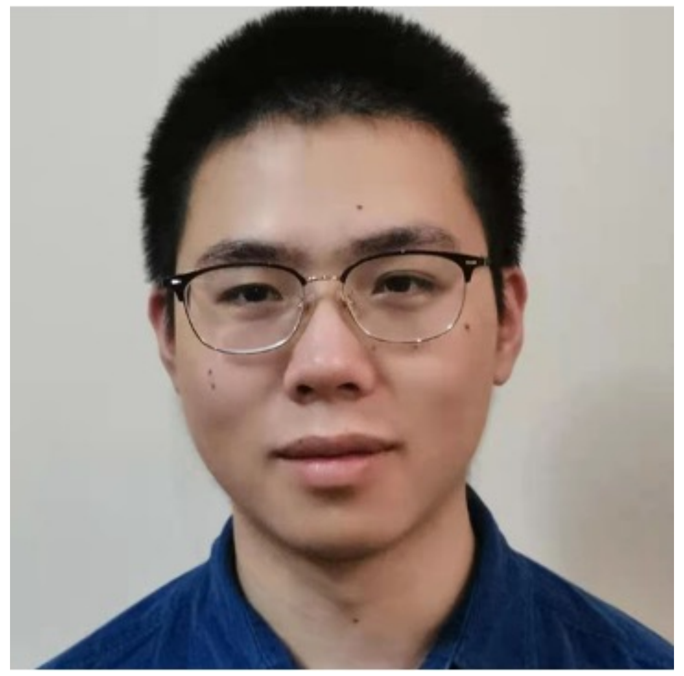
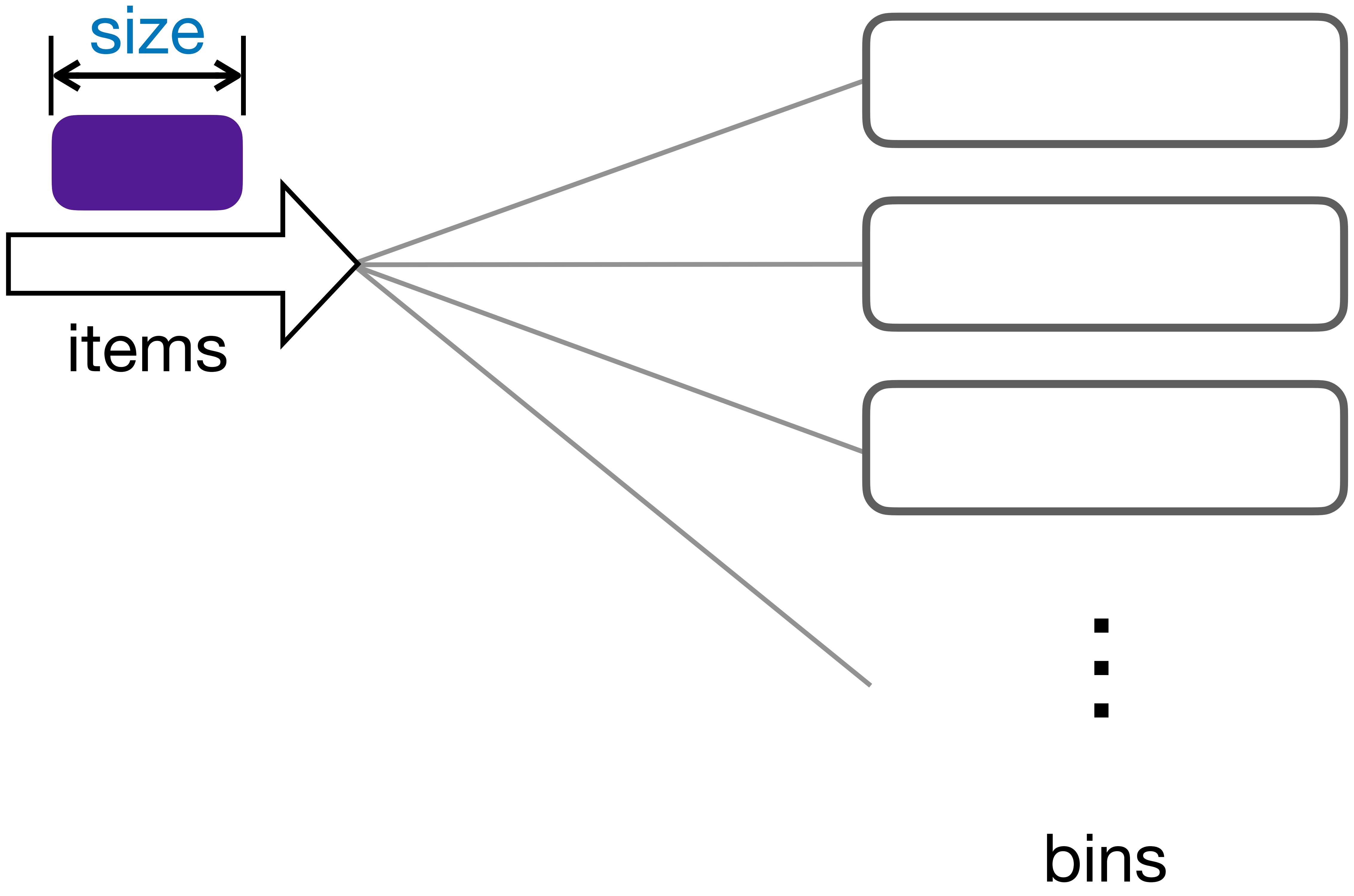
Near-Optimal Stochastic Bin-Packing in Large Service Systems with Time-Varying Item Sizes



