By pricing A, C at \$10 above the cost and B at \$10 below the cost, the seller makes a total profit of \$30. On the contrary, if no item is allowed to price below its cost, it is easy to verify that the maximum profit of the seller is at most \$20. .

To formally study the problem of pricing loss leaders, in[BBCH07], Balcan, Blum, Chan and Hajiaghayi proposed two reasonable theoretical models: the *discount model* and the *coupon model*. The *discount model* is the most direct one: it assumes that the profit the seller collect from a bundle of items is the sum of the profit margin on each item in the bundles. One drawback of the discount model is that it does not make sense to assign a negative profit margin when the margin cost of each item is 0 (e.g, for the HIGHWAY-PRICING problem). To address this, the authors also propose the *coupon model* which assumes that the profit on each bundle is at least 0; i.e., if the sum of the profit margin on each item in the bundle is negative, then the seller has profit 0 on that bundle. This model also assumes that a customer is interested in a particular set of items and will not purchase a superset even if it is cheaper, which is true for problems such as HIGHWAY-PRICING where the driver is only interested in travelling a particular path and would not like to travel additional stretches to save tolls. One interesting fact shown in [BBCH07, BB06] is that the maximum profit under either the coupon model or discount model can be as large as $\Omega(\log n)$ times the maximum profit when only positive profit margin prices are allowed. Such a gap of $\Omega(\log n)$ holds even for the HIGHWAY-PRICING problem when all the drivers have the same valuation (budget).

Given the possibility of making more money by pricing loss leaders properly, studying the problem of pricing loss leaders is formulated as an open problem in [BB06] where the authors ask: "what kind of approximation guarantees are achievable if one allows the seller to price some items below their margin cost?" Later in [BBCH07], it is further suggested that "Obtaining constant factor appropriation algorithms in the coupon model for general GRAPH-VERTEX-PRICING problem and the HIGHWAY-PRICING with arbitrary valuations seems believable but very challenging. "

Our Main Results: In this paper, we prove strong negative results for several pricing problems with loss leaders. We show that it is NP-hard to get better than $O(\log\log\log n)$-approximation, under either the coupon or discount model, even for the simple setting when $k = 3$, i.e., each customer is at most interested in 3 of the items. This improves a super-constant UG-hardness[1] result [Wu11] for the same problem. As for the case of GRAPH-VERTEX-PRICING (i.e, when $k = 2$) and HIGHWAY-PRICING problem, contrary to the suggestion of [BBCH07], we show a *super-constant* hardness of approximation result under the coupon model assuming Khot's Unique Games Conjecture [Kho02]. Previously, an APX-hardness result is known for the HIGHWAY-PRICING problem [ERRS09].

It is interesting to compare our hardness results with the known approximation algorithms for the corresponding problem using positive profit margin prices only. For the general pricing problem, there is a 4-approximation algorithm when $k = 2$ and $\frac{1}{3e}$-approximation algorithm for $k = 3$ [BB06]. As for the highway problem, there exists a PTAS [GR11]. All of the three problems have (at least) a constant approximation algorithm for positive profit margin prices while our corresponding hardness results for pricing loss leaders are (at least) super-constant. Conceptually, our result indicates the problem of pricing loss leaders is substantially harder.

It is worth mentioning that we assume the seller knows all the valuation of all the buyers. While this assumption may seems to be too strong, our results imply the inherent hardness of pricing loss leaders even when the seller fully understands the buyers.

[1] UG-hard means NP-hard assuming the Unique Games Conjecture

2 Definitions and Our Results

In this section, we formally define all the pricing problems and state our main results.

2.1 Notations

The following notation is used repeatedly in the rest of the paper. For q being a positive integer, we define:

- $\mathbb{Z}_q$: the set $\{0, 1 \ldots, q-1\}$.
- $[x]_q$: the remainder of x divided by q
- $\oplus_q$: addition of integers (or integer vectors) modulo q.
- For any statement ω, $I(\omega) \to \{0,1\}$ is the indicator function of whether w is correct (when $I(w) = 1$) or not (when $I(w) = 0$).
- $\lfloor x \rfloor$: for any $x \in \mathbb{R}$, $\lfloor x \rfloor$ is the largest integer less than or equal to x.
- $\lceil x \rceil$: for any $x \in \mathbb{R}$, $\lceil x \rceil$ is the smallest integer greater than or equal to x.

2.2 Problem Definitions

The item pricing problem can be thought of as defined on a multi-hypergraph and therefore it is also called the VERTEX-PRICING problem.

Definition 2.1 (VERTEX-PRICING)**.** An instance $\mathcal{I}(G(V,E), \{b_e \mid e \in E\}, \{w_e \mid e \in E\})$ of VERTEX-PRICING is characterized by a multi-hypergraph $G(V,E)$. Here each vertex corresponds to an item and each hyper-edge corresponds to the bundle of items a customer is interested in. For each edge $e \in E$, there is an associated budget $b_e > 0$ and a weight w_e. When G is a k-hypergraph, we call the corresponding problem VERTEX-PRICING$_k$.

The goal is to find a pricing function $f : V \to \mathbb{R}$ so as to maximize the profit. As we have discussed, there are mainly two kinds of profit models considered previously. The first profit model is the discount model. Given a vertex pricing instance $\mathcal{I}(G(V,E), \{b_e \mid e \in E\}, \{w_e \mid e \in E\})$ and a price function $f : V \to \mathbb{R}$:

Definition 2.2 (profit under the discount model)**.**

$$\mathbf{profit}_{\mathcal{I}}(f) = \sum_e I\left(f(e) \le b_e\right) \cdot w_e \cdot f(e)$$

where $f(e) := \sum_{v \in e} f(v)$.

Under the above model, the seller may lose money to the buyer when $\sum_{v \in e} f(v) < 0$. The coupon model assumes the seller would have at least profit 0 from each buyer.

Definition 2.3 (profit under the coupon model)**.**

$$\mathbf{profit}^+_{\mathcal{I}}(f) = \sum_e I(0 \le f(e) \le b_e) \cdot w_e \cdot f(e)$$

where $f(e) := \sum_{v \in e} f(v)$.

Now given a vertex pricing instance $\mathcal{I}(G(V,E), \{b_e \mid e \in E\}, \{w_e \mid e \in E\})$, we can study the problem of maximizing the profit under the following three settings. The first one is the widely studied one when the seller want to price each item with a *positive* profit margin:

Definition 2.4. (Positive price model) $\mathrm{Opt}(\mathcal{I}) = \max_{f:V \to \mathbb{R}^+} \mathbf{profit}_{\mathcal{I}}(f)$.

When we allow a real-valued price function, we can maximize the profit under either the coupon or discount model.

Definition 2.5. (Discount Model) $\mathrm{Opt}^D(\mathcal{I}) = \max_{f:V \to \mathbb{R}} \mathbf{profit}_{\mathcal{I}}(f)$.

Definition 2.6. (Coupon Model) $\mathrm{Opt}^C(\mathcal{I}) = \max_{f:V \to \mathbb{R}} \mathbf{profit}^+_{\mathcal{I}}(f)$.

We also consider the following HIGHWAY-PRICING problem.

Definition 2.7 (HIGHWAY-PRICING)**.** Let $V = \{0, 1, 2, \ldots, n\}$. G is an n-edge line with $e_i = (i-1, i)$ for $i = 1, 2, \ldots, n$. We are given a set of intervals $I_1, I_2, \ldots I_m$ where each interval is specified by $I_j = [s_j, t_j]$ for $s_j, t_j \in V$ with an associated budget b_j and weight w_j. The goal is to properly price each e_i with a price function $p : [n] \to \mathbb{R}$ so as to maximize the total revenue.

The item of the HIGHWAY-PRICING problem is the segments of a line graph. Alternatively, we can think of the problem as finding a function that assign a price on $f : V \to \mathbb{R}$ and the toll on $\{i, i+1\}$ is defined as $p(i) = f(i+1) - f(i)$. Under above formulation, it is easy to see that the approximability of the HIGHWAY-PRICING is equivalent to the VERTEX-PRICING$_2$ on bipartite graph.

Lemma 2.8. *Consider the profit maximization problem under the coupon model. If* VERTEX-PRICING$_2$ *on bipartite graph is hard to approximate to factor α , then* HIGHWAY-PRICING *problem is also hard to approximate to factor α under the coupon model.*

Proof: Let $\mathcal{I}(G(V,E), \{b_e \mid e \in E\}, \{w_e \mid e \in E\})$, $V := V_L \cup V_R$ be an instance of VERTEX-PRICING$_2$ and assuming that G is a bipartite graph. We construct an instance of HIGHWAY-PRICING $\mathcal{J}(G(V,E), \{b_e \mid e \in E\}, \{w_e \mid e \in E\})$, $V := V_L \cup V_R$ as follows.

The vertex set remains the same. We align all the vertices in V_L to the left of the vertices in V_R in a line. Then for every edges $e = (v_1, v_2)$ in $\mathcal{I}$, we also add a driver interested in the interval between v_1 and v_2.

We claim that $\mathrm{Opt}^+(\mathcal{I}) = \mathrm{Opt}^+(\mathcal{J})$. This is because if $f_L : V_L \mapsto \mathbb{R}$, $f_R : V_R \mapsto \mathbb{R}$ is a pair of pricing functions for $\mathcal{I}$ then $(-f_L, f_R)$ is a pair of pricing functions for $\mathcal{J}$ such that $\mathbf{profit}^+_{\mathcal{I}}(f_L, f_R) = \mathbf{profit}^+_{\mathcal{J}}(-f_L, f_R)$, and the argument is reversible. ∎

2.3 Main Results

Our first result is a NP-hardness result for VERTEX-PRICING$_3$. Note here that although we defined the pricing problem in a weighted graph, all of our hardness results also hold for unweighted multi-graph. This can be shown in a similar way as in [KKMS09, Wu11]. Basically, it involves discretizing the weights to be multiples of a suitably small number and then replacing weighted edges by parallel edges.

Theorem 2.9 (hardness of VERTEX-PRICING$_3$)**.** *It is NP-hard to distinguish the following two cases for the* VERTEX-PRICING$_3$ *problem.*

- $\mathrm{Opt}^D(\mathcal{I}) \geq \Omega(\log\log\log n)$,
- $\mathrm{Opt}^C(\mathcal{I}) \leq O(1)$.

Notice that $\mathrm{Opt}^C(\mathcal{I}) \geq \mathrm{Opt}^D(\mathcal{I})$, we immediately get the following two corollaries.

Corollary 2.10. VERTEX-PRICING$_3$ *under the discount model is NP-hard to approximate to factor* $\Omega(\log\log\log n)$*.*

Corollary 2.11. VERTEX-PRICING$_3$ *under the coupon model is NP-hard to approximate to factor* $\Omega(\log\log\log n)$*.*

Our second result is a UG-hardness result for VERTEX-PRICING$_2$.

Theorem 2.12 (hardness of VERTEX-PRICING$_2$)**.** VERTEX-PRICING$_2$ *under the coupon model is UG-hard to approximate to any constant factor, even when the graph is bipartite.*

Applying Lemma 2.8, we also get the same hardness of approximation result for the HIGHWAY-PRICING problem.

Theorem 2.13 (hardness of HIGHWAY-PRICING)**.** HIGHWAY-PRICING *under the coupon model is UG-hard to approximate to any constant factor.*

3 Overview of our proof

In this section, we give a high level overview of our proof. We assume some familiarity with the proof of MAX 2-LIN$_q$ in [KKMO07] and MAX 3-LIN$_q$ in [Hås01]. Since the approximability of HIGHWAY-PRICING is equivalent to that of VERTEX-PRICING$_2$ on bipartite graph, we would only describe our proof for VERTEX-PRICING$_2$ and VERTEX-PRICING$_3$.

