The sparse vector recovery problem

Sparse Vector Recovery Problem: Given a matrix $A \in \mathbb{R}^{n \times N}$, with $n < N$, and a vector $y \in \mathbb{R}^n$, find a $k$-sparse vector $x \in \mathbb{R}^N$ such that

$$y = Ax$$

There exists efficient algorithm recovering $x$ if $A$ exhibits the Restricted Isometry Property ($RIP$).
Restricted Isometry Property (RIP)

Definition
Given $k < n$ and $0 < \delta < 1$, a matrix $A \in \mathbb{R}^{n \times N}$ is $(k, \delta)$-RIP if, for any $k$-sparse vector $x \in \mathbb{R}^n$,

$$(1 - \delta) \|x\|_2 \leq \|Ax\|_2 \leq (1 + \delta) \|x\|_2$$

Ideally, a matrix that exhibits strong RIP has

- large $k$
- small $\delta$
Theorem (Candes, Romberg and Tao)

If $A$ is $(2k, \delta)$-RIP for some $\delta < \sqrt{2} - 1$, we can find an $k$-sparse $x$ efficiently by solving

$$\min_{a \in \mathbb{R}^n} \|a\|_1 \quad \text{subject to } Aa = y$$

Above result is very useful in compressing $N$ dimension $k$-sparse signals into a $n$ dimensional vectors by constructing a matrix $A$ that exhibits good RIP.
On Constructing RIP Matrices

For $N = \text{poly}(n)$,

- Best deterministic constructions can achieve $k \leq n^{0.50001}$ by Bourgain et al. (2011)
- Randomized construction can give $k = \Omega(n/\text{polylog}(n))$ by sampling a random Bernoulli matrix or a random Gaussian matrix.

Randomized construction is much better than deterministic construction!
RIP Certification

Definition

(RIP Certification Problem) Given a matrix $M$

- (Exact Version) Decide whether the matrix satisfies $(k, \delta)$-RIP.
- (Approximate Version) Decide whether a matrix satisfies $(k_1, \delta_1)$-RIP or does not satisfy $(k_2, \delta_2)$-RIP.
  - We only need to have $\delta \geq \sqrt{2} - 1$ for most applications

Motivation:

- Randomized constructions are much better than deterministic construction.
- Certifying RIP of a randomly sampled matrix becomes important.
- "An alternate approach, and one of interest in its own right, is to work on improving the time it takes to verify that a given matrix (possibly one of a special form) obeys the RIP." – Terry Tao
Previous Work (1): Hardness of Exact Certification

Given $\delta$, $k$, and a matrix $M$ as input, decide if $M$ satisfy $(k, \delta)$-RIP.

- Bandeira et al. (2013) proved that it is NP-hard
- Tillmann and Pfetsch (2014) proved that it is co-NP-hard
- Both results works when $\delta = 1 - o_n(1)$. 
Previous Work (2): Inapproximability of RIP Certification

Koiran and Zouzias (2011, 2012) show inapproximability results by various hardness assumptions of the hidden clique problem and the densest $k$-subgraph problem

- All the results state that it is hidden clique/DKS-hard to distinguish $(k, \delta_1)$-RIP from $(k, \delta_2)$-RIP for some $\delta_1 < \delta_2$.
- In almost all of the cases, $\delta_1, \delta_2$ are $\in o_n(1)$
- Exception:
  - No polynomial time algorithm can distinguish matrices that satisfy the $(k, \frac{\kappa}{2})$-RIP from matrices that do not satisfy the $(k, \kappa)$-RIP

where $\kappa \left( \leq \frac{\sqrt{5}}{3} \right)$ is an unknown constant depending on the correctness of certain hardness assumption of densest $k$-subgraph.
Our Results

Theorem

For any $0 \leq \delta \leq 1$ and arbitrary large constant $C$, there exists $k$ such that, given a matrix $M$ it is Small-Set-Expansion-hard to distinguish between:

- (Highly RIP) $M$ is $(k, \delta)$-RIP.
- (Far away from RIP) $M$ is not $(\frac{k}{C}, 1 - \delta)$-RIP.