Joint work with Yige Hong (CMU) and Qiaomin Xie (UW–Madison)

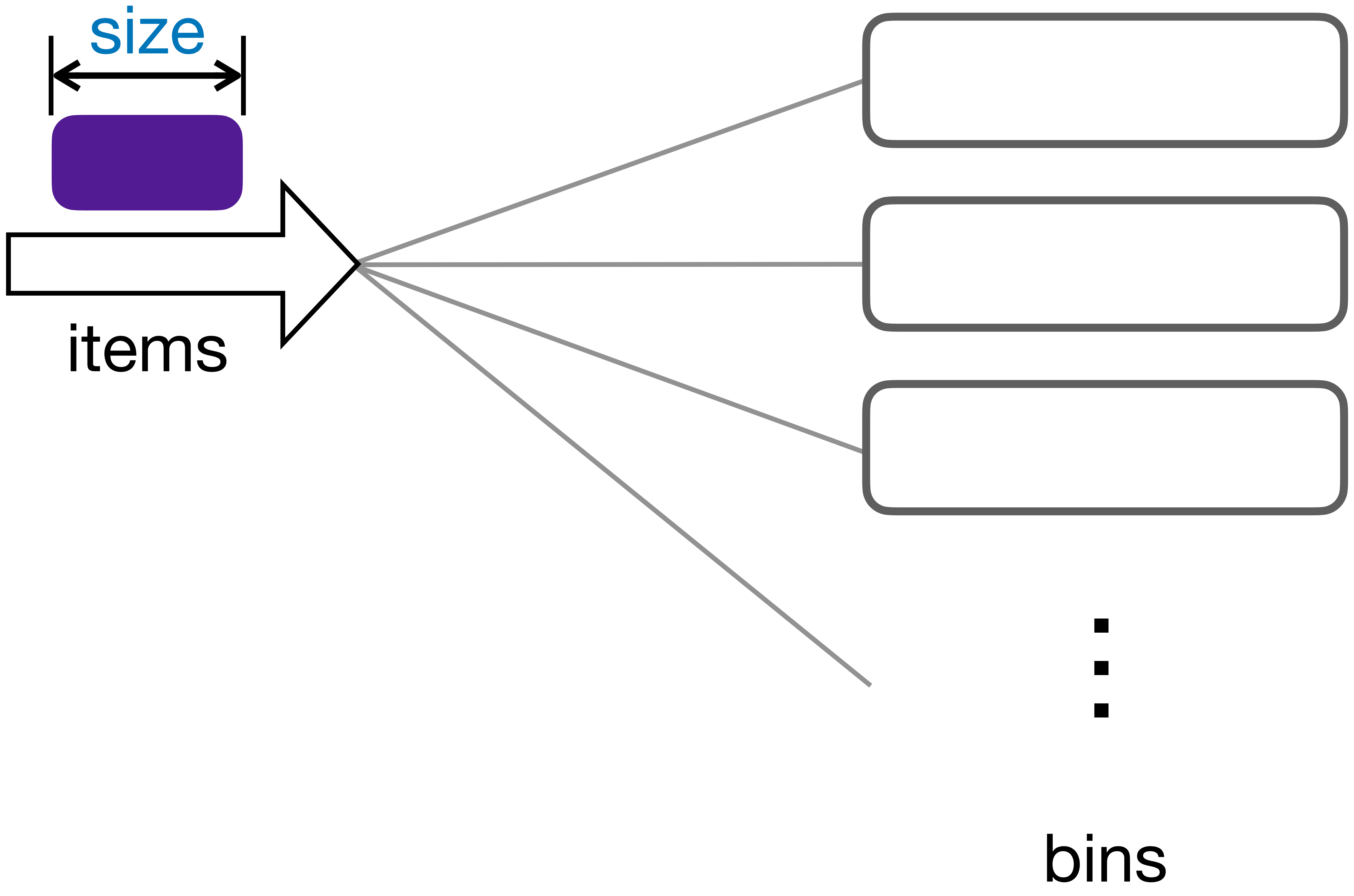
Weina Wang
Carnegie Mellon University

The problem



The problem