The work of [KKMO07] establishes a connection between "Dictator Test" and "Hardness of approximation" of CSPs assuming the Unique Games Conjecture. The pricing problem can be viewed as a CSP with a generalized payoff function. Therefore, our hardness results for the VERTEX-PRICING is based on building a proper Dictator Test. In addition, for the VERTEX-PRICING$_3$ problem, since our result only assumes P$\neq$ NP, we need to combine the Dictator Test with a PCP construction that Håstad uses to obtain the NP-hardness result for MAX 3-LIN$_q$ [Hås01].

Roughly speaking, a Dictator Test for VERTEX-PRICING is just a instance of the VERTEX-PRICING problem defined over the vertex set $\mathbb{Z}_q^n$ where n is thought of as a large number. A pricing to these items is as a function defined over $f : \mathbb{Z}_q^n \to \mathbb{R}$. A Dictator function is functions that only depend on one of its coordinates. The Dictator Test is a VERTEX-PRICING instance with the following properties:

- (completeness) There exists some one dimensional real function $h : \mathbb{R} \to \mathbb{R}$ such that $f(x) = h(x_i)$ has a high profit c for every $i \in [n]$.
- (soundness) Any function that depends on a lot of its coordinates will have at most profit s.

By the reduction in [KKMO07], Dictator Test with above property would establish that it is UG-hard to distinguish whether a given instance of vertex pricing has profit above c or below s (which implies a s/c hardness of approximation result).

For example, to construct a Dictator Test for VERTEX-PRICING$_2$ under the coupon model, it is enough to specify a distribution over $x, y \in \mathbb{Z}_q^n, b \in \mathbb{R}^+$. Here we add a customer interested in x, y with budget b and the weight is the probability mass on (x, y, b). The profit of price function $f : \mathbb{Z}_q^n \to \mathbb{R}$ is $\mathbf{E}_{x,y,b}[I(0 \le f(x) + f(y) \le b) \cdot (f(x) + f(y))]$. The main task is to construct such a distribution which has a good profit for dictator functions and has a low profit for all functions that depend on a lot of coordinates.

3.1 The Dictator Test for vertex-pricing$_2$

The Dictator Test for VERTEX-PRICING$_2$ is, in a sense, similar to the Dictator Test that is used in [KKMO07] to obtain a hardness result of MAX 2-LIN$_q$. MAX 2-LIN$_q$ is a problem of solving a linear system over $\mathbb{Z}_q$ where each equation only depends on two variables. The main construction of the Dictator Test (as a instance of MAX 2-LIN$_q$ over $\mathbb{Z}_q^n$) is described as follows: choose x to be uniformly random from $\mathbb{Z}_q^n$ and y is generated by adding "ϵ noise" as follows: for every i, $\mathbf{y}_i = \mathbf{x}_i$ with probability $1 - \epsilon$ and $\mathbf{y}_i$ is set to be a random element in $\mathbb{Z}_q$ with probability ϵ. Then the Dictator Test will add a equation $f(\mathbf{x}) - f(\mathbf{y}) = 0$.

Since our objective function is of the form $I(0 \le f(\mathbf{x})+f(\mathbf{y}) \le b)\cdot(f(\mathbf{x})+f(\mathbf{y}))$, we first construct $\mathbf{x}$ uniformly random from $[q]^n$ and $\mathbf{y}_i = [b-\mathbf{x}_i]_q$ where budget b is randomly chosen from $2, 4, 8, \ldots, 2^k$ where $k := \log \sqrt{q}$. We add $2^k/b$ edges between $\mathbf{x}$ and $\mathbf{y}$. Notice that if we use the Dictator price function $f(\mathbf{x}) = \mathbf{x}_i - q/2$ and $f(\mathbf{y}) = \mathbf{y}_i - q/2$, then with probability at least $1 - 1/\sqrt{q} \ge 1/2$, we have $f(\mathbf{x})+f(\mathbf{y}) = b$. Therefore, the profit of the dictator function is at least $1/2 \cdot 1/k \cdot \sum_{i=1}^{k} 2^k/2^i = \Omega(2^k)$.

On the other hand, we manage to show that functions that depend on a lot of coordinates cannot have a profit significantly better than the constant price function that assign the same price to every item. It is easy to verify that for any constant price function, the profit on the above instance is at most $O(2^k/k)$. This gives us a $\Omega(\log q)$ gap between the profit of Dictator function and functions that depends on a lot of its coordinates. Note that q can be an arbitrarily large constant in this construction.

Technically, the main body of the proof is to show that functions that depends on a lot of coordinates behave like constant functions. We use the general approach of [KKMO07]. For simplicity, suppose the pricing function is integer valued $f : \mathbb{Z}_q^n \to [q]$ and suppose we have a customer interested in x, y with budget q, then the profit can be written as $\sum_{0 \le i+j \le q} (i+j) f_i(x) f_j(x)$. One of the difficulty we face is that there can be $\Omega(q^2)$ terms in the sum while the analysis in [KKMO07] usually generate x, y with noise rate ϵ and this would bound the sum by $q^{2-\epsilon}$. To see why this is the case, let us recall how the analysis in [KKMO07] proceeds. The goal is to show that functions which depend on many coordinates satisfy a small fraction of equations in the MAX 2-LIN$_q$ instance. This is achieved by writing the fraction of equations satisfied by a function f in terms of its *Noise stability*, and using the invariance principle of [MOO05] to show that if f depends on many co-ordinates then the Noise stability of f is essentially the same as the Noise stability of a related function $\tilde{f}$ which takes as input gaussian random variables rather than $\mathbb{Z}_q$-valued random variables. It is known by a result of [Bor85] that the function $\tilde{f}$ for which the noise stability is maximum in the gaussian domain is the half-space with the appropriate measure, and [KKMO07] use estimates about the noise stability of this function to prove their result.

We follow the same approach as [KKMO07] of writing the profit in terms of the Noise stability of the pricing function. It turns out that if the noise is small then the expression for profit is quite large when written for the half-space function in the gaussian domain. Our main technical contribution is to get around this issue by introducing a large noise ($1 - o(1)$ noise) and carefully analyzing the noise stability of pairs of half-spaces in the gaussian domain (Lemma 3.4).

Another technical challenge in our proof is that we need our hardness result hold even for VERTEX-PRICING$_2$ on bipartite graph, therefore we use a bipartite version of the invariance principle due to [DMR09].

3.1.1 Main Technical Lemma

Here we state the main technical lemma used in the reduction for VERTEX-PRICING$_2$.

Definition 3.1. Let φ be the probability density function of the standard gaussian i.e. $\varphi(t) := \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$.

Definition 3.2. Let N be the cumulative distribution function of the standard gaussian i.e. $N(t) := \int_t^\infty \varphi(x)dx$. Equivalently, $N(t) := \mathbf{Pr}[X \ge t]$ where X is a standard gaussian random variable.

Definition 3.3. (Gaussian noise stability of half-spaces) $\Lambda_\rho(\mu, \nu) := \mathbf{Pr}[X \ge t \ \ and \ \ Y \ge s]$ where $t := N^{-1}(\mu)$, $s := N^{-1}(\nu)$ and X, Y are standard Gaussians with $\mathbf{E}[XY] = \rho$.

Lemma 3.4. *Let $1/(q \log q) \le \mu \le 1$, $\rho \le (\log q)^{-(1/2+\epsilon)}$, $k \le \log q$ and $\{\nu_1, \nu_2, \dots, \nu_k\}$ be such that $\sum_{i=1}^k \nu_i \le 1$. Then for q large enough,*

$$\sum_{i=1}^k \Lambda_\rho(\mu, \nu_i) = O(\mu)$$

3.2 The Dictator Test for vertex-pricing$_3$

Our construction for the VERTEX-PRICING$_3$ is based on Håstad's seminal result of MAX 3-LIN$_q$ [Hås01] and a Dictator Test for VERTEX-PRICING$_3$ that is previously introduced in [Wu11]. Håstad essentially construct "Matching-Dictator Test" on two functions for $f\colon \mathbb{Z}_q^K \to \mathbb{Z}_q$, $g\colon \mathbb{Z}_q^L \to \mathbb{Z}_q$ and $\pi\colon L \to K$. The test is defined by a distribution over $\mathbf{x} \in \mathbb{Z}_q^K, \mathbf{y} \in \mathbb{Z}_q^L, \boldsymbol{z} \in \mathbb{Z}_q^L$ with the check $f(x) + g(y) + g(z) = 0 \mod q$. Håstad's Test has the following completeness and soundness promises:

- If $f(x) = x_i$ and $g(y) = y_j$ such that $\pi(i) = j$, then f and g passes with probability $1 - \epsilon$.
- If f and g are far from being a pair of matching dictator functions, then they behave like constant functions.

Our proof essentially use the same distribution of x, y, z and add $\lfloor \sqrt{q} \rfloor$ buyers such that they are interested in $x, y, z \oplus_q \lfloor q/t \rfloor \cdot (1, 1, ...1)$ with budget q/t for every $t \in \lfloor \sqrt{q} \rfloor$. It is easy to verify that for every $i \in [q]$ and $f(x) = x_i - q/3, g(y) = y_i - q/3$ would have $\log q$ times more profit than setting f, g to be a constant. The main body of the work is to show if f, g are far from being "matching" dictator functions" , then they just behave like being constant functions.

Notice that in [Wu11], the author manages to construct such a test for $K = L$ and $f = g$, which suffices to give a hardness result assuming the Unique Games Conjecture. Technically speaking, in [Wu11], the author used the *invariance principle* [MOO05] to analyze the profit of functions that depends on a lot of coordinates. However we can not use the invariance here directly partly because it requires pairwise independent distributions. Also the projection instead of bijection in our test make it hard for us to use the same analysis. We also found it hard to directly use the Fourier Analysis with complex function basis by which Håstad proved the hardness result for Max 3-LIN$_q$. This is because our objective function is less symmetric compared with the objective function of Max 3-LIN$_q$. Instead we use the Efron-Stein Decomposition combining with a Håstad style decoding. Such a proof also avoid the use of the "invariance principle" which we view as a simplification of the proof in [Wu11]. Our proof is inspired by a recent work [OWZ11] which also uses the same method to generalize Håstad's Max 3-LIN$_q$ result to the integer domain without using the complex Fourier analysis.

3.3 Open problems

We show that graph-vertex-pricing and highway-pricing are UG-hard to approximate to any constant factor under the Coupon model. It would be interesting to prove a similar result for the discount model. Our techniques fall short of achieving this because of the necessity to introduce very large noise as explained in Section 3.1.

4 Mathematical Tools

4.1 Gaussians

Lemma 4.1.

$$N^{-1}(\mu) = \Theta(\sqrt{\log(1/\mu)})$$

Proof: Let $t = N^{-1}(\mu)$. Use the well known fact that $N(t) \sim \varphi(t)/t$ along with the definition of φ and $N(t) = \mu$. ∎

4.2 Tools from Discrete Fourier Analysis

We recall some standard definitions from the discrete Fourier analysis (see, e.g., [Rag09]). We will be considering functions of the form $f : \mathbb{Z}_q^n \to \mathbb{R}$. The set of all functions $f : \mathbb{Z}_q^n \to \mathbb{R}$ forms an inner product space with inner product

$$\langle f, g \rangle = \mathop{\mathbf{E}}_{\mathbf{x} \sim \mathbb{Z}_q^n}[f(\mathbf{x}) \cdot g(\mathbf{x})],$$

where $\mathbf{x} \sim \mathbb{Z}_q^n$ means that $\mathbf{x}$ is chosen uniformly at random from $\mathbb{Z}_q^n$. We also write $\|f\|_2 = \sqrt{\langle f, f \rangle}$ as usual.

