Various Notions of Compactness

Yao-Liang Yu
Department of Computing Science
University of Alberta
Edmonton, AB T6G 2E8, Canada
yaoliang@cs.ualberta.ca

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Abstract

We document various notions of compactness, with some of their useful properties. Our main reference is [Engelking, 1989]. For counterexamples, refer to [Steen and Seebach Jr., 1995] while for background, see the excellent textbooks [Willard, 2004] or [Munkres, 2000]. Some of the proofs are taken freely from the internet. The writer expresses his gratitude to all sources.

1 Definitions

Our domain in this note will always be some topological space $(\mathcal{X}, \tau)$. Recall the usual definition of compactness:

Definition 1 (Compact) *$K \subseteq \mathcal{X}$ is compact if every open cover of it has a finite subcover*[1].

As usual we may have "countable" versions[2]:

Definition 2 (Countably Compact) *$K \subseteq \mathcal{X}$ is countably compact*[3] *if every countable open cover of it has a finite subcover.*

Definition 3 (Lindelöf) *$K \subseteq \mathcal{X}$ is Lindelöf if every open cover of it has a countable subcover.*

Of course one immediately sees that K is compact iff K is countably compact and Lindelöf. The contrapositive of the definition also tells us that K is (countably) compact iff any (countable) collection of closed sets with finite intersection property has nonempty intersection.

Definition 4 (Limit point Compact) *$K \subseteq \mathcal{X}$ is limit point compact if every infinite subset of it has a limit point.*

Recall that $x \in \mathcal{X}$ is a limit point of **the set** K if every neighbourhood of x intersects $K - \{x\}$. For a **net** $\{x_\lambda\}_{\lambda \in \Lambda}$ we define x as its cluster point[4] if for every neighbourhood $\mathcal{O}$ of x, the index set $\{\lambda : x_\lambda \in \mathcal{O}\}$ is cofinal in Λ. It can be shown that x is a cluster point of the net $\{x_\lambda\}_{\lambda \in \Lambda}$ iff some subnet $\{x_{\lambda_\gamma}\}_{\gamma \in \Gamma}$ converges to x. Similarly x is a limit point of the set K iff there exists some net $x \notin \{x_\lambda\}_{\lambda \in \Lambda} \subseteq K$ that converges to x. We will also need the notion of ω-limit point. A point $x \in \mathcal{X}$ is an ω-limit point of the set K if every neighbourhood of x intersects K at infinitely many points[5]. Apparently ω-limit point is *bona fide* a limit point while the converse is true only in T_1 space. Note that there is no need to define ω-cluster point (it coincides with cluster point).

[1] Whenever we say "a" (or "an") we mean at least one; if it is exactly one, we will say so explicitly.
[2] By countable we mean finite or countably infinite.
[3] While it is tempting to call countably compact as σ-compact, the latter has been used in the literature with a different meaning: countable union of compact sets.
[4] We have deliberately used limit point for sets and cluster point for nets, to distinguish the two notions a bit.
[5] One can further subdivide "infinity" in this definition to derive more refined notions.

Theorem 1 *$K \subseteq \mathcal{X}$ is countably compact iff every infinite subset of it has an ω-limit point iff every sequence in it has a cluster point.*

Proof: Suppose every sequence has a cluster point and let A be an infinite set, then we can find a sequence (with distinct elemets) in A whose cluster point clearly is an ω-limit point of A. On the other hand, suppose every infinite subset has an ω-limit point. Consider any sequence $\{x_n\}$, let its distinct elements be $\{x_{n_m}\}$, which we assume is an infinite set (otherwise we are done). Apparently any ω-limit point of $\{x_{n_m}\}$ is a cluster point of $\{x_n\}$.