- Each arriving item needs to be assigned to a bin



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- Each arriving item needs to be assigned to a bin
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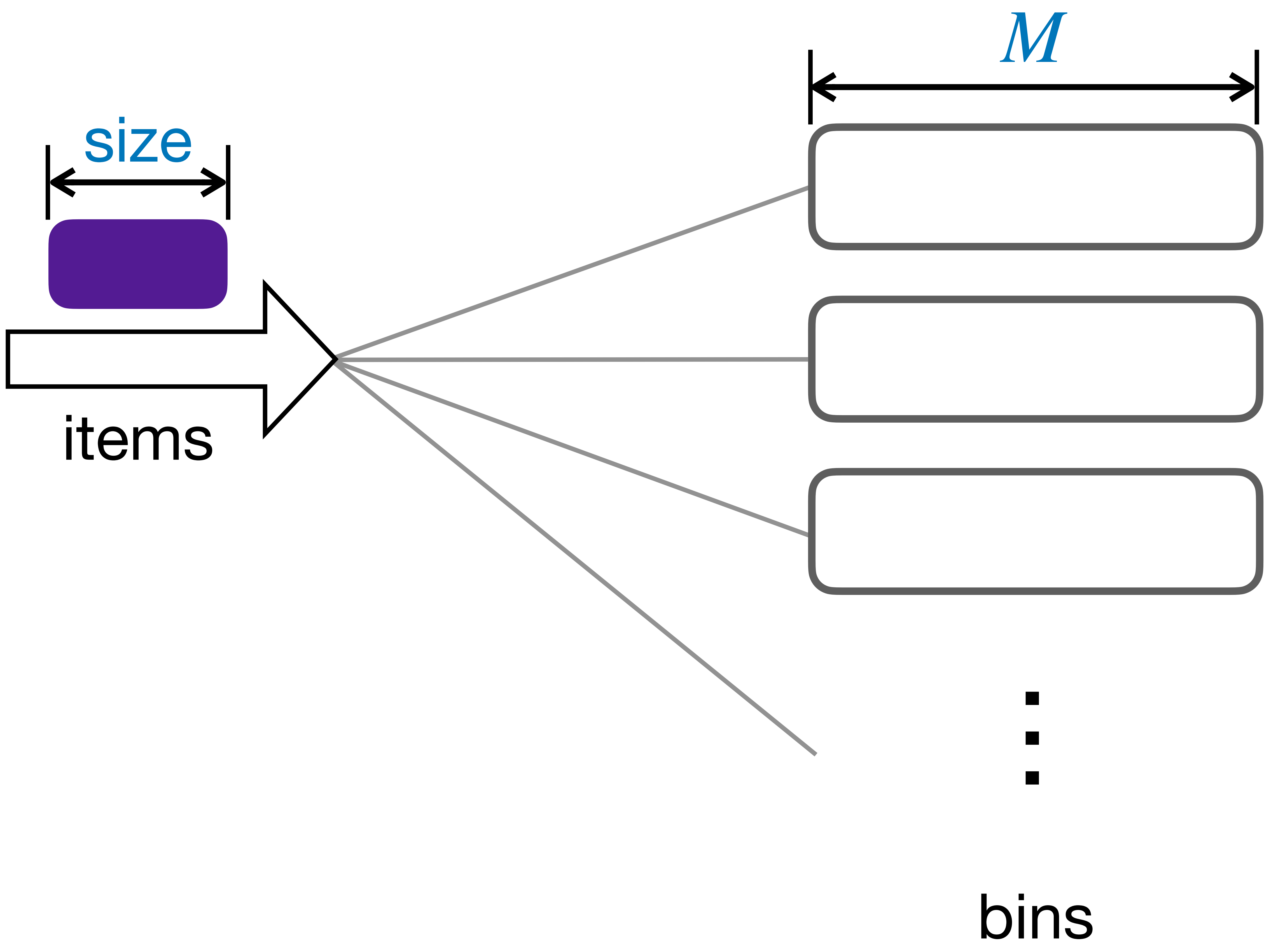

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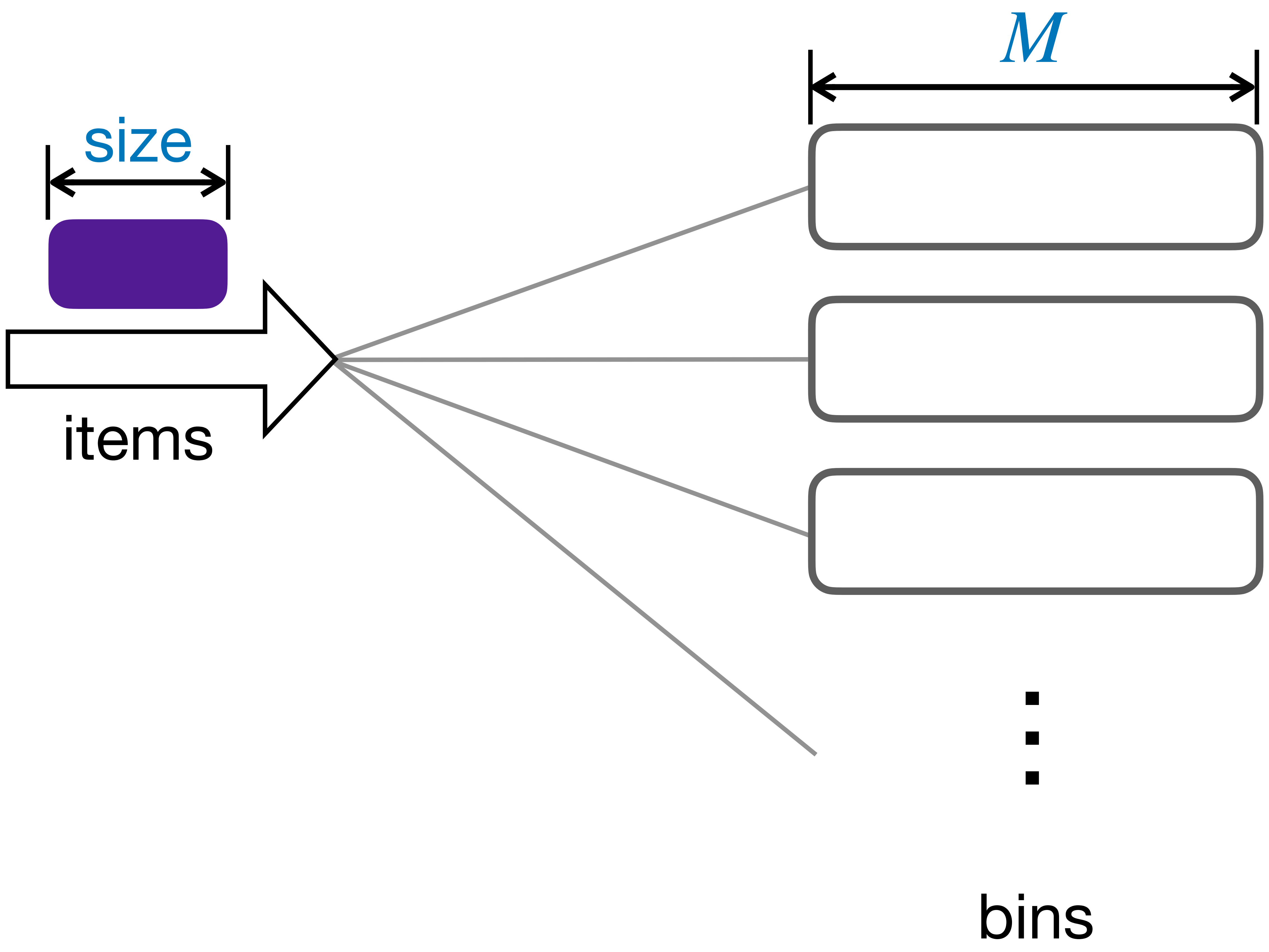