The following Efron–Stein decomposition theorem is well-known; see e.g. [KKMO07].

Theorem 4.2. *Any $f : \mathbb{Z}_q^n \to \mathbb{R}$ can be uniquely decomposed into sum of functions*

$$f(x) = \sum_{S \subseteq [n]} f^S(x),$$

where

- *$f^S(x)$ depends only on $x_S = (x_i, i \in S)$,*
- *for every $S \subseteq [n]$, for every S' such that $S \setminus S' \neq \emptyset$, and for every $y \in \mathbb{Z}_q^n$, it holds that*

$$\mathop{\mathbf{E}}_{\mathbf{x}}[f^S(\mathbf{x}) | \mathbf{x}_{S'} = y_{S'}] = 0.$$

We also need define the noise operator as follows:

Definition 4.3. For $x \in \mathbb{Z}_q^n$, we define random variable $\mathbf{y} \sim_\rho x$ if $\mathbf{y}$ is generated as follows: for each coordinate $i \in [n]$, independently we set $\mathbf{y}_i = x_i$ with probability ρ and uniformly random in $[q]$ with probability $1 - \rho$. For functions $f : \mathbb{Z}_q^n \to \mathbb{R}$, define the noise operator T_ρ to be

$$T_\rho f(x) = \mathop{\mathbf{E}}_{\mathbf{y} \sim_\rho x}[f(\mathbf{y})].$$

Definition 4.4 (influence)**.** For function $f : \{-1,1\}^n \to \mathbb{R}$, we define the influence of the i-th coordinate $\mathbf{Inf}_i f$ as

$$\mathbf{Inf}_i f = \sum_{S \ni i} \|f^S\|_2^2$$

Definition 4.5 (low-degree influence)**.** For function $f : \{-1,1\}^n \to \mathbb{R}$, we define the k-degree influence of the i-th coordinate $\mathbf{Inf}_i^k f$ as

$$\mathbf{Inf}_i^k f = \sum_{S \ni i,\ |S| \leq k} \|f^S\|_2^2$$

Following facts are well known.

Fact 4.6. $\|T_\rho f^S\|_2^2 = \rho^{|S|}\|f^S\|_2^2$.

Fact 4.7. *For* $f : \mathbb{Z}_q^n \mapsto \mathbb{Z}_q$

$$\sum_{i=1}^{n} \mathbf{Inf}_i(T_{1-\eta} f) \le 1/\eta.$$

Fact 4.8. *For* $f : \mathbb{Z}_q^n \mapsto \mathbb{Z}_q$

$$\sum_{i=1}^{n} \mathbf{Inf}_i^k(f) \le k$$

Definition 4.9. For a function $f : \mathbb{Z}_q^n \mapsto \mathbb{Z}_q$ and $a \in \mathbb{Z}_q$, let $f^a : \mathbb{Z}_q^n \mapsto \mathbb{R}$ be defined as $f^a(x) := 1$ if $f(x) = a$ and $f^a(x) := 0$ otherwise.

Definition 4.10. The noise stability of f and g at ρ is defined to be $\mathbb{S}_\rho(f, g) := \langle f, T_\rho g\rangle$.

Theorem 4.11. *[DMR09]*

Fix $q \ge 2$ and $0 < \rho < 1$. Then for any $\delta > 0$ there is a $\tau = \tau(\rho, \delta, q) > 0$ small enough and $k = k(\rho, \delta, q)$ large enough such that if $f, g : [q]^n \mapsto [0, 1]$ are any functions satisfying $E[f] = \mu$, $E[g] = \nu$ and $\min(\mathbf{Inf}_i^k(f), \mathbf{Inf}_i^k(g)) \le \tau$ for all $i = 1 \ldots n$, then

$$\mathbb{S}_\rho(f, g) \le \Lambda_\rho(\mu, \nu) + \delta$$

4.3 Unique Games

Definition 4.12. A UNIQUE-GAMES instance $\mathcal{U}\left(G(U, W, E), [n], \{\pi_{uw}\}_{(u,w)\in E}\right)$ consists of a regular bipartite graph $G(U, W, E)$ and for each edge $e = (u, w) \in E$ a permutation $\pi_{uw} : [n] \mapsto [n]$. The algorithm needs to assign a label from $[n]$ to every vertex $u \in U \cup V$. For a given labeling $L = U \cup W \mapsto [n]$, an edge $e = (u, w) \in E$ is said to be satisfied if $L(w) = \pi_{uw}(L(u))$. The goal is to find a labeling which satisfies the maximum fraction of edges possible.

Conjecture 4.13. *[Kho02] Given a* UNIQUE-GAMES *instance $\mathcal{U}$ let $OPT(\mathcal{U})$ denote the maximum fraction of edges satisfiable by any assignment. Then for every $\epsilon, \eta > 0$ there is a label size $[n]$ large enough so that given a unique games instance $\mathcal{U}\left(G(U, W, E), [n], \{\pi_{uv}\}_{(u,v)\in E}\right)$ it is NP-hard to distinguish between the following two cases:-*

- $OPT(\mathcal{U}) = 1 - \epsilon$
- $OPT(\mathcal{U}) \le \eta$

5 UG-hardness of Bipartite graph-vertex-pricing

5.1 Main technical lemma

Lemma 5.1. *Let $1/(q \log q) \le \mu \le 1$, $\rho \le (\log q)^{-(1/2+\epsilon)}$, $k \le \log q$ and $\{\nu_1, \nu_2, \ldots, \nu_k\}$ be such that $\sum_{i=1}^{k} \nu_i \le 1$. Then for q large enough,*

$$\sum_{i=1}^{k} \Lambda_\rho(\mu, \nu_i) = O(\mu)$$

Proof:

Fix $1 \leq i \leq k$. Let $t := N^{-1}(\mu)$, $s_i := N^{-1}(\nu_i)$ and (X, Y) be standard Gaussians with $\mathbf{E}[XY] = \rho$.

$$\begin{aligned}
\Lambda_\rho(\mu, \nu_i) &= \mathbf{Pr}[X \geq t \ , \ Y \geq s_i] \\
&= \mathbf{Pr}[Y \geq s_i \mid X \geq t] \cdot \mathbf{Pr}[X \geq t] \\
&= \mu \cdot \mathbf{Pr}[Y \geq s_i \mid X \geq t] \\
&\leq \mu \cdot (\mathbf{Pr}[Y \geq s_i \mid X \in [t, 2t]] + \mathbf{Pr}[X \geq 2t \mid X \geq t])
\end{aligned}$$

Using $N(x) \sim \varphi(x)/x$ and $N(t) = \mu$ we get

$$\mathbf{Pr}[X \geq 2t \mid X \geq t] = O(\mu^2 \log(1/\mu))$$

Thus,

$$\begin{aligned}
\sum_{i=1}^{k} \Lambda_\rho(\mu, \nu_i) &\leq \sum_{i=1}^{k} \mu \cdot (\mathbf{Pr}[Y \geq s_i \mid X \in [t, 2t]] + \mathbf{Pr}[X \geq 2t \mid X \geq t]) \\
&= \mu \cdot O(\mu^2 \log^2(1/\mu)) + \sum_{i=1}^{k} \mu \cdot \mathbf{Pr}[Y \geq s_i \mid X \in [t, 2t]] \\
&= O(\mu) + \mu \cdot \sum_{i=1}^{k} \mathbf{Pr}[Y \geq s_i \mid X \in [t, 2t]]
\end{aligned}$$

Thus, it suffices to show that $\sum_{i=1}^{k} \mathbf{Pr}[Y \geq s_i \mid X \in [t, 2t]] = O(1)$. Let Z be a standard gaussian independent of X. For a fixed i we have,

$$\begin{aligned}
\mathbf{Pr}[Y \geq s_i \mid X \in [t, 2t]] &= \mathbf{Pr}[\rho X + \sqrt{1-\rho^2} Z \geq s_i \mid X \in [t, 2t]] \\
&\leq \mathbf{Pr}[Z \geq (s_i - 2\rho t)/\sqrt{1-\rho^2}] \\
&\leq \mathbf{Pr}[Z \geq s_i - (2\rho t/\sqrt{1-\rho^2})] \\
&\leq \mathbf{Pr}[Z \geq s_i - 4\rho t]
\end{aligned}$$

where we used $\rho^2 < 3/4$.

Using Fact 4.1 we have $t = O(\sqrt{\log(1/\mu)}) = O(\sqrt{\log q})$. Since $\rho \leq (\log q)^{-(1/2+\epsilon)}$, we have that $4\rho t = O((\log q)^{-\epsilon})$.

Now $\mathbf{Pr}[Z \geq a - b] \leq \mathbf{Pr}[Z \geq a] + b\varphi(a - b)$. Also, $\varphi(a - b) = 1/\sqrt{2\pi} e^{-(a-b)^2/2} = O(e^{-a^2/2} \cdot e^{-b^2/2} \cdot e^{ab}) = O(\varphi(a) \cdot e^{ab})$. Thus, $\mathbf{Pr}[Z \geq s_i - 4\rho t] \leq \mathbf{Pr}[Z \geq s_i] + \rho t \cdot \varphi(s_i) \cdot e^{O(\rho t s_i)} \leq \nu_i + \rho t \cdot \varphi(s_i) \cdot e^{O((\log q)^{-\epsilon} s_i)}$. We consider two cases.

- If $s_i \leq (\log q)^{\epsilon/2}$, then $\rho t \cdot \varphi(s_i) \cdot e^{O((\log q)^{-\epsilon} s_i)} \leq \rho t \cdot O(\varphi(s_i)) = \rho t \cdot O(s_i N(s_i)) = O((\log q)^{-\epsilon} \nu_i s_i) = O(\nu_i)$.

- If $s_i \geq (\log q)^{\epsilon/2}$, then $\nu_i + \rho t \cdot \varphi(s_i) \cdot e^{O((\log q)^{-\epsilon} s_i)} \leq O(\varphi(s_i)) + O(\rho t \cdot \varphi(s_i) \cdot e^{s_i}) = e^{-\Omega(s_i^2)} = O(e^{-\Omega((\log q)^{\epsilon})})$.

Putting everything together we have,

$$\begin{aligned}
\sum_{i=1}^{k} \mathbf{Pr}[Y \geq s_i \mid X \in [t, 2t]] &\leq \sum_{i=1}^{k} \mathbf{Pr}[Z \geq s_i - 4\rho t] \\
&= \sum_{i: s_i \leq (\log q)^{\epsilon/2}} \mathbf{Pr}[Z \geq s_i - 4\rho t] + \sum_{i: s_i \geq (\log q)^{\epsilon/2}} \mathbf{Pr}[Z \geq s_i - 4\rho t] \\
&= \sum_{i: s_i \leq (\log q)^{\epsilon/2}} O(\nu_i) + \sum_{i: s_i \geq (\log q)^{\epsilon/2}} O(e^{-\Omega((\log q)^{\epsilon})}) \\
&\leq O(k e^{-\Omega((\log q)^{\epsilon})}) + \sum_{i: s_i \leq (\log q)^{\epsilon/2}} O(\nu_i) = O(1)
\end{aligned}$$

where the last line uses $k \leq \log q$ and $\sum_{i=1}^{k} \nu_i \leq 1$.

∎

5.2 Dictatorship Test

We will create an instance of GRAPH-VERTEX-PRICING where the vertex set consists of two disjoint hypercubes L, R where $L = R = \mathbb{Z}_q^n$. The instance will have the property that dictator pricing functions have good profit in the coupon model. On the other hand, if there is a pair of pricing functions $f_L : L \mapsto \mathbb{R}$, $f_R : R \mapsto \mathbb{R}$ which has sufficiently high profit then we will show that f_L and f_R have a common influential co-ordinate.