Suppose $\mathcal{X}$ is countably compact, let $\{x_n\}$ be a sequence in $\mathcal{X}$. Denote $B_n := \text{cl}(\{x_i\}_{i=n}^{\infty})$, hence $\exists x \in \cap_{n=1}^{\infty} B_n$. Any neighbourhood $\mathcal{O} \ni x$ must intersect infinitely many elements of the sequence $\{x_n\}$ (again we omit the uninteresting case where the sequence only has finitely many distinct elements) since otherwise it would imply that $\cap_{n=1}^{\infty} B_n = \emptyset$. We have used the fact that for any open set $\mathcal{O}$, $\mathcal{O} \cap A = \emptyset \iff \mathcal{O} \cap \text{cl}(A) = \emptyset$. On the other hand, suppose every sequence has a cluster point, and let B_n be a sequence of closed sets that satisfy the finite intersection property. We can choose $x_n \in B_1 \cap \ldots \cap B_n$. Any cluster point of $\{x_n\}$, say x, must belong to $\cap_{n=1}^{\infty} B_n$. ■

It is apparent from the above theorem that countably compact implies limit point compact. The converse is true only in T_1 space. On the other hand, this theorem also tell us that countably compact is implied by the next notion of compactness.

Definition 5 (Sequentially Compact) *$K \subseteq \mathcal{X}$ is sequentially compact if every sequence of it has a convergent subsequence.*

As the name suggests, sequential compactness is a sequence version of compactness, since the latter can be equivalently defined as every net has a convergent subnet.

Definition 6 (Pseudo-Compact) *$K \subseteq \mathcal{X}$ is pseudo-compact if every real-valued continuous function defined on it is bounded.*

Pseudo-compactness is an important notion due to the next theorem.

Theorem 2 *Pseudo-compact sets are exactly those on which all real-valued functions attain their supremum and infimum.*

Proof: Let $f : \mathcal{X} \mapsto \mathbb{R}$ be bounded, and assume its supremum $s := \sup f$ is not attained. Define the continuous function $g : \mathbb{R} - \{s\} \mapsto \mathbb{R}$ as $g(x) = \frac{1}{x-s}$. Note that since s is not attained, $f : \mathcal{X} \mapsto \mathbb{R} - \{s\}$ remains continuous, hence $g \circ f : \mathcal{X} \mapsto \mathbb{R}$ is again continuous, but it is not bounded. ■

2 Properties

The following properties are interesting to us:

(a) closed subspace of * is *;

(b) * as a subspace of some T_2 space is closed;

(c) product of * is *;

(d) continuous image of * is *;

(e) lower(upper)-semicontinuous functions on * attains their lower(upper) bound, and the minimizers constitute a *.

As we will see, the last property (e), holds for countably compact sets, which, in fact, amazed the writer and caused him to write this note.

Our first theorem can be found in any good general topology book (for instance, [Willard, 2004]):

Theorem 3 *Compact satisfies (a)-(e) (where in (c) the products can be arbitrary).*

Theorem 4 *Countably compact satisfies (a), (d), (e) but does not satisfy (b), or (c) even for finite products.*

Proof: We only prove the validity of (e). Let $f : \mathcal{X} \mapsto \mathbb{R}$ be l.s.c and $\mathcal{X}$ be countably compact. We first show that $c := \inf_{x\in\mathcal{X}} f(x)$ is finite. Suppose not, then the countable collection of closed sets $\{x \in \mathcal{X} : f(x) \leq -n\}$ satisfies the finite intersection property hence has nonempty intersection, *i.e.* $\exists x \in \mathcal{X}, \text{s.t. } f(x) = -\infty$, contradiction. Next consider the countable collection of closed sets $\{x \in \mathcal{X} : f(x) \leq c + 1/n\}$ which again satisfies the finite intersection property hence has nonempty intersection, call it $\mathcal{M} := \{x \in \mathcal{X} : f(x) = c\}$. Apparently, M is closed hence countably compact. ■

Note that the product of a countably compact space and a countably compact k-space is countably compact [Engelking, 1989, Theorem 3.10.13]. Also the product of a countably compact space and a sequentially compact space is countably compact [Engelking, 1989, Theorem 3.10.36].