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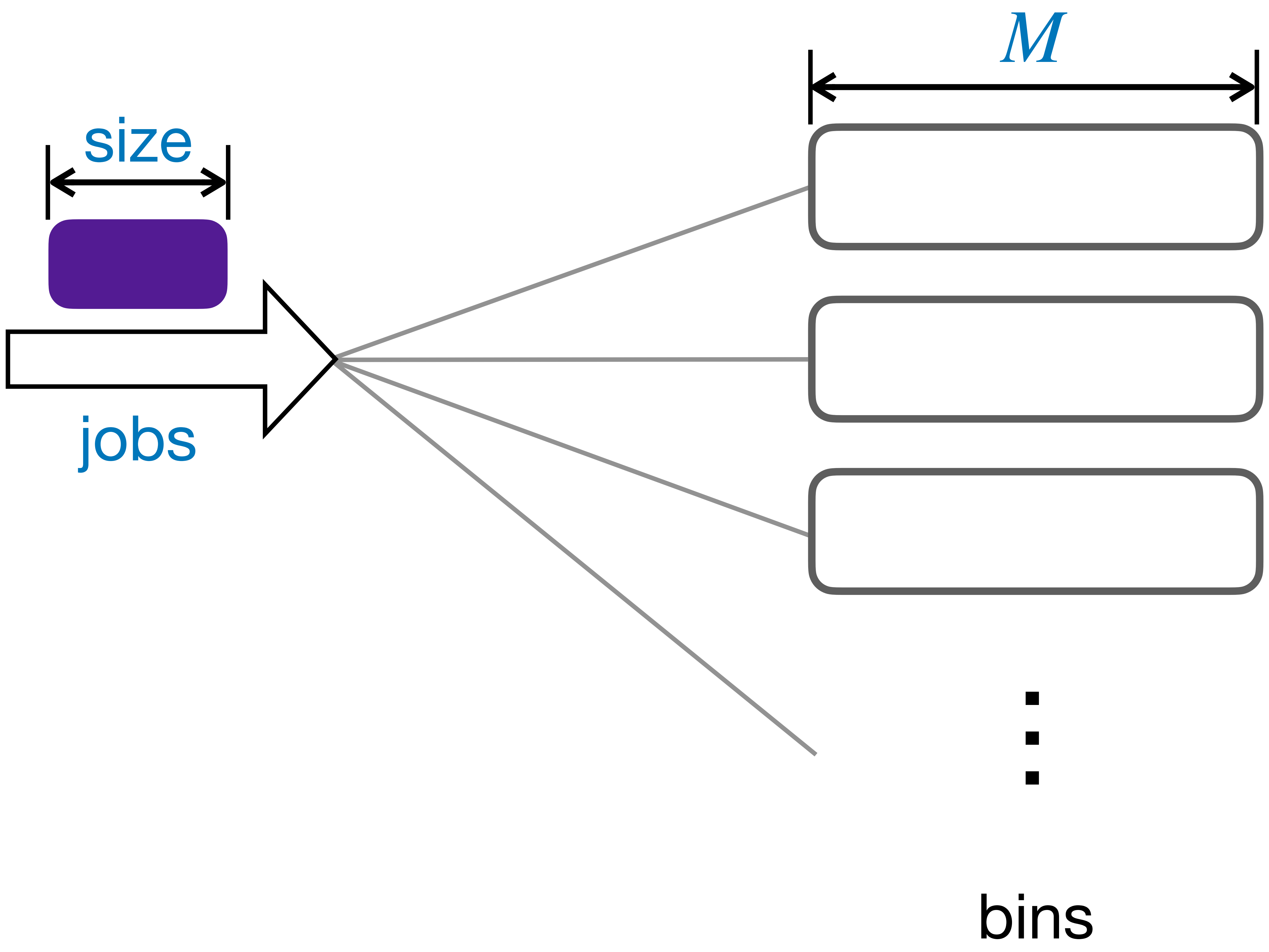

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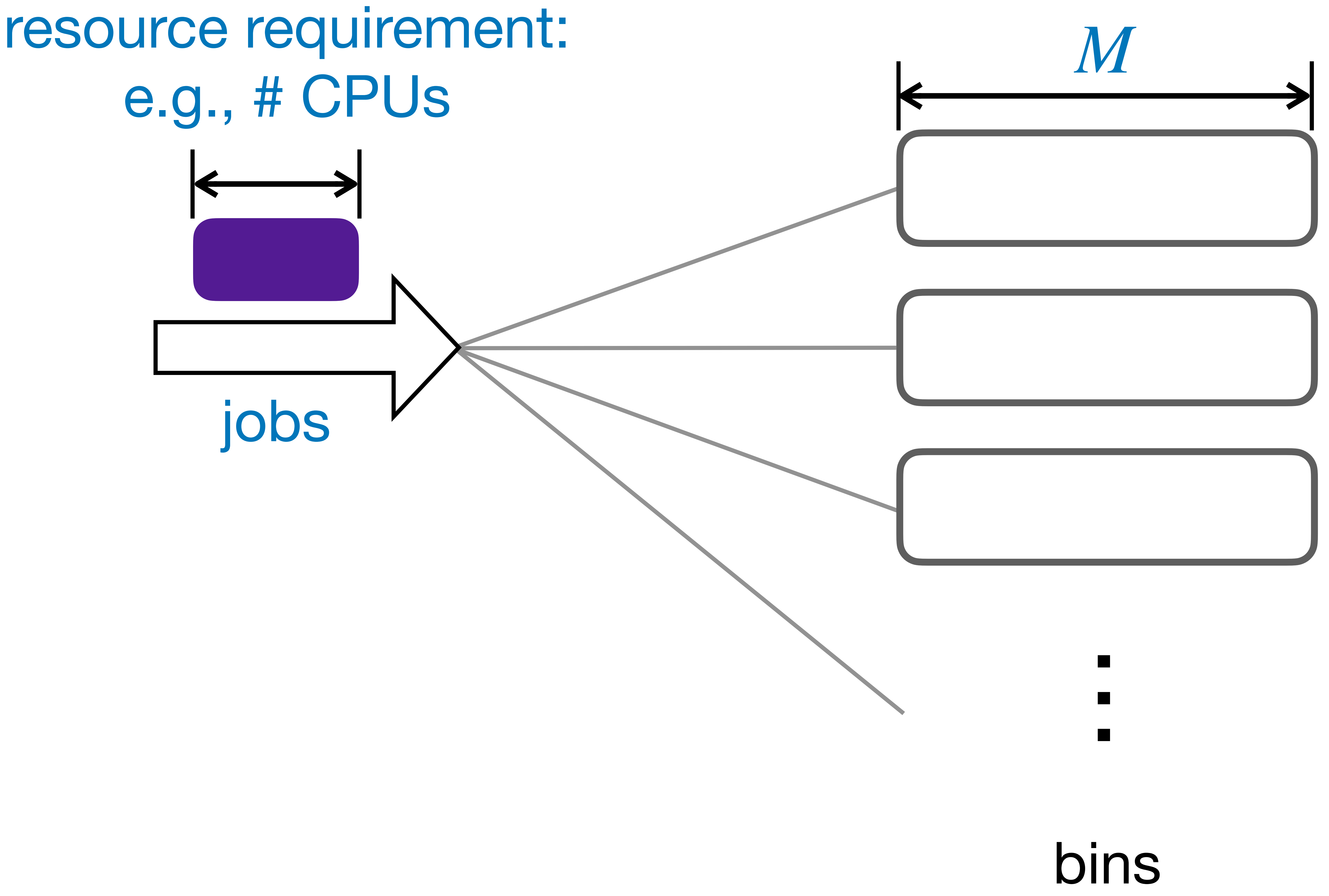