Formally, we will describe the GRAPH-VERTEX-PRICING instance $\mathcal{I}(G(V, E), \{b_e \mid e \in E\}, \{w_e \mid e \in E\})$ where $V = L \cup R$ as above and E is given by the randomized procedure in Figure 1. The weight w_e of an edge e corresponds to the probability with which it was generated. Note that the total weight of all edges is at most $2t$.

For $\mathbf{x} \in \mathbb{Z}_q^n$, we will denote by $\mathbf{x}_L$ as its copy in L and $\mathbf{x}_R$ as its copy in R.

Theorem 5.2.

- **Completeness:** *Let $f : L \cup R \mapsto \mathbb{R}$ be of the form*

$$f(\mathbf{x}_L) = f(\mathbf{x}_R) = \mathbf{x}_i$$

for some $i \in [n]$ then $\mathbf{profit}_{\mathcal{I}}^{+}(f) = \Omega(\rho t \log t) = \Omega(t(\log q)^{1/3})$

- **Soundness:** *Let $f_L : L \mapsto \mathbb{R}, f_R : R \mapsto \mathbb{R}$. Then there is a $\tau = \tau(q)$ small enough and $k = k(q)$ large enough such that if*

$$\min(\mathbf{Inf}_i^k(f_L), \mathbf{Inf}_i^k(f_R)) \leq \tau$$

for all $i \in [n]$ then $\mathbf{profit}_{\mathcal{I}}^{+}(f_L, f_R) = O(t)$

Figure 1: Dictatorship test for GRAPH-VERTEX-PRICING

.

1. Generate $\mathbf{x} \in \mathbb{Z}_q^n$ uniformly at random and $\mathbf{x}' \sim_\rho \mathbf{x}$ where $\rho := (\log q)^{-2/3}$.
2. For $j \in [\log q]$, let $\mathbf{y}^j \in \mathbb{Z}_q^n$ be defined as $\mathbf{y}^j := (\vec{2^j} - \mathbf{x})_q$, where $\vec{2^j} := \{2^j, \ldots, 2^j\} \in \mathbb{Z}_q^n$.
3. Add $t/2^j$ hyper edges between $(\mathbf{x}'_L, \mathbf{y}^j_R)$ each of budget 2^j for every $j \in \{1, 2, \ldots, k\}$ where $k := \log t$, $t := \sqrt{q}$.

Proof:

Completeness: Let $f(\mathbf{x}_L) = f(\mathbf{x}_R) = \mathbf{x}_i$. For a hyper edge e, let $f(e) := f(\mathbf{y}_R) + f(\mathbf{x}_L)$.

$$\begin{aligned}\mathbf{profit}^+_{\mathcal{I}}(f) &= \sum_e I(0 \le f(e) \le b_e) \cdot w_e \cdot f(e) \\ &= \mathop{\mathbf{E}}_{\mathbf{x}, \mathbf{x}' \sim_\rho \mathbf{x}} \left[\sum_{j=1}^{k} I(0 \le \mathbf{y}^j_i + \mathbf{x}'_i \le 2^j) \cdot \frac{(\mathbf{y}^j_i + \mathbf{x}'_i) \cdot t}{2^j} \right]\end{aligned}$$

Now note that $\mathbf{y}^j_i + \mathbf{x}_i \in \{2^j, 2^j + q\}$. Furthermore, whenever $\mathbf{x}_i \ge 2^j$ it holds that $\mathbf{y}^j_i + \mathbf{x}_i = q + 2^j$. This happens for all j with probability at least $1 - t/q \ge 1/2$ over the choice of $\mathbf{x}$. With probability at least ρ over the choice of $\mathbf{x}'$, we have $\mathbf{x}'_i = \mathbf{x}_i$. Thus, with probability at least $\rho/2$ we have $\mathbf{y}^j_i + \mathbf{x}'_i = 2^j$.

Thus, $\mathbf{profit}^+_{\mathcal{I}}(f) \ge \Omega(\rho t k) = \Omega(\rho t \log t) = \Omega(t (\log q)^{1/3})$.

Soundness:

For a hyper-edge e, let $f(e) := f(\mathbf{y}_R) + f(\mathbf{x}_L)$.

We first show that it suffices to work with $\mathbb{Z}_q$-valued pricing functions.

Definition 5.3. ($\mathbb{Z}_q$-valued pricing)

Let $\mathcal{I}(G(V,E), \{b_e \mid e \in E\}, \{w_e \mid e \in E\})$ be an instance of VERTEX-PRICING or HIGHWAY-PRICING. For a function $f : V \mapsto \mathbb{Z}_q$, the price for an edge $e \in E$ is defined as $f(e) := (\sum_{v \in e} f(v))_q$. Given the price of each edge, $\mathbf{profit}_{\mathcal{I}}(f)$ and $\mathbf{profit}^+_{\mathcal{I}}(f)$ are defined in the usual manner.

Lemma 5.4. *Let $f'_L := \lfloor f_L \rfloor$, $f'_R := \lfloor f_R \rfloor$ then*

$$\mathbf{profit}^+_{\mathcal{I}}(f_L, f_R) \le \mathbf{profit}^+_{\mathcal{I}}(f'_L, f'_R) + 2 \cdot \sum_{e \in E} w_e = \mathbf{profit}^+_{\mathcal{I}}(f'_L, f'_R) + O(t)$$

Proof:

Let $f'(e) := f'_R(\mathbf{y}) + f'_L(\mathbf{x})$

For every edge $e \in E$, $f'(e) \leq f(e) \leq f'(e) + 2$. Thus,

$$\begin{aligned}
\textbf{profit}^+_{\mathcal{I}}(f_L, f_R) &= \textstyle\sum_e I(0 \leq f(e) \leq b_e) \cdot w_e \cdot f(e) \\
&\leq \textstyle\sum_e I(0 \leq f'(e) \leq b_e) \cdot w_e \cdot (f'(e) + 2) \\
&\leq \textbf{profit}^+_{\mathcal{I}}(f'_L, f'_R) + 2 \cdot \textstyle\sum_{e \in E} w_e
\end{aligned}$$

∎

Lemma 5.5. *Let f_L, f_R be integral pricing functions for the instance given by the dictatorship test, and let $f'_L := (f_L)_q$, $f'_R := (f_R)_q$. Then*

$$\textbf{profit}^+_{\mathcal{I}}(f_L, f_R) \leq \textbf{profit}^+_{\mathcal{I}}(f'_L, f'_R) + \textbf{profit}^+_{\mathcal{I}}(f'_L - q, f'_R)$$

Proof:

Fix an edge $e = (\mathbf{x}_L, \mathbf{y}_R)$ with budget b_e. f gets a non-zero profit on e if and only if $0 \leq f(e) \leq b_e$. Since $f'_R(\mathbf{y}) + f'_L(\mathbf{x}) = (f_R(\mathbf{y}) + f_L(\mathbf{x}))_q$, we must have $f'_R(\mathbf{y}) + f'_L(\mathbf{x}) \in \{f(e), f(e) + q\}$. In either case, one of (f'_L, f'_R) and $(f'_L - q, f'_R)$ has the same profit as f on e, which immediately implies the lemma.

∎

Thus, it suffices to prove the soundness for $f_L : L \mapsto \mathbb{Z}_q, f_R : R \mapsto \mathbb{Z}_q$.

We now arithmetize the profit.

$$\begin{aligned}
\textbf{profit}^+_{\mathcal{I}}(f_L, f_R) &= \sum_{e \in E : 0 \leq f(e) \leq b_e} w_e \cdot f(e) \\
&= \mathop{\mathbf{E}}_{\mathbf{x}, \mathbf{x}' \sim_\rho \mathbf{x}} \left[\sum_{j=1}^{k} I(0 \leq f_R(\mathbf{y}^j) + f_L(\mathbf{x}') \leq 2^j) \cdot \frac{(f_R(\mathbf{y}^j) + f_L(\mathbf{x}')) \cdot t}{2^j} \right] \\
&= \mathop{\mathbf{E}}_{\mathbf{x}, \mathbf{x}' \sim_\rho \mathbf{x}} \left[\sum_{j=1}^{k} \sum_{0 \leq b+a \leq 2^j} f_R^b(\mathbf{y}^j) \cdot f_L^a(\mathbf{x}') \cdot \frac{(b+a) \cdot t}{2^j} \right] \\
&= \sum_{j=1}^{k} \sum_{0 \leq b+a \leq 2^j} \mathop{\mathbf{E}}_{\mathbf{x}} \left[f_R^b(\mathbf{y}^j) \cdot (T_\rho f_L)^a(\mathbf{x}) \cdot \frac{(b+a) \cdot t}{2^j} \right]
\end{aligned}$$

Let $g_j : \mathbb{Z}_q^n \mapsto \mathbb{Z}_q$ be defined as $g_j(\mathbf{x}) := f_R(\mathbf{y}^j)$ where $\mathbf{y}^j$ is as in Step 2 of the Test. It is easy to see that $\mathbf{E}[g_j^a] = E[f_R^a]$ for all $a \in \mathbb{Z}_q$. This gives,

$$\textbf{profit}^+_{\mathcal{I}}(f_L, f_R) = t \cdot \sum_{j=1}^{k} \sum_{0 \leq b+a \leq 2^j} \mathbb{S}_\rho(f_L^a, g_j^b) \frac{(b+a)}{2^j}$$

For $l \in \{1, 2, \ldots, k\}$ Let

$$F^a_{j,l}(\mathbf{x}) := \sum_{b:2^{l-1}<b+a\leq 2^l} g^b_j(\mathbf{x})$$

Equivalently, $F^a_{j,l}(\mathbf{x}) = 1$ if $2^{l-1} < g_j(\mathbf{x}) + a \leq 2^l$ and 0 otherwise.

It is clear that $\sum_{l=1}^k \mathbf{E}[F^a_{j,l}] \leq 1$. Also, for every l, $\mathbf{E}[F^a_{j,l}]$ is independent of j since $\mathbf{E}[g^b_j] = \mathbf{E}[f^b_R]$ is independent of j for each $b \in \mathbb{Z}_q$. Let $\mu^a := E[f^a_L]$ and $\nu^a_l := \mathbf{E}[F^a_{j,l}]$.

Note that $\mathbf{Inf}^k_i(F^a_{j,l}) \leq \mathbf{Inf}^k_i(f_R) \leq \tau$ for all $i \in [n]$. Similarly, $\mathbf{Inf}^k_i(f^a_L) \leq \mathbf{Inf}^k_i(f_L) \leq \tau$.