This is the first hardness result that applies for any $0 < \delta < 1$ (including $\sqrt{2} - 1$).
Corollaries

As corollaries, we have that

**Corollary**

*Given a matrix \( M \) and \( k \), it is small-set-expansion-hard to distinguish whether the matrix is \((k, \delta)\)-RIP or not \((k, 1 - \delta)\)-RIP.*

**Corollary**

*Given a fixed \( \delta \) and matrix \( M \), it is small-set-expansion-hard to get a constant approximation for the smallest \( k \) such that \( M \) exhibits \((k, \delta)\)-RIP.*
Small-Set-Expansion Conjecture

Definition
Given a graph $d$-regular graph $G(V, E)$, we define

$$
\phi_G(S) = \frac{|E(S, V - S)|}{d \cdot \min(|S|, |V - S|)}
$$

$$
\phi_G(\delta) = \min_{S, |S| \leq \delta |V|} \phi_G(S)
$$

Conjecture (Raghavendra and Steurer 2010)

For every $\epsilon > 0$, $\exists 0 \leq \delta \leq \frac{1}{2}$, such that it is NP-hard to distinguish between:

- $\phi_G(\delta) \leq \epsilon$
- $\phi_G(\delta) \geq 1 - \epsilon$
Proof Overview (1)

- If $A$ is the adjacency matrix of a $d$ regular graph, we consider the matrix $M$ such that $M^T M = I - \frac{1}{d} A = L$ for RIP certification.

- (Completeness of the Reduction) If there is a small set $S$ with expansion less than $\epsilon$, then $\phi_G(S) = \frac{\|Mx_S\|_2^2}{\|x_S\|_2^2} \leq \epsilon$, where $x_S \in \{0, 1\}^n$ is the indicator vector on $S$. This gives us $\|Mx_S\|_2 \leq \sqrt{\epsilon} \|x_S\|_2$, which suggests that $M$ is far away from RIP.
Proof Overview (2)

- (Soundness of the Reduction) show that if \( \exists \) a k-sparse \( x \in \mathbb{R}^n \) such that
  \[
  \frac{x^T M^T M x}{\|x\|_2^2} \leq (1 - \Omega(1))
  \]
  then we can find a small set \( S \) such that \( \phi(S) \) is also bounded away from 1. This uses the Sparse Cheeger’s Inequality
Sparse Cheeger’s Inequality

We prove the following Cheeger’s Inequality on sparse vectors.

Theorem
(Sparse Cheeger’s Inequality) Let $A$ be the adjacency matrix of a $d$-regular graph $G$, and $L = I - \frac{1}{d}A$ be its normalized Laplacian matrix. For any $\delta \leq 1/2$, we have that

$$\lambda_\delta \leq \phi_G(\delta) \leq \sqrt{(2 - \lambda_\delta)\lambda_\delta}$$

where $\lambda_\delta = \min_{\|x\|_0 = \delta} \frac{x^T L x}{x^T x}$
Comparison with Cheeger’s Inequality

Theorem

Let $A$ be the adjacency matrix of a graph $G$, and $L = I - \frac{1}{d} A$ be its normalized Laplacian matrix. We have that

$$\frac{\lambda_2}{2} \leq \phi(G) \leq \sqrt{2\lambda_2}$$

where

$$\lambda_2 = \min_{x \in \mathbb{R}^n} \frac{\|x^T L x\|_2}{\|x\|_2^2}$$

is the second smallest eigenvalue of $L$.

It must be noted that the relation between $\lambda_2$ and $\phi_\delta(G)$ is tighter in this case.
Proof of Sparse Cheeger’s Inequality

- Lower bound of $\phi_\delta(G)$ is easy to prove

\[
\phi_\delta(G) = \min_{S \subseteq V, S \leq \delta n} \phi(S) = \min_{x \in \{0,1\}^n, \|x\|_0 \leq \delta} \frac{x^T L x}{x^T x} \geq \lambda_\delta
\]

- Upper bound is called hard direction. Here, we assume we are given the vector $x$ that gives us $\frac{x^T L x}{x^T x} = \lambda_\delta$.

- The same randomized rounding as the proof of Cheeger’s Inequality, we can create a cut set in the graph, and that the expansion of the cut is restricted.
Concluding Remarks

Summary:

▶ We have proved that RIP certification is hard to approximate in a strong sense assuming the Small-Set-Expansion Hypothesis
▶ We developed a variant of Cheeger’s inequality for sparse vectors

Future directions:

▶ It will be interesting to see if RIP certification is hard even when the matrix satisfies certain natural properties such as coherence
▶ It will also be interesting to prove NP/UG-hardness, because correctness of the Small-Set-Expansion Hypothesis is uncertain
▶ Subexponential algorithm for RIP certification