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- Each arriving job needs to be assigned to a server
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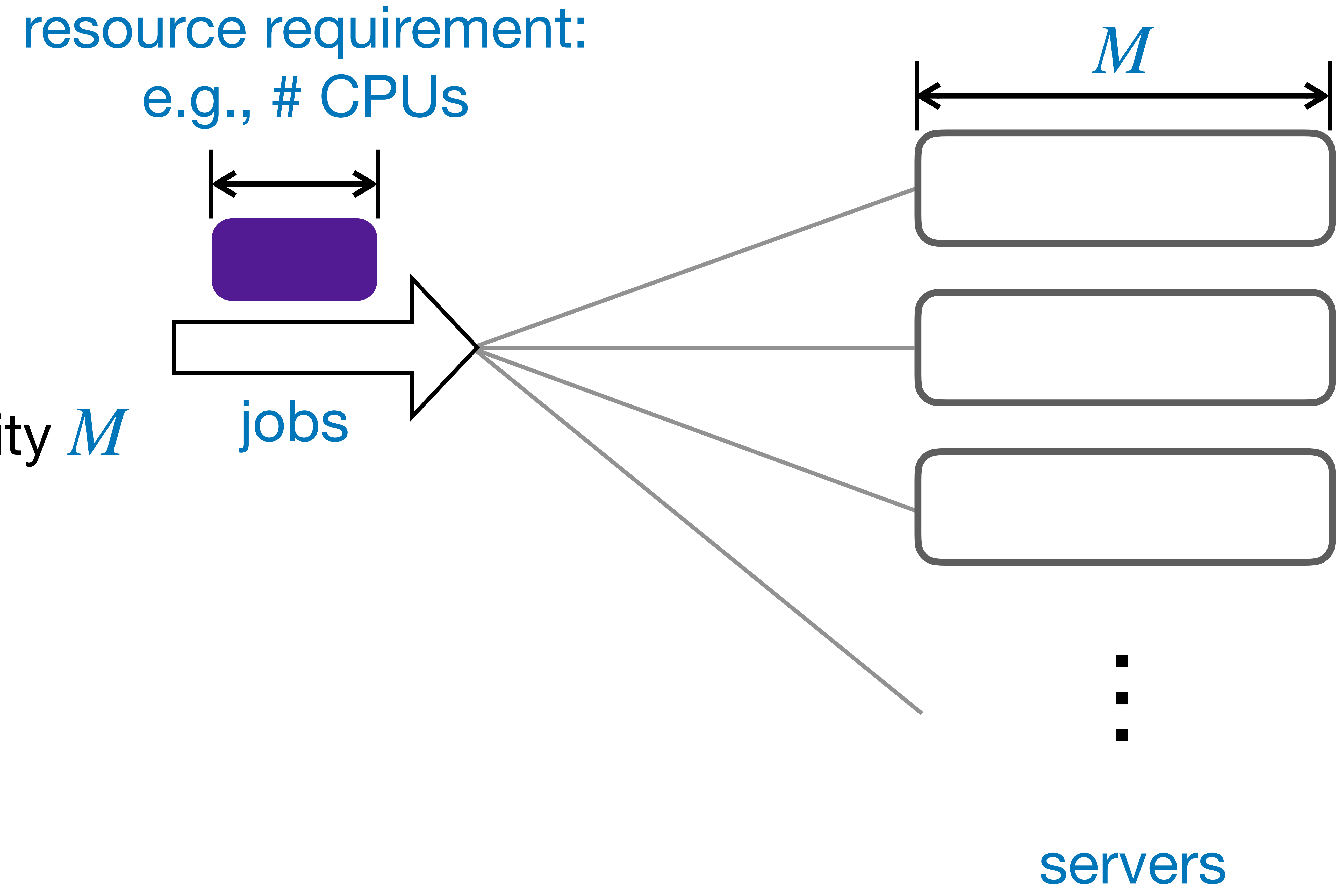