We thus have,

$$\begin{aligned}
& \mathbf{profit}^+_{\mathcal{I}}(f_L, f_R) \\
= \; & t \cdot \sum_{j=1}^k \sum_{0\leq b+a\leq 2^j} \mathbb{S}_\rho(f^a_L, g^b_j)\frac{(b+a)}{2^j} \\
\leq \; & t \cdot \sum_{j=1}^k \sum_{a\in\mathbb{Z}_q} \sum_{l=1}^j \mathbb{S}_\rho(f^a_L, F^a_{j,l})\frac{2^l}{2^j} \\
\leq \; & t \cdot \textstyle\sum_{a\in\mathbb{Z}_q} \displaystyle\sum_{j=1}^k \sum_{l=1}^j \Lambda_\rho(\mu^a, \nu^a_l)\frac{2^l}{2^j} + o(1) && \text{(Choose } \delta = (tqk^2)^{-1} \text{ in Theorem 4.11)} \\
= \; & t \cdot \textstyle\sum_{a\in\mathbb{Z}_q} \displaystyle\sum_{l=1}^k \Lambda_\rho(\mu^a, \nu^a_l) \sum_{j=l}^k \frac{2^l}{2^j} + o(1) \\
\leq \; & 2t \cdot \textstyle\sum_{a\in\mathbb{Z}_q} \displaystyle\sum_{l=1}^k \Lambda_\rho(\mu^a, \nu^a_l) + o(1) \\
= \; & 2t \cdot \textstyle\sum_{a\in\mathbb{Z}_q:\mu_a\geq(q\log q)^{-1}} \displaystyle\sum_{l=1}^k \Lambda_\rho(\mu^a, \nu^a_l) \\
& +2t \cdot \textstyle\sum_{a\in\mathbb{Z}_q:\mu_a\leq(q\log q)^{-1}} \displaystyle\sum_{l=1}^k \Lambda_\rho(\mu^a, \nu^a_l) \\
\leq \; & 2t \cdot \textstyle\sum_{a\in\mathbb{Z}_q:\mu_a\geq(q\log q)^{-1}} \displaystyle\sum_{l=1}^k \Lambda_\rho(\mu^a, \nu^a_l) \\
& +2t \cdot \textstyle\sum_{a\in\mathbb{Z}_q:\mu_a\leq(q\log q)^{-1}} \displaystyle\sum_{l=1}^k \mu^a \\
\leq \; & 2t \cdot \textstyle\sum_{a\in\mathbb{Z}_q:\mu_a\geq(q\log q)^{-1}} \displaystyle\sum_{l=1}^k \Lambda_\rho(\mu^a, \nu^a_l) + O(t) && \text{(Using } k \leq \log q\text{)} \\
\leq \; & 2t \cdot \textstyle\sum_{a\in\mathbb{Z}_q} O(\mu^a) + O(t) && \text{(Using Lemma 5.1)} \\
= \; & O(t)
\end{aligned}$$

■

Figure 2: Reduction from UNIQUE-GAMES to GRAPH-VERTEX-PRICING

.

1. Pick a random edge $e = (u, w) \in E'$.
2. Generate $\mathbf{x} \in \mathbb{Z}_q^n$ uniformly at random and $\mathbf{x}' \sim_\rho \mathbf{x}$ where $\rho := (\log q)^{-(2/3)}$.
3. For $j \in [\log q]$, let $\mathbf{y}^j \in \mathbb{Z}_q^n$ be defined as $\mathbf{y}^j := (\vec{2^j} - \mathbf{x})_q$, where $\vec{2^j} := \{2^j, \ldots, 2^j\} \in \mathbb{Z}_q^n$.
4. Add $t/2^j$ edges between $\big((u, \mathbf{x}'), (w, \pi_{uw}(\mathbf{y}^j))\big)$ each of budget 2^j for each $j \in \{1, 2, \ldots, k\}$ where $k := \log t$, $t := \sqrt{q}$ and $\pi(x) := (x_{\pi_1}, x_{\pi_2}, \ldots, x_{\pi_n})$.

5.3 Reduction from UNIQUE-GAMES

Given a bipartite UNIQUE-GAMES instance $\mathcal{U}(G'(U, W, E'), [n], \{\pi_e\}_{e\in E})$ we create an instance of GRAPH-VERTEX-PRICING $\mathcal{I}(G(V, E), \{b_e \mid e \in E\}, \{w_e \mid e \in E\})$ where $V = U \times \mathbb{Z}_q^n \cup W \times \mathbb{Z}_q^n$ and E is defined by the randomized procedure in Figure 2. The weight w_e of an edge $e \in E$ corresponds to the probability with which it was generated and the budgets are as specified in the procedure.

Theorem 5.6. • **Completeness:** *If $OPT(\mathcal{U}) \geq 1 - \epsilon$ then there is an assignment of prices $f : V \mapsto \mathbb{R}$ such that* $\mathbf{profit}_{\mathcal{I}}^+(f) = \Omega(\rho t \log t) = \Omega(t(\log q)^{1/3})$.

- **Soundness:** *There is an $\eta = \eta(q)$ small enough such that if $OPT(\mathcal{U}) \leq \eta$ then for every assignment of prices $f : V \mapsto \mathbb{R}$,*
$$\mathbf{profit}_{\mathcal{I}}^+(f) = O(t)$$

Since q can be arbitrarily large, this implies Theorem 2.12.

Proof:

For a fixed edge $e = (u, w) \in E'$, the instance $\mathcal{I}$ of Figure 2 restricted to $u \times \mathbb{Z}_q^n \cup w \times \mathbb{Z}_q^n$ is same as the one constructed by Figure 1 up to renumbering of labels according to π_{uw}. Formally, let $L = R = \mathbb{Z}_q^n$, let $f_L : L \mapsto \mathbb{R}$ be defined as $f_L^e(\mathbf{x}) := f(u, \mathbf{x})$ and $f_R^e : R \mapsto \mathbb{R}$ be defined as $f_R(\mathbf{x}) := f(w, \pi_{uw}(\mathbf{x}))$. Then it is clear that

$$\mathbf{profit}_{\mathcal{I}}^+(f) = \mathop{\mathbf{E}}_{e\in E'}[\mathbf{profit}_{\mathcal{I}}^+(f_L^e, f_R^e)]$$

where $\mathbf{profit}_{\mathcal{I}}^+(f_L^e, f_R^e)$ refers to the profit of (f_L^e, f_R^e) on $\mathcal{I}$ restricted to e.

- **Completeness:**

  Let $L : U \cup W \mapsto [n]$ be a labeling which satisfies $1 - \epsilon$ fraction of the constraints. We define the pricing function $f : V \mapsto \mathbb{R}$ as $f(u, \mathbf{x}) := \mathbf{x}_{L(u)}$.

If e is satisfied by L, then $f_L^e(\mathbf{x}) = f_R^e(\mathbf{x}) = \mathbf{x}_i$ for some $i \in [n]$. By Theorem 5.2 we get that $\mathbf{profit}_{\mathcal{I}}^+(f_L^e, f_R^e)$ is at least $\Omega(\rho t \log t)$.

Thus, the overall profit is at least $(1-\epsilon)\Omega(\rho t \log t) = \Omega(\rho t \log t)$.

- **Soundness:**

  We will show that if $\mathbf{profit}_{\mathcal{I}}^+(f) = \omega(t)$ then there is a randomized labeling strategy to the UNIQUE-GAMES instance which in expectation satisfies more than η fraction of the edges.

  Note that the profit for any $e = (u, w) \in E'$ is bounded by $O(t \log t)$. So if $\mathbf{profit}_{\mathcal{I}}^+(f) = \omega(t)$ then for at least $1/(\log t)$ fraction of the edges $e \in E'$ we have $\mathbf{profit}_{\mathcal{I}}^+(f_L^e, f_R^e) = \omega(t)$. For such e by Theorem 5.2 we have that $\mathbf{Inf}_i^k(f_L^e)$ and $\mathbf{Inf}_i^k(f_R^e)$ are both larger than τ for some $i \in [n]$. By definition of f_L^e and f_R^e this implies $\mathbf{Inf}_i^k(f_u)$ and $\mathbf{Inf}_{\pi_{uw}(i)}^k(f_v)$ are both larger than τ where f_u is f restricted to $u \times \mathbb{Z}_q^n$.

  For each $u \in U \cup W$, let

  $$\mathbf{Inf}(u) := \{i \in [n] \mid \mathbf{Inf}_i(f_u) \geq \tau\}$$

  The labeling strategy is to assign for each $u \in U \cup W$ a label independently and uniformly at random from $\mathbf{Inf}(u)$. If $\mathbf{Inf}(u)$ is empty, assign an arbitrary label to u. By Fact 4.8 (and since we can work with $\mathbb{Z}_q$-valued functions) , the size of $\mathbf{Inf}(u)$ is at most k/τ for each u.

  The above analysis shows that the expected fraction of edges satisfied by this labeling is at least

  $$\frac{1}{\log t} \cdot \left(\frac{\tau}{k}\right)^2$$

  Since this quantity depends only on q, we can choose $\eta = \eta(q)$ small enough so that more than η fraction of the edges are satisfied. This completes the proof.

■

6 NP-Hardness of pricing loss leaders

In the following section, we prove our Theorem 2.9. We first describe the matching Dictator Test; then we use that as a gadget to make a hardness reduction from the LABEL-COVER problem.

6.1 Test for pricing loss leaders

Let $K, L \in \mathbb{Z}^+$ and $L > K$. For $\pi : [L] \to [K]$ being a projection, we define the VERTEX-PRICING$_3$ instance corresponding to the Dictator Test $\mathcal{T}_\pi$ on $V = [q]^L \cup [q]^K$ in Figure 3.
Below are two key properties of $\mathcal{T}_\pi$.

Theorem 6.1 (completeness)**.** *For every $j \in [L]$ and $i = \pi(j)$, if we set $f(t) = t_i - q/3$ for $t \in \mathbb{Z}_q^K$ and $g(r) = r_j - q/3$ for $r \in \mathbb{Z}_q^L$, then* $\mathbf{profit}_{\mathcal{T}_\pi}(f, g) \geq \Omega(\log q)$.

Figure 3: Dictator test $\mathcal{T}_\pi$

.

Test $\mathcal{T}_\pi$.

1. Generate $\mathbf{x}$ to be uniformly random from $\mathbb{Z}_q^K$.
2. Generate $\mathbf{y}$ to be a uniformly random from $\mathbb{Z}_q^L$.
3. For each $j \in [L]$ and $i = \pi(j)$, set

$$\boldsymbol{z}_j = \begin{cases} q - (\mathbf{x}_i + \mathbf{y}_j) & \text{if } \mathbf{x}_i + \mathbf{y}_j \le q \\ 2q - (\mathbf{x}_i + \mathbf{y}_j) & \text{if } \mathbf{x}_i + \mathbf{y}_j > q \end{cases}$$

4. Let $\mathbf{x}' \sim_{1-\epsilon} x$, $\mathbf{y}' \sim_{1-\epsilon} y$ and $\boldsymbol{z}' \sim_{1-\epsilon} z$ for $\epsilon = 1/q$
5. Randomly generate a integer $k \in [\sqrt{q}]$,
6. Let $\boldsymbol{z}'' = \boldsymbol{z}' + \vec{1} \cdot \lfloor \sqrt{q}/k \rfloor$ and add a customer interested in three items $\mathbf{x}', \mathbf{y}', \boldsymbol{z}''$ with budget $\lfloor \sqrt{q}/k \rfloor$

Proof. It is easy to check with probability at least 1/3, we have that $\mathbf{x}_i + \mathbf{y}_j \le q$ for randomly and independently generated $\mathbf{x}_i$ and $\mathbf{y}_j$ after the second step in Figure 3. Therefore for these $\mathbf{x}_i, \mathbf{y}_j$, we have $\boldsymbol{z}_j = q - \mathbf{x}_i - \mathbf{y}_j$ at the third step.

Since $\mathbf{x}'_i, \mathbf{y}'_j$ and $\boldsymbol{z}'_j$ is generated by perturbing $\mathbf{x}_i, \mathbf{y}_j, \boldsymbol{z}_j$ with probability at most $1/q$, by union bound we know that with probability at least $1/3 - 3/q$, we still have $\mathbf{x}'_i + \mathbf{y}'_j + \boldsymbol{z}'_j = q$. Also since $\boldsymbol{z}'_j$ follows the uniform distribution on $\mathbb{Z}_q$, we know with probability at least $1/3 - 3/q - 1/\sqrt{q} \ge 1/4$ it holds that $\boldsymbol{z}'_j \le q - \sqrt{q}$. Let us call these $\mathbf{x}', \mathbf{y}', \boldsymbol{z}'$ "good".