Theorem 5 *Sequentially compact satisfies (a), (d), (e) and (c) for countably infinite products (but fails for $\aleph_1$ products), but does not satisfy (b).*

Proof: For (c), see [Engelking, 1989, Theorem 3.10.35].

To show (b) is invalid, consider the ordered space $[0, \aleph_1)$, which is easily seen to be Hausdorff and first countable (local basis for $b < \aleph_1$: $(a, b], \forall a < b$). For any infinite subset A of $[0, \aleph_1)$, extract a countable subset, say $B \subseteq A$, then define $s = \sup B$ which we know exists. The interval $[0, s]$ is compact hence limit point compact, which proves that $B \subseteq [0, s]$ has a limit point in $[0, s]$ hence $A \supseteq B$ also has a limit point in $[0, \aleph_1)$. Combining everything we know $[0, \aleph_1)$ is sequentially compact (but not compact). $[0, \aleph_1)$ as a subspace of the compact Hausdorff space $[0, \aleph_1]$ is clearly not closed (on the opposite, open). ■

Theorem 6 *Limit point compact satisfies (a) but does not satisfy (b), (c) (even for finite products), (d) or (e).*

Proof: (a) is easy hence omitted.

To show why (d) is invalid, Take $\mathbb{R}$ with its usual topology and $\{0, 1\}$ with the trivial topology. Then $\mathbb{R} \times \{0, 1\}$ is easily seen to be limit point compact but the projection (which is continuous) to $\mathbb{R}$ is not limit point compact ($\mathbb{R}$ with its usual topology is not limit point compact, take, say $\mathbb{N}$ as the infinite set).

For (c), see [Steen and Seebach Jr., 1995, Example 112].

For (e), we show that limit point compactness does not even imply pseudo-compactness. Take the natural numbers $\mathbb{N}$ equipped with the discrete topology, and consider the product space $\{0, 1\} \times \mathbb{N}$ where the space $\{0, 1\}$ is equipped with the trivial topology. It is easy to see every non-empty subset of the product space has a limit point (for $(0, n)$ is a limit point of $(1, n), \forall n \in \mathbb{N}$). The projection $\pi : \{0, 1\} \times \mathbb{N} \mapsto \mathbb{N}$ is clearly continuous but $\mathbb{N}$ with discrete topology is not pseudo-compact, hence $\{0, 1\} \times \mathbb{N}$ cannot be pseudo-compact either (see the next theorem). ■

Theorem 7 *Pseudo-compact satisfies (d) but does not satisfy (a), (b), (c) (even for finite products) or (e).*

Proof: We only show why (e) is false. Consider the natural numbers $\mathbb{N}$ equipped with the particular point topology[6], where the particular point is, say, 0. Any function $f : (\mathbb{N}, \tau_p) \mapsto \mathbb{R}$ is continuous iff f is constant while f is l.s.c. iff $f(x) \leq f(0), \forall x$. Hence $(\mathbb{N}, \tau_p)$ is pseudo-compact but apparently does not satisfy (e). ■

Note however that the product of a pseudo-compact space and a pseudo-compact k-space is pseudo-compact [Engelking, 1989, Theorem 3.10.26]. Also the product of a pseudo-compact space and a sequentially compact space is pseudo-compact [Engelking, 1989, Theorem 3.10.37].

Theorem 8 *Lindelöf satisfies (a) and (d) but does not satisfy (b), (c) (even for finite products), or (e).*

Proof: For (b), it is enough to observe that even second countable subspace of a Hausdorff space need not be closed. (Take (0,1) as a subspace of $\mathbb{R}$.)

For (c), see [Munkres, 2000, Examples 4 and 5, § 30].

For (e), use the same counterexample as that for limit point compact. ■

Finally, we summarize this section in Table 1.