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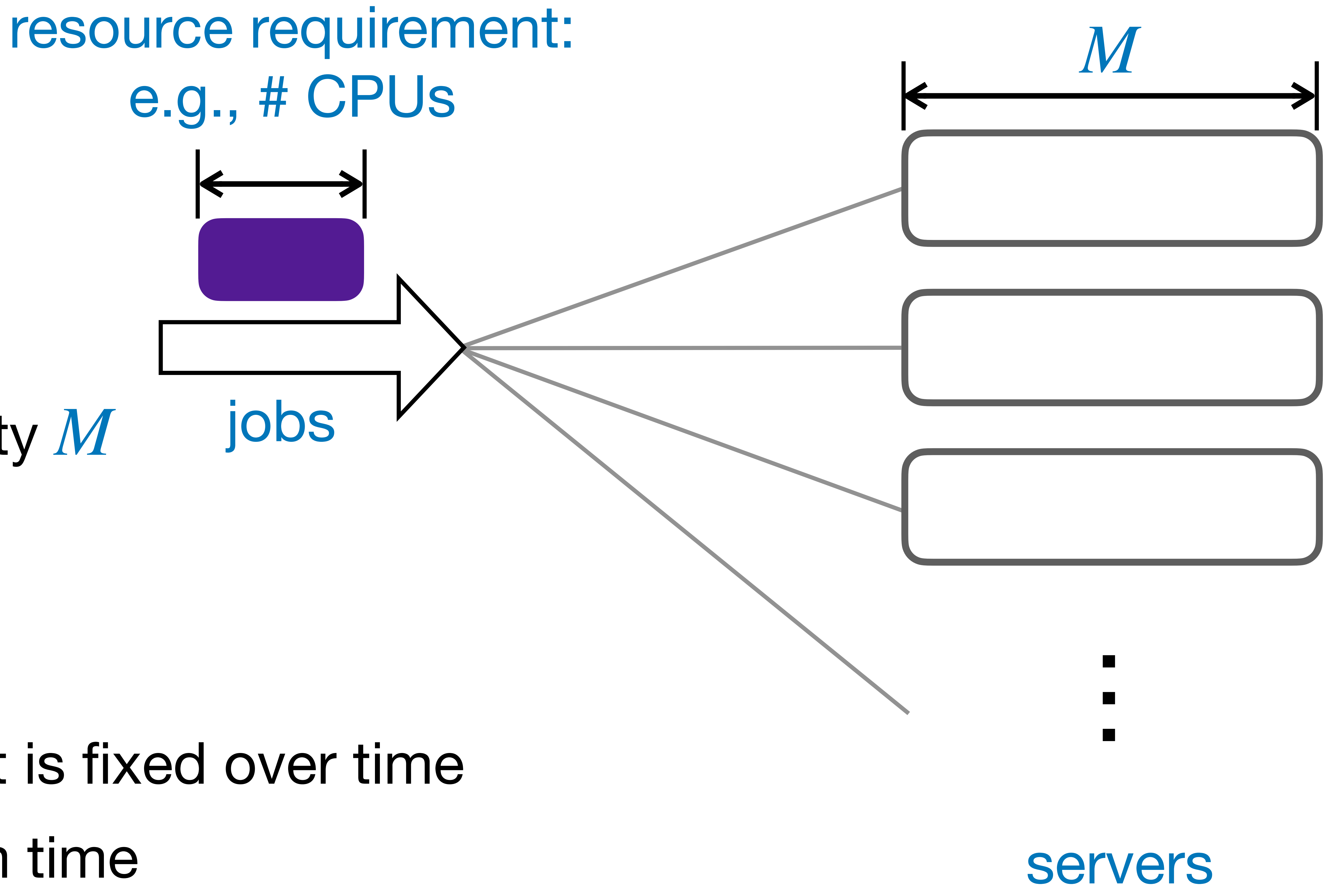

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