For "good" $\mathbf{x}', \mathbf{y}', \boldsymbol{z}'$, we know that $f(\mathbf{x}') = \mathbf{x}_i - q/3$ and $g(\mathbf{y}') = \mathbf{y}_j - q/3$ and $g(\boldsymbol{z}' \oplus_q + \vec{1} \cdot \lfloor \sqrt{q}/k \rfloor) = \boldsymbol{z}_j + \lfloor \sqrt{q}/k \rfloor - q/3$ (since $\boldsymbol{z}'_j \le q - \sqrt{q}$). Thus, $f(\mathbf{x}') + g(\mathbf{y}') + g(\boldsymbol{z}' + \vec{1} \cdot \lfloor \sqrt{q}/k \rfloor) = \lfloor \sqrt{q}/k \rfloor$. Therefore, for good $\mathbf{x}', \mathbf{y}', \boldsymbol{z}'$, we made $\lfloor \sqrt{q}/k \rfloor$ on the buyer interested in $\mathbf{x}', \mathbf{y}', \boldsymbol{z}' + \vec{1} \cdot \lfloor \sqrt{q}/k \rfloor$. Since we have at least 1/4 "good" $\mathbf{x}', \mathbf{y}', \boldsymbol{z}'$, we made profit $1/4 \cdot 1/\sqrt{q} \cdot \sum_k \lfloor \sqrt{q}/k \rfloor = \Omega(\log q)$ on them.

We also need to bound the negative profit as f and g can also take negative value. This is when the case that $f(\mathbf{x}) + f(\mathbf{y}) + f(\boldsymbol{z} + \vec{1} \cdot \lfloor \sqrt{q}/k \rfloor) < 0$. We claim that we may only lose money in one of the following two cases:

1. $\mathbf{x}_i \ne \mathbf{x}'_i$ or $\mathbf{y}_j \ne \mathbf{y}'_j$ or $\boldsymbol{z}_j \ne \boldsymbol{z}'_j$.
2. $\mathbf{x}_i + \mathbf{y}_j <= \sqrt{q}$.

To verify this, if case 1 and case 2 do not happen, then $\mathbf{x} = \mathbf{x}'$, $\mathbf{y} = \mathbf{y}'$ and $\boldsymbol{z} = \boldsymbol{z}'$ and $\mathbf{x}_i + \mathbf{y}_j > \sqrt{q}$. When $\sqrt{q} < \mathbf{x}_i + \mathbf{y}_j \le q$,we know that $f(\mathbf{x}) + f(\mathbf{y}) + f(\boldsymbol{z} + \vec{1} \cdot \lfloor \sqrt{q}/k \rfloor) = \lfloor \sqrt{q}/k \rfloor$; when $\mathbf{x}_i + \mathbf{y}_j > q$, we know that $f(\mathbf{x}) + f(\mathbf{y}) + f(\boldsymbol{z} + \vec{1} \cdot \lfloor \sqrt{q}/k \rfloor) \ge \mathbf{x}_i + \mathbf{y}_j - q + [\boldsymbol{z}_j + \lfloor \sqrt{q}/k \rfloor]_q > 0$.

By union bound we know that case 1 happens with probability at most $3/q$; and case 2 could only happen when $x_i + y_j \leq \sqrt{q}$ and this also happens with probability at most $\frac{1}{q}$ (because we must have both x, y less than $\sqrt{q}$ which occur with probability $1/q$). Overall, we know that only for $4/q$ fraction of the x, y, z generated, we can possibly lose money. Also since $f, g >= -q/3$ by definition, we can lose money for at most q on each customer. Therefore, we can at most lose profit $4/q \cdot q <= 4$.

Overall the profit we have on f and g is still $\Omega(\log q)$. $\square$

Theorem 6.2 (soundness)**.** *If for some function $f : \mathbb{Z}_q^K \to \mathbb{R}$ and $g : \mathbb{Z}_q^L \to \mathbb{R}$, we have $\mathbf{profit}_{\mathcal{T}}^{+}(f, g) \geq 12$, then we can have a (randomized) way of decoding f in to a coordinate $i_f \in [K]$ and g into a coordinate $j_g \in [L]$ such that the*

$$Pr(\pi(j_g) = i_f) \geq 1/q^6.$$

In addition, the decoding of f is independent of g or π; i.e., there is one decoding procedure that works for all possible π, g. Similarly the decoding procedure of g is independent of f and π

Proof. First, we can assume that the pricing function is integer with profit loss 3 simply by taking the integer part of f and g.. To see this, for any fixed $\mathbf{x}', \mathbf{y}', \boldsymbol{z}''$, we have some *real* pricing function $f(\mathbf{x}') + g(\mathbf{y}') + g(\boldsymbol{z}'')$, then $\lfloor f(\mathbf{x}') \rfloor + \lfloor g(\mathbf{y}') \rfloor + \lfloor g(\boldsymbol{z}'') \rfloor \geq f(\mathbf{x}') + g(\mathbf{y}') + g(\boldsymbol{z}'') - 3$. Therefore, we have that

$$\mathbf{profit}^{+}(\lfloor f \rfloor, \lfloor g \rfloor) \geq \mathbf{profit}^{+}(f, g) - 3.$$

Further, we restrict the range of f, g by modulo q as in Definition 5.3. For any function f, we define $\tilde{f} = [\lfloor f \rfloor]_q$ $\tilde{g} = [\lfloor g \rfloor]_q$. Following lemma illustrates the relationship between the profit of using $\lfloor f \rfloor, \lfloor g \rfloor$ and $\tilde{f}, \tilde{g}$.

Lemma 6.3. $\mathbf{profit}_{\mathcal{I}_\pi}^{+}(\lfloor f \rfloor, \lfloor g \rfloor) \leq \mathbf{profit}_{\mathcal{I}_\pi}^{+}(\tilde{f}, \tilde{g}) + \mathbf{profit}_{\mathcal{I}_\pi}^{+}(\tilde{f} - q/3, \tilde{g} - q/3) + \mathbf{profit}_{\mathcal{I}_\pi}^{+}(\tilde{f} - 2q/3, \tilde{g} - 2q/3)$

Proof. We know that if for some buyer who is interested in $\mathbf{x}', \mathbf{y}', \boldsymbol{z}''$ with budget $\lfloor \sqrt{q}/k \rfloor$, then if $f(\mathbf{x}') + g(\mathbf{y}') + g(\boldsymbol{z}'') \leq \lfloor \sqrt{q}/k \rfloor$. Then it must be the case that $0 < \tilde{f}(\mathbf{x}') + \tilde{g}(\mathbf{y}') + \tilde{g}(\boldsymbol{z}'') < \lfloor \sqrt{q}/k \rfloor$ or $q < \tilde{f}(\mathbf{x}') + \tilde{g}(\mathbf{y}') + \tilde{g}(\boldsymbol{z}'') < q + \lfloor \sqrt{q}/k \rfloor$ or $2q < \tilde{f}(\mathbf{x}') + \tilde{g}(\mathbf{y}') + \tilde{g}(\boldsymbol{z}'') < 2q + \lfloor \sqrt{q}/k \rfloor$. Therefore, at least one of the pricing strategy among $(\tilde{f}, \tilde{g})$, $(\tilde{f} - q/3, \tilde{g} - q/3)$ or $(\tilde{f} - 2q/3, \tilde{g} - 2q/3)$ will have the same profit as (f, g) on $\mathbf{x}', \mathbf{y}', \boldsymbol{z}''$. $\square$

It remains to bound $\mathbf{profit}_{\mathcal{T}}^{+}(\tilde{f}, \tilde{g}) + \mathbf{profit}_{\mathcal{T}}^{+}(\tilde{f} - q/3, \tilde{g} - q/3) + \mathbf{profit}^{+}(\tilde{f} - 2q/3, \tilde{g} - 2q/3)$. We will only show how to bound $\mathbf{profit}^{+}(\tilde{f}, \tilde{g}) \leq 3$ and the other proof is similar.

Let us also introduce the notion $\tilde{f}_i : \mathbb{Z}_q^L \to \{0, 1\}$ as indicator function of whether $\tilde{f} = i$. We similarly define $\tilde{g}_i = i$. We also write $\tilde{f}_i = \sum \tilde{f}_i^S$ and $\tilde{g}_i = \sum g_i^S$ as the Efron-Stein Decomposition of $\tilde{f}_i, \tilde{g}_i$. .

We can represent the $\mathbf{profit}^{+}(\tilde{f}, \tilde{g})$ as follows:

$$\mathbf{profit}^+_{\mathcal{T}_\pi}(\tilde{f},\tilde{g}) \le \mathop{\mathbf{E}}_{\mathbf{x}',\mathbf{y}',\boldsymbol{z}'',k}\left[\sum_{0<i+j+l\le\lfloor\sqrt{q}/k\rfloor}(i+j+l)\cdot\tilde{f}_i(\mathbf{x}')\tilde{g}_j(\mathbf{y}')\tilde{g}_l(\boldsymbol{z}'')\right] \tag{6.1}$$

Now we plug in the Efron-Stein Decomposition of $\tilde{f}_i,\tilde{g}_j,\tilde{g}_l$:

$$(6.1) = \mathop{\mathbf{E}}_{k}\left[\sum_{0<i+j+l\le\lfloor\sqrt{q}/k\rfloor}\ \sum_{T_1,T_2\subseteq[L],S\subseteq[K],}(i+j+l)\mathop{\mathbf{E}}_{\mathbf{x}',\mathbf{y}',\boldsymbol{z}''}[\tilde{f}_i^S(\mathbf{x}')\tilde{g}_j^{T_1}(\mathbf{y}')\tilde{g}_l^{T_2}(\boldsymbol{z}'')\right] \tag{6.2}$$

We know that $\mathbf{x}'$ is independent of $\mathbf{y}'$ and $\mathbf{x}'$ is independent of $\boldsymbol{z}''$. By the second property of Efron-Stein Decomposition, we must have that $T_1=T_2=T$ as otherwise ,$\mathbf{E}_{\mathbf{x}',\mathbf{y}',\boldsymbol{z}''}[\tilde{f}_i^S(x)\tilde{g}_j^{T_1}(y)\tilde{g}_l^{T_2}(\boldsymbol{z}'')]=0$. For the similar reason if we write the set $\pi(T)=\{\pi(j)\mid j\in T\}$, then we must also have $S\subseteq\pi(T)$. We know then

$$\begin{aligned}
(6.2) &= \mathop{\mathbf{E}}_{k}\left[\sum_{0<i+j+l\le\lfloor\sqrt{q}/k\rfloor}\ \sum_{\substack{T\subseteq[L]\\ S\subseteq\pi(T)}}(i+j+l)\mathop{\mathbf{E}}_{\mathbf{x}',\mathbf{y}',\boldsymbol{z}'}[\tilde{f}_i^S(\mathbf{x}')\tilde{g}_j^{T}(\mathbf{y}')\tilde{g}_l^{T}(\boldsymbol{z}'+\vec{1}\cdot\lfloor\sqrt{q}/k\rfloor)]\right] && (6.3)\\
&= \mathop{\mathbf{E}}_{k}\left[\sum_{0<i+j+l\le\lfloor\sqrt{q}/k\rfloor}\ \sum_{\substack{T\subseteq[L]\\ S=\emptyset}}(i+j+l)\mathop{\mathbf{E}}_{\mathbf{x}',\mathbf{y}',\boldsymbol{z}'}[\tilde{f}_i^S(\mathbf{x}')\tilde{g}_j^{T}(\mathbf{y}')\tilde{g}_l^{T}(\boldsymbol{z}'+\vec{1}\cdot\lfloor\sqrt{q}/k\rfloor)]\right] && (6.4)\\
&+ \mathop{\mathbf{E}}_{k}\left[\sum_{0<i+j+l\le\lfloor\sqrt{q}/k\rfloor}\ \sum_{\substack{T\subseteq[L]\\ \emptyset\subsetneq S\subseteq\pi(T)}}(i+j+l)\mathop{\mathbf{E}}_{\mathbf{x}',\mathbf{y}',\boldsymbol{z}'}[\tilde{f}_i^S(\mathbf{x}')\tilde{g}_j^{T}(\mathbf{y}')\tilde{g}_l^{T}(\boldsymbol{z}'+\vec{1}\cdot\lfloor\sqrt{q}/k\rfloor)]\right] && (6.5)
\end{aligned}$$

In the second equality above, we divide (6.3) into two parts. (6.4) is when $S=\emptyset$ and the (6.5) is when $S\neq\emptyset$ and we will bound the two parts individually.