[6]The particular point topology on a set $\mathcal{X}$ is defined with respect to a particular point $p \in \mathcal{X}$ such that $A \subseteq \mathcal{X}$ is open iff $A \ni p$ or $A = \emptyset$.

Notions \ Properties	(a)	(b)	(c)	(d)	(e)
Compact	✓	✓	✓	✓	✓
Sequentially Compact	✓	×	✓	✓	✓
Countably Compact	✓	×	×	✓	✓
Limit Point Compact	✓	×	×	×	×
Pseudo Compact	×	×	×	✓	×
Lindelöf	✓	×	×	✓	×

Table 1: Properties owned by different notions of compactness.

3 Relations

We discuss in this section the relations between various notions of compactness.

Theorem 9 *Limit point compactness implies sequential compactness in first countable spaces.*

Proof: Consider a sequence $\{x_n\}$ (w.l.o.g., with distinct elements), which by assumption has a limit point x. We now construct a subsequence that converges to x. Since the space is first countable, let $\{\mathcal{N}_n\}$ be the collection of local countable bases at x. Choose (arbitrarily) $x_{n_1} \in \mathcal{N}_1$, and choose $x_{n_m} \in \cap_{i=1}^m \mathcal{N}_i$. We have $x_{n_m} \to x$. ■

The previous theorem can be strengthened a bit:

Theorem 10 *Every sequential space that is limit point compact is sequentially compact.*

Proof: Let $\{x_n\}$ be a sequence (w.l.o.g. with distinct elements) in the limit point compact sequential space $\mathcal{X}$. Let x be a limit point of the (infinite) set $A := \{x_n\}$, therefore $x \in \mathrm{cl}(A - \{x\})$. It follows that $A - \{x\}$ is not closed hence not sequentially closed either, so $\exists \{y_n\} \subset A - \{x\}$ that converges to some point $y \notin A - \{x\}$. Rearranging the sequence $\{y_n\}$ we get a convergent subsequence of $\{x_n\}$. ■

Theorem 11 *Every pseudo-compact normal space is countably compact.*

Proof: Suppose not, then $\exists \{x_n\} := S$ which does not have a limit point, hence S is closed and discrete (for if $\{x\} \subseteq S$ is not open then x is a limit point of S). Therefore the function $f : S \mapsto \mathbb{N}$ defined as $f(x_n) = n$ is continuous hence by the Tietze extension theorem it can be extended to the whole space (which is normal by assumption). But the continuous extension is not bounded, contradicting pseudo-compactness. ■

Theorem 12 *Every sequentially compact metric space is totally bounded hence second countable.*

Proof: Let $\mathcal{X}$ be a sequentially compact metric space, which we suppose is not totally bounded, then $\exists\ \epsilon > 0$ s.t. $\mathcal{X}$ cannot be covered by finite many balls with radius smaller than ϵ. Construct a sequence as follows: choose $x_1 \in \mathcal{X}$ arbirrarily, and choose $x_n \in \mathcal{X} - \cup_{i=1}^{n-1} B(x_i, \epsilon)$, where $B(x_i, \epsilon)$ is the ball centred at x_i with radius ϵ. Since $\mathcal{X}$ by assumption is sequentially compact, $\exists$ subsequence $\{x_{n_i}\} \mapsto x \in \mathcal{X}$, which is impossible since the tails of $\{x_i\}$ by construction are at least ϵ apart.