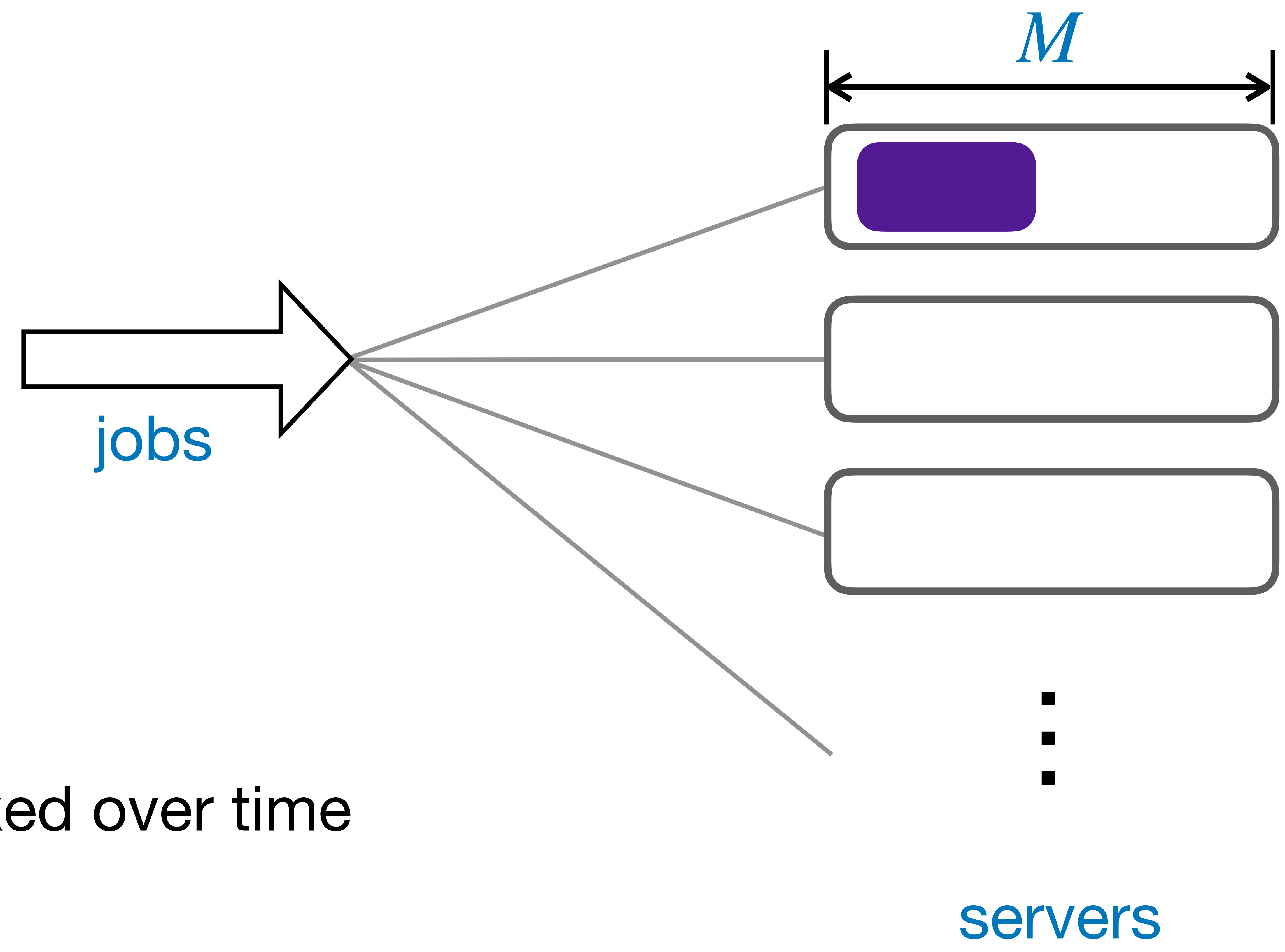

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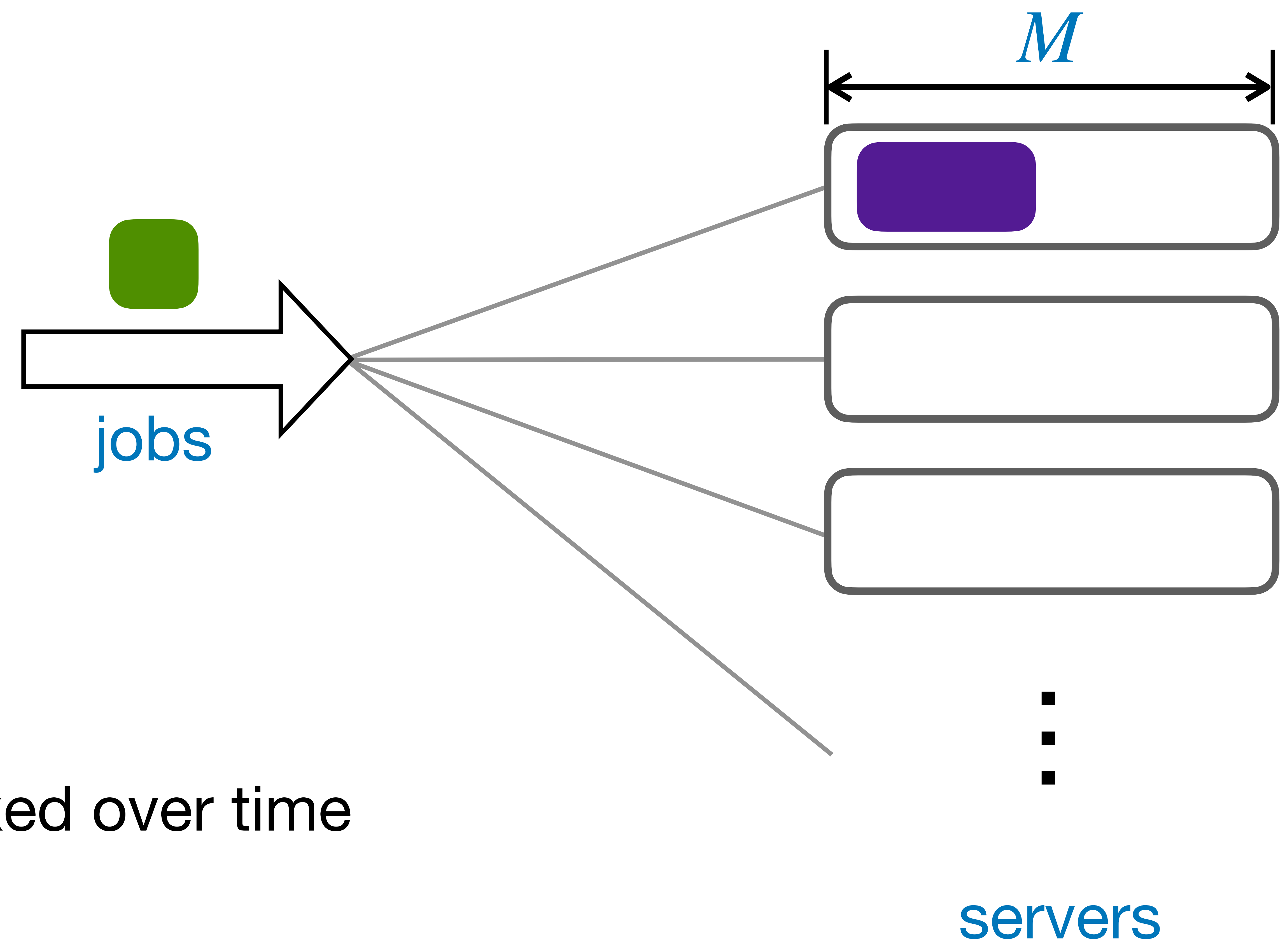

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