Case i) First we prove that (6.4) ≤ 2. Notice that $\tilde{f}_i^{\emptyset}(\mathbf{x}')$ is a constant, say $\tilde{f}_i^{\emptyset}$. Thus (6.4) is equal to

$$\mathop{\mathbf{E}}_{k}\left[\sum_{0<i+j+l\le\lfloor\sqrt{q}/k\rfloor}\ \sum_{T\subseteq[L]}(i+j+l)\ \tilde{f}_i^{\emptyset}\mathop{\mathbf{E}}_{\mathbf{y}',\boldsymbol{z}'}[g_j^{T}(\mathbf{y}')\tilde{g}_l^{T}(\boldsymbol{z}'+\vec{1}\cdot\lfloor\sqrt{q}/k\rfloor)]\right] \tag{6.6}$$

A crucial observation is that above expression can be viewed as the the profit of $\lfloor\tilde{f}\rfloor,\lfloor\tilde{g}\rfloor$on Dictator test $\mathcal{T}'$ defined in Figure 4.The test $\mathcal{T}'_\pi$ is different from $\mathcal{T}_\pi$ only in the generation of $\mathbf{x}'$ which is set to be independent with $\mathbf{y}'$ and $\boldsymbol{z}''$.Then when we calculate the profit as in equation (6.3), we would get that $\mathbf{E}_{\mathbf{x}',\mathbf{y}',\boldsymbol{z}''\sim\mathcal{T}'_\pi}[\tilde{f}_i^S(\mathbf{x}')\tilde{g}_j^{T_1}(\mathbf{y}')\tilde{g}_l^{T}(\boldsymbol{z}'')]\neq 0$ only when $S=\emptyset$ and $T_1=T_2$. This is exactly the same as (6.6). It remains to bound the profit on $\mathcal{T}'_\pi$.

The next important observation on $\mathcal{T}'_\pi$ here is that actually $\mathbf{y}',\boldsymbol{z}'$ and $\mathbf{y}',\boldsymbol{z}''$ has the same marginal distribution. Therefore, we can further simplify the test as $\mathcal{T}''_\pi$ defined in Figure 5. As for the test $\mathcal{T}''_\pi$, for every fixed $\mathbf{x}',\mathbf{y}',\boldsymbol{z}'$, and suppose $\sqrt{q}/(k_0+1)<\tilde{f}(x)+\tilde{g}(y)+\tilde{g}(z)\le\sqrt{q}/k_0$. Then such a

Figure 4: Test $\mathcal{T}'_\pi$

Test $\mathcal{T}'$.

1. generate $\mathbf{y}', \boldsymbol{z}', k$ according to their margin distribution on $\mathcal{T}$.
2. **set $\mathbf{x}'$ to be uniformly over $\mathbb{Z}_q^K$ independent with $\mathbf{y}', \boldsymbol{z}', k$.**
3. let us write $\boldsymbol{z}'' = \boldsymbol{z}' + \vec{1} \cdot \lfloor \sqrt{q}/k \rfloor$ and add a customer interested in three items $\mathbf{x}', \mathbf{y}', \boldsymbol{z}''$ with budget $\lfloor \sqrt{q}/k \rfloor$

Figure 5: Test $\mathcal{T}''_\pi$

Test $\mathcal{T}''$.

1. generate $\mathbf{y}', \boldsymbol{z}', k$ according to their marginal distribution on $\mathcal{T}$.
2. **set $\mathbf{x}'$ to be uniformly over $\mathbb{Z}_q^K$ independent with $\mathbf{y}', \boldsymbol{z}', k$.**
3. add a customer interested in three items $\mathbf{x}', \mathbf{y}', \boldsymbol{z}'$ with budget $\lfloor \sqrt{q}/k \rfloor$.

pricing function will only have profit when $k \leq (k_0 + 1)$ and for that fixed $\mathbf{x}', \mathbf{y}', \boldsymbol{z}'$, the expected profit conditioned on k (being randomly generated from $[\sqrt{q}]$) is at most $\frac{1}{\sqrt{q}} \cdot \sqrt{q}/k_0 \cdot (k_0 + 1) \leq 2.$.

Overall, we proved that

$$(6.4) = \mathbf{profit}^+_{\mathcal{T}'_\pi}(\lfloor \tilde{f} \rfloor, \lfloor \tilde{g} \rfloor) = \mathbf{profit}^+_{\mathcal{T}''_\pi}(\lfloor \tilde{f} \rfloor, \lfloor \tilde{g} \rfloor) \leq 2.$$

Case ii) It remains to bound (6.5). Let us prove this by contradiction. We will show that if (6.5) ≥ 1, then there exists a way of decoding f and g as described in Theorem 6.2.

We know that

$$(6.5) = \mathop{\mathbf{E}}_{k} \left[\sum_{0 < i+j+l \leq \lfloor \sqrt{q}/k \rfloor} \sum_{\substack{T \subseteq [L] \\ \emptyset \subsetneq S \subseteq \pi(T)}} (i+j+l) \mathop{\mathbf{E}}_{\mathbf{x}', \mathbf{y}', \boldsymbol{z}'} [\tilde{f}_i^S(\mathbf{x}') \tilde{g}_j^T(\mathbf{y}') \tilde{g}_l^T(\boldsymbol{z}' + \vec{1} \cdot \lfloor \sqrt{q}/k \rfloor)] \right] \tag{6.7}$$

Notice that $\mathbf{x}' \sim_{1-\epsilon} \mathbf{x}$, $\mathbf{y}' \sim_{1-\epsilon} \mathbf{y}'$ and $\boldsymbol{z}' \sim_{1-\epsilon} \boldsymbol{z}$, by the definition of the noise operator, we have that

$$(6.7) = \mathop{\mathbf{E}}_{k} \left[\sum_{0 < i+j+l \leq \lfloor \sqrt{q}/k \rfloor} \sum_{\substack{T \subseteq [L] \\ \emptyset \subsetneq S \subseteq \pi(T)}} (i+j+l) \mathop{\mathbf{E}}_{\mathbf{x}, \mathbf{y}, \boldsymbol{z}} [T_{1-\epsilon} \tilde{f}_i^S(\mathbf{x}) T_{1-\epsilon} \tilde{g}_j^T(\mathbf{y}) T_{1-\epsilon} \tilde{g}_l^T(\boldsymbol{z} + \vec{1} \cdot \lfloor \sqrt{q}/k \rfloor)] \right] \tag{6.8}$$

Let us write $\tilde{f}^{\pi(T)} = \sum_{\emptyset \subsetneq S \subseteq \pi(T)} \hat{\tilde{f}}(S)$. Then we know

$$(6.8) = \mathop{\mathbf{E}}_{k}\left[\sum_{0<i+j+l\leq\lfloor\sqrt{q}/k\rfloor}\sum_{T\subseteq[L]}(i+j+l)\mathop{\mathbf{E}}_{\mathbf{x},\mathbf{y},\boldsymbol{z}}[T_{1-\epsilon}\tilde{f}_i^{\pi(T)}(\mathbf{x})T_{1-\epsilon}\tilde{g}_j^T(\mathbf{y})T_{1-\epsilon}\tilde{g}_l^T(\boldsymbol{z}+\vec{1}\cdot\lfloor\sqrt{q}/k\rfloor)]\right]$$

Using Cauchy Inequality in the expectation of above formula and noticing that $\mathbf{x},\mathbf{y}$ and $\mathbf{y},\boldsymbol{z}$ are independent, we have

$$\begin{aligned}(6.8) \leq& \mathop{\mathbf{E}}_{k}\left[\sum_{0<i+j+l\leq\lfloor\sqrt{q}/k\rfloor}(i+j+l)\sum_{T\subseteq[L]}\sqrt{\mathop{\mathbf{E}}_{\mathbf{x}}[T_{1-\epsilon}\tilde{f}_i^{\pi(T)}(\mathbf{x})^2]\mathop{\mathbf{E}}_{\mathbf{y}}[T_{1-\epsilon}\tilde{g}_j^T(\mathbf{y})^2]\mathop{\mathbf{E}}_{\boldsymbol{z}}[T_{1-\epsilon}\tilde{g}_l^T(\boldsymbol{z}+\vec{1}\cdot\lfloor\sqrt{q}/k\rfloor)^2]}\right]\\ =& \mathop{\mathbf{E}}_{k}\left[\sum_{0<i+j+l\leq\lfloor\sqrt{q}/k\rfloor}(i+j+l)\sum_{T\subseteq[L]}\|T_{1-\epsilon}\tilde{f}_i^{\pi(T)}\|_2\cdot\|T_{1-\epsilon}\tilde{g}_j^T\|_2\cdot\|T_{1-\epsilon}\tilde{g}_l^T\|_2\right]\end{aligned} \tag{6.9}$$

We can further use Cauchy inequality to bound the inside sum for every i, j, l :

$$\begin{aligned}&\sum_{T\subseteq[L]}\|T_{1-\epsilon}\tilde{f}_i^{\pi(T)}\|_2\cdot\|T_{1-\epsilon}\tilde{g}_j^T\|_2\cdot\|T_{1-\epsilon}\tilde{g}_l^T\|_2\\ \leq& \sqrt{\sum_{T\subseteq[L]}\|T_{1-\epsilon}\tilde{f}_i^{\pi(T)}\|_2^2\cdot\|T_{1-\epsilon}\tilde{g}_j^T\|_2^2\sum_{T\subseteq[L]}\|T_{1-\epsilon}\tilde{g}_l^T\|_2^2}\end{aligned} \tag{6.10}$$

Notice that $\sum_{T\subseteq[L]}\|T_{1-\epsilon}\tilde{g}_l^T\|_2^2 = \|T_{1-\epsilon}\tilde{g}_l\|_2^2 \leq 1$. Therefore, we have that

$$1 \leq (6.5) \leq (6.9) \leq \sum_{0<i+j+l\leq\sqrt{q}}(i+j+l)\sqrt{\sum_{T\subseteq[L]}\|T_{1-\epsilon}f_i^{\pi(T)}\|_2^2\cdot\|T_{1-\epsilon}g_j^T\|^2}$$

Thus there must exist some i_0, j_0 such that

$$\sum_{T\subseteq[L]}\|T_{1-\epsilon}\tilde{f}_{i_0}^{\pi(T)}\|_2^2\cdot\|T_{1-\epsilon}\tilde{g}_{j_0}^T\|^2 = \sum_{\substack{T\subseteq[L]\\ \emptyset\subsetneq S\subseteq\pi(T)}}(1-\epsilon)^{|S|+|T|}\|\tilde{f}_{i_0}^S\|_2^2\|\tilde{g}_{j_0}^T\|_2^2 \geq 1/q^5.$$

It is easy to verify that $\sum_{i\in\mathbb{Z}_q, S\subseteq[K]}\|\tilde{f}_i^S\|_2^2 = 1$ and $\sum_{j\in\mathbb{Z}_q, T\subseteq[L]}\|\tilde{g}_j^T\|_2^2 = 1$. Below is the randomized decoding procedure for f and g. For f, we sample (i, S) with probability $\|\tilde{f}_i^S\|_2^2$ and randomly output a $m_f \in S$. Similarly for g, we randomly sample (j, T) with probability $\|g_j^T\|_2^2$ and randomly output a coordinate n_g in T.