It is obvious that every totally bounded metric space is second countable: take the basis as $\cup_{n\in\mathbb{N}} \cup_{i\in I_n} B(x_i, \frac{1}{n})$, where for a fixed $n \in \mathbb{N}$, the index set I_n can be chosen to be finite since the space is by assumption totally bounded. ■

Again, we summarize our findings in Figure 1. Note that locally compact separable space need not be σ-compact: take $\mathcal{X} := \mathcal{Y} - y$, where y is any point in $\mathcal{Y} := \{0,1\}^{\mathbb{R}}$. As an open subspace of the compact separable space $\mathcal{Y}$, $\mathcal{X}$ is also separable and locally compact. Suppose $\mathcal{X}$ is σ-compact, then y would be a G_δ: $\mathcal{X} = \mathcal{Y} - y = \cup_n K_n$, where $\{K_n\}$ is a compact covering of $\mathcal{X}$ (also note that $\mathcal{Y}$ is Hausdorff so that K_n is closed). Therefore $y = \cap_n \mathcal{O}_n$, where $\mathcal{O}_n$ are basic open sets in $\mathcal{Y}$. For a fixed $n \in \mathbb{N}$, $\mathcal{O}_n = \prod_{i\in\mathbb{R}} \mathcal{O}_{n,i}$, where only finitely many $\mathcal{O}_{n,i}$ has cardinality 1 (*i.e.* $\mathcal{O}_{n,i} = \{0\}$ or $\mathcal{O}_{n,i} = \{1\}$), index these by I_n (hence

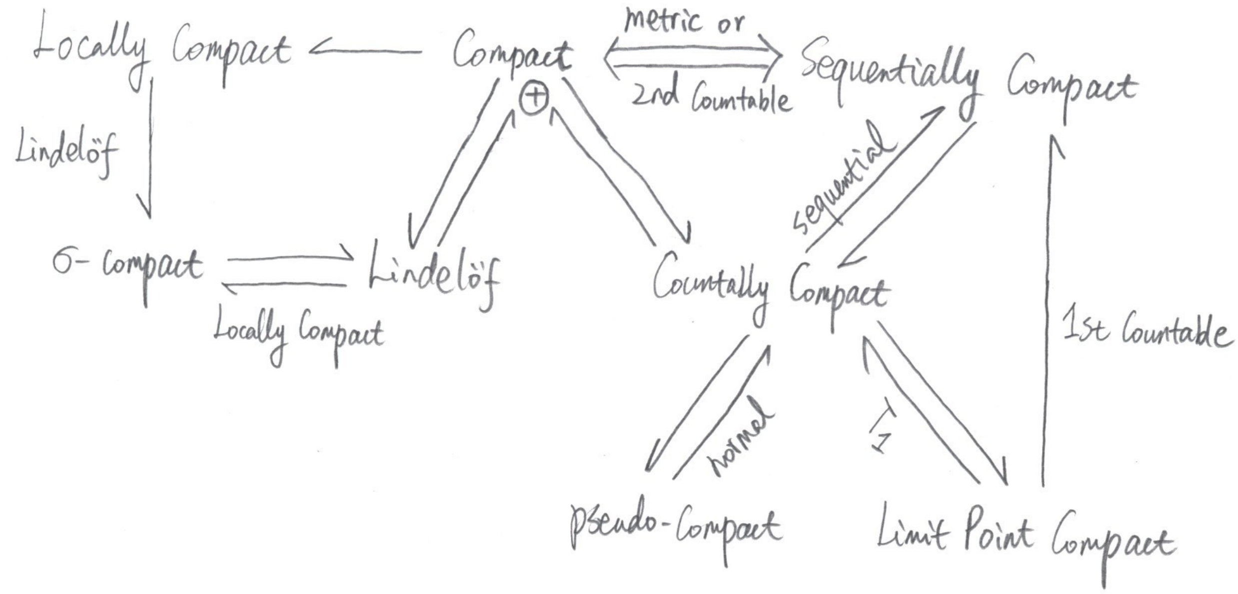

Figure 1: Relations between various notions of compactness

$|I_n| < \infty$). Let $j \in \mathbb{R} - \cup_{n\in\mathbb{N}} I_n$, and consider the point x satisfying $x_i = y_i, \forall i \in \mathbb{R} - \{j\}$ and $x_j \neq y_j$. By construction, $x \in \cap_n \mathcal{O}_n$, contradiction.

Lastly we mention that we apparently have not covered all notions of compactness which have been developed in the literature, such as paracompactness, realcompactness, metacompactness, etc. One should consult [Engelking, 1989] or [Steen and Seebach Jr., 1995] in case s/he hasn't had enough fun.

Appendix: Sequential Space and k-space

We recall the definitions of sequential space and k-space. The maim aim of this appendix is solely for the sake of the writer, as he is not yet familiar with sequential spaces and k-spaces at the time of writing.

Recall that $\mathcal{O} \subseteq \mathcal{X}$ is called open if every net in $\mathcal{X}$ converging to some point in $\mathcal{O}$ is eventually in $\mathcal{O}$; while $\mathcal{C} \subseteq \mathcal{X}$ is called closed if every converging net in $\mathcal{C}$ converges to some point in $\mathcal{C}$. It is easy to show that the complement of a closed set $\mathcal{C}$ is open: suppose not, then there exists some net $\{x_\lambda\}_{\lambda\in\Lambda} \subseteq \mathcal{X}$ converging to $x \in \mathcal{X} - \mathcal{C}$ but not eventually in $\mathcal{X} - \mathcal{C}$, hence we can extract a subnet $\{x_{\lambda_\gamma}\}_{\gamma\in\Gamma} \subseteq \mathcal{C}$ that converges to x too. Since $\mathcal{C}$ is closed we arrive at the contradiction that $\mathcal{C} \cap \mathcal{X} - \mathcal{C} = \{x\}$. Similarly the complement of an open set is closed.

By changing nets to sequences we get the definition for sequentially open (closed) sets. Again, the complement of a sequentially open (closed) set $\mathcal{O}$ is closed (open): suppose not, then there exists some sequence $\{x_n\} \subseteq \mathcal{X} - \mathcal{O}$ converging to $x \in \mathcal{O}$; since $\mathcal{O}$ is open we arrive at the contradiction that the tail of the sequence $\{x_n\}$ is both in $\mathcal{O}$ and $\mathcal{X} - \mathcal{O}$. Trivially, open (closed) sets are sequentially open (closed) while the converse in general is not true hence motivates the next definition:

Definition 7 (Sequential space) *Topological space $(\mathcal{X}, \tau)$ is a sequential space iff every sequentially open (closed) set is open (closed).*

For a given set A, we define its sequential closure as

$$\mathrm{scl}(A) := \{x \in \mathcal{X} : x \leftarrow \{a_n\} \subseteq A\}.$$

Obviously $\mathrm{scl}(\emptyset) = \emptyset$, $A \subseteq \mathrm{scl}(A) \subseteq \mathrm{cl}\, A$. Moreover $\mathrm{scl}(A \cup B) = \mathrm{scl}(A) \cup \mathrm{scl}(B)$, hence in general (even in sequential spaces) $\mathrm{scl}(\mathrm{scl}(A)) \neq \mathrm{scl}(A)$ as otherwise it would imply $\mathrm{scl}(\cdot) = \mathrm{cl}(\cdot)$. Topological spaces that do satisfy $\mathrm{scl}(\cdot) = \mathrm{cl}(\cdot)$ are called Fréchet-Urysohn. Sequential closure is inherited by subspaces in the sense that if $\mathcal{S}$ is a subspace of $\mathcal{X}$, then $\forall A \subseteq \mathcal{S}, \mathrm{scl}_{\mathcal{S}}(A) = \mathcal{S} \cap \mathrm{scl}_{\mathcal{X}}(A)$, hence it follows that Fréchet-Urysohn spaces are exactly those whose subspaces are all sequential. Easily we know that every first-countable space is Fréchet-Urysohn hence sequential: fix $x \in \mathrm{cl}(A)$, take $x_i \in A \cap \mathcal{O}_1 \cap \ldots \cap \mathcal{O}_i$, where $\{\mathcal{O}_i\}$ denotes the local basis at x, hence $x_i \to x$ and $x \in \mathrm{scl}(A)$.