Then the probability that $\pi(n_g) = m_f$ is at least

$$\mathbf{Pr}(\pi(n_g) = m_f) \geq \sum_{\emptyset\subsetneq S\subseteq\pi(T), T\subseteq[L]}\frac{\|\tilde{f}_{i_0}^S\|_2^2\cdot\|\tilde{g}_{j_0}^T\|^2}{T} \tag{6.11}$$

Above we only count the probability when (i_0, S) and (j_0, T) are selected such that $\emptyset \subsetneq S \subseteq \pi(T)$. Then we know that for randomly picked elements $m_f \in S$ and $n_g \in T$, with probability at least $1/T$, we have $\pi(n_g) = m_f$.

Also notice that $1/|T| \geq \epsilon \cdot (1-\epsilon)^{|T|}$. Since $\epsilon = 1/q$, we have that

$$\mathbf{Pr}(\pi(n_g) = m_f) \geq (6.11) \geq \sum_{\emptyset \subsetneq S \subseteq \pi(T), T \subseteq [L]} \epsilon(1-\epsilon)^T \|\tilde{f}_{i_0}^S\|_2^2 \cdot \|\tilde{g}_{j_0}^T\|^2 \geq 1/q^6. \tag{6.12}$$

□

6.2 Hardness reduction

The starting point of our hardness reduction is the following Label-Cover problem.

Definition 6.4 (Label Cover)**.** A Label Cover $\mathcal{L}(G(U,V,E),[L],[K],\{\pi_e | e \in E\})$ is a constraint satisfaction problem defined as follows. $G(U,V,E)$ is a bipartite graph whose vertices represent variables and edges represent the constraints. The goal is to assign to each vertex in U a label from the set $[L]$ and to each vertex in V a label from the set $[K]$ for positive integers $L > K$. The constraint on an edge $e = (u,v) \in E$ is described by a projection map $\pi_e : [L] \to [K]$. A labeling $\sigma : U \to [L], \sigma : V \to [K]$ satisfies the constraint on edge $e = (u,v)$ if and only if $\pi_e(\sigma(v)) = \sigma(u)$. Let $\mathrm{Opt}(\mathcal{L})$ denote the maximum fraction of constraints that can be satisfied by any labeling :

$$\mathrm{Opt}(\mathcal{L}) := \max_{\substack{\sigma: U \to [L] \\ \sigma: V \to [K]}} \frac{1}{|E|} \cdot |\{e \in E | L \text{ satisfies } e\}|.$$

Following theorem states the hardness of approximating Label-Cover.

Theorem 6.5 ([MR10])**.** *For some positive constant $c > 0$, it is NP-hard to distinguish a label cover problem vertices of n vertices and alphabet size $K, L \leq \sqrt{\log n}$.*

- *YES Case:* $\mathrm{Opt}(\mathcal{L}) = 1$.
- *NO Case:* $\mathrm{Opt}(\mathcal{L}) \leq \delta$ *for* $\delta = 1/(\log\log n)^c$.

Given a Label-Cover instance $\mathcal{L}(G(U,V,E),[L],[K],\{\pi_e | e \in E\})$, we construct a vertex pricing instance $\mathcal{I}$ with its vertices defined over $(U \times [q]^L \cup V \times [q]^R)$ for $q = (\log\log n)^{c/10}$. The construction of edges and budget is described in in Figure 6. It is easy to verify the reduction is in polynomial time.We identify each item by (w,r) for $w \in U, r \in [q]^L$ or $w \in V, r \in [q]^K$. Let us denote the corresponding pricing function to be $\{f_u : [q]^L \to \mathbb{R} | u \in U\} \cup \{f_v : [q]^K \to \mathbb{R} \mid v \in V\}$: we price items (w,r) by $f_w(r)$. We will prove that the reduction has the following properties(Theorem 6.6 and Theorem 6.7). Combining Theorem 6.6, 6.7, 6.5, one can immediately obtain Theorem 2.9. It remains to prove the following two properties of the reduction.

Theorem 6.6 (completeness of the reduction)**.** *If there is a labelling that satisfies every edge for $\mathcal{L}$, then* $\mathrm{Opt}^D(\mathcal{I}) \geq \Omega(\log q)$.

Figure 6: Hardness reduction from Label-Cover

1. randomly sample an edge $e = (u, v) \in E$.
2. sample $\mathbf{x}', \mathbf{y}', \boldsymbol{z}'', k$ according to the $\mathcal{T}_{\pi_e}$
3. add a customer interested in $(v, \mathbf{x}')$,$(u, \mathbf{y}')$, $(u, \boldsymbol{z}'')$ with budget k.

Proof. If there is a labelling $\sigma : U \to [L], V \to [K]$, then we can simply use the following pricing function: for $w \in U \cup V$, we use the price function $f_w(t) = t_{\sigma(w)} - q/3$.

By the completeness property of $\mathcal{T}_{\pi_e}$, we know that such a pricing strategy will have profit $\Omega(\log q)$. □

Theorem 6.7 (soundness of the reduction)**.** *If* $\mathrm{Opt}^C(I) \geq 13$*, then there is a labelling that satisfies more than* $1/q^7 \geq \delta$ *fraction of the edges.*

Proof. (soundness) Suppose $\mathrm{Opt}^C(\mathcal{I}) \geq 13$, notice that the maximum profit is at most $\sqrt{q}$ with each customer, then by an average argument, we know that for $1/q$ fraction of the edges (u, v) picked, we have that f_u, f_v has expected profit at least $13 - 1/\sqrt{q} > 12$. Let us call these (u, v) to be good.

Then by Theorem 6.2, there is way of decoding the f_u, f_v into coordinate i_u, i_v with the promise that $\mathbf{Pr}(\pi(i_u) = i_v) \geq 1/q^6$. Then if we just label each "good" edge (u, v) with i_u, i_v, such a labelling will satisfy at least $1/q \cdot 1/q^6 = 1/q^7$ fraction of the edges. □

7 Acknowledgement

We thank an anonymous reviewer for helpful comments.

The first author thanks Konstantin Makarychev for introducing him to the problem.

The second author thanks Yuan Zhou and Pinyan Lu for valuable discussion on pricing loss leaders for the highway problem.

References

[AGU10] Joyce van Loon Alexander Grigoriev and Marc Uetz. On the complexity of the highway pricing problem. *SOFSEM 2010: Theory and Practice of Computer Science*, pages 465–476, 2010. 1

[BB06] Maria-Florina Balcan and Avrim Blum. Approximation algorithms and online mechanisms for item pricing. In Joan Feigenbaum, John C.-I. Chuang, and David M. Pennock, editors, *ACM Conference on Electronic Commerce*, pages 29–35. ACM, 2006. 1, 1

[BBCH07] Maria-Florina Balcan, Avrim Blum, T.-H. Hubert Chan, and MohammadTaghi Hajiaghayi. A theory of loss-leaders: Making money by pricing below cost. In Xiaotie Deng and Fan Chung Graham, editors, *WINE*, volume 4858 of *Lecture Notes in Computer Science*, pages 293–299. Springer, 2007. 1, 1

[BK06] P. Briest and P. Krysta. Single-minded unlimited supply pricing on sparse instances. In *Proceedings of the seventeenth annual ACM-SIAM symposium on Discrete algorithm*, pages 1093–1102. ACM, 2006. 1

[Bor85] C. Borell. Geometric bounds on the ornstein-uhlenbeck velocity process. *Z. Wahrsch. Verw. Gebiete*, 70(1):1–13, 1985. 3.1

[DHFS06] E.D. Demaine, M.T. Hajiaghayi, U. Feige, and M.R. Salavatipour. Combination can be hard: Approximability of the unique coverage problem. In *Proceedings of the seventeenth annual ACM-SIAM symposium on Discrete algorithm*, pages 162–171. ACM, 2006. 1

[DMR09] Irit Dinur, Elchanan Mossel, and Oded Regev. Conditional hardness for approximate coloring. *SIAM J. Comput.*, 39(3):843–873, 2009. 3.1, 4.11

[ERRS09] K. Elbassioni, R. Raman, S. Ray, and R. Sitters. On profit-maximizing pricing for the highway and tollbooth problems. *Algorithmic Game Theory*, pages 275–286, 2009. 1, 1

[ESZ07] K. Elbassioni, R. Sitters, and Y. Zhang. A quasi-PTAS for profit-maximizing pricing on line graphs. *ESA 2007*, 2007. 1

[GHK+05] Venkatesan Guruswami, J.D. Hartline, A.R. Karlin, David Kempe, Claire Kenyon, and F. McSherry. On profit-maximizing envy-free pricing. In *Proceedings of the sixteenth annual ACM-SIAM symposium on Discrete algorithms*, pages 1164–1173. Society for Industrial and Applied Mathematics, 2005. 1

[GR11] F. Grandoni and T. Rothvo\ss. Prizing on paths: A PTAS for the highway problem. *SODA*, 2011. 1, 1

[GS10] I. Gamzu and D. Segev. A sublogarithmic approximation for highway and tollbooth pricing. *Automata, Languages and Programming*, pages 582–593, 2010. 1

[GVLSU06] A. Grigoriev, J. Van Loon, R. Sitters, and M. Uetz. How to sell a graph: Guidelines for graph retailers. In *Graph-Theoretic Concepts in Computer Science*, pages 125–136. Springer, 2006. 1

[Hås01] Johan Håstad. Some optimal inapproximability results. *Journal of the ACM*, 48(4):798–859, 2001. 3, 3.2

[HK05] J.D. Hartline and V. Koltun. Near-optimal pricing in near-linear time. *Algorithms and Data Structures*, pages 422–431, 2005. 1

[Kho02] Subhash Khot. On the power of unique 2-prover 1-round games. In *STOC*, pages 767–775, 2002. 1, 4.13

[KKMO07] Subhash Khot, Guy Kindler, Elchanan Mossel, and Ryan O'Donnell. Optimal inapproximability results for max-cut and other 2-variable csps? *SIAM J. Comput.*, 37(1):319–357, 2007. 3, 3.1, 4.2

[KKMS09] Rohit Khandekar, Tracy Kimbrel, Konstantin Makarychev, and Maxim Sviridenko. On hardness of pricing items for single-minded bidders. In Irit Dinur, Klaus Jansen, Joseph Naor, and José D. P. Rolim, editors, *APPROX-RANDOM*, volume 5687 of *Lecture Notes in Computer Science*, pages 202–216. Springer, 2009. 1, 2.3

[MOO05] Elchanan Mossel, Ryan O'Donnell, and Krzysztof Oleszkiewicz. Noise stability of functions with low influences: invariance and optimality. In *FOCS*, pages 21–30, 2005. 3.1, 3.2

[MR10] Dana Moshkovitz and Ran Raz. Two-query pcp with subconstant error. *J. ACM*, 57(5), 2010. 6.5

[OWZ11] Ryan O'Donnell, Yi Wu, and Yuan Zhou. Hardness of max-2lin and max-3lin over integers, reals, and large cyclic groups. In *Proc. 26th CCC*, 2011. 3.2

[Rag09] Prasad Raghavendra. *Approximating NP-hard problems: efficient algorithms and their limits.* PhD thesis, University of Washington, 2009. 4.2

[Wu11] Yi Wu. Pricing loss leaders can be hard. In *Proc.\ 2nd Innovation in Computer Science*, 2011. 1, 1, 2.3, 3.2