Theorem 13 *Let $\mathcal{X}$ be sequential and $\mathcal{Y}$ be topological, then $f : \mathcal{X} \mapsto \mathcal{Y}$ is continuous iff $x_i \to x \implies f(x_i) \to f(x)$.*

Proof: Take any closed subset $B \subseteq \mathcal{Y}$, let $x \in \mathrm{scl}(f^{-1}(B))$, then $\exists x_i \in f^{-1}(B)$ s.t. $x_i \to x$. By assumption $B \ni f(x_i) \to f(x)$ hence $x \in f^{-1}(B)$, implying that $f^{-1}(B) = \mathrm{scl}(f^{-1}(B))$ is closed. ■

It follows easily from the above proposition that any map from one sequential space into another sequential space is continuous iff its inverse image of any sequentially closed (open) set is sequentially closed (open).

Proposition 1 *If every sequence in a topological space $\mathcal{X}$ converges to at most one point, then $\mathcal{X}$ is T_1. Moreover, if $\mathcal{X}$ is first-countable, then $\mathcal{X}$ is actually T_2.*

Proof: Let $y \in \mathrm{cl}(\{x\})$, then the (constant) sequence $\{x\}_i$ converges to both y and x, hence $x = y$ follows that $\{x\}$ is closed. ■

Proposition 2 *Every quotient space of a sequential spaces is also sequential.*

Proof: Let $f : \mathcal{X} \mapsto \mathcal{Y}$ be a quotient map and $\mathcal{X}$ be sequential. Let $A \subseteq \mathcal{Y}$ be sequentially closed. To show A is closed, it is enough to show $f^{-1}(A)$ is closed since f is a quotient map. Let $\{x_n\} \subseteq f^{-1}(A)$ converge to x, then $f(x_n) \to f(x)$ since f is continuou, hence $f(x) \in A$, *i.e.* $x \in f^{-1}(A)$. ■

We mention a few other properties of sequential space without proof. Open (closed) subspace of a sequential space is sequential, while continuous image or product of sequential spaces is *not* sequential. It can be shown that $\mathcal{X}$ is sequential iff it is a quotient space of some metric space iff it is a quotient space of some first countable space. For examples and counterexmaples, refer to [Engelking, 1989].

Definition 8 (k-space) *$\mathcal{X}$ is called a k-space if any $A \subseteq \mathcal{X}$ is open (closed) iff $A \cap \mathcal{K}$ is open (closed) in $\mathcal{K}$, for all compacta $\mathcal{K} \subseteq \mathcal{X}$.*

Note that sequential spaces are k-spaces: suppose $A \subseteq \mathcal{X}$ is not closed (hence not sequentially closed since $\mathcal{X}$ is sequential), then $\exists \{x_n\} \subseteq A$ such that $x_n \to x \notin A$. The set $\mathcal{K} := \{x_n\} \cup \{x\}$ is compact but $\mathcal{K} \cap A = \{x_n\}$ is not closed. Similarly, subspaces of a k-space need *not* be a k-space, but quotient spaces of a k-space are k-spaces, in fact $\mathcal{X}$ is a k-space iff $\mathcal{X}$ is a quotient space of some locally compact space[7].

Continuous image of a second-countable, first-countable, Fréchet-Urysohn, sequential, k-space need *not* be a second-countable, first-countable, Fréchet-Urysohn, sequential, k-space, respectively. There exists counterexample that shows the product of a first countable space and a Fréchet-Urysohn space need not be a k-space, whence (finite) products of k-spaces, sequential spaces, Fréchet-Urysohn spaces are not k-spaces, sequential spaces, Fréchet-Urysohn spaces, respectively. Countable products of second-countable, first-countable spaces are second-countable, first-countable, respectively.

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[7]Note that continuous image of a locally compact space need *not* be locally compact: take a discrete space (which apparently is locally compact), but any topological space with the same domain is its continuous image (under the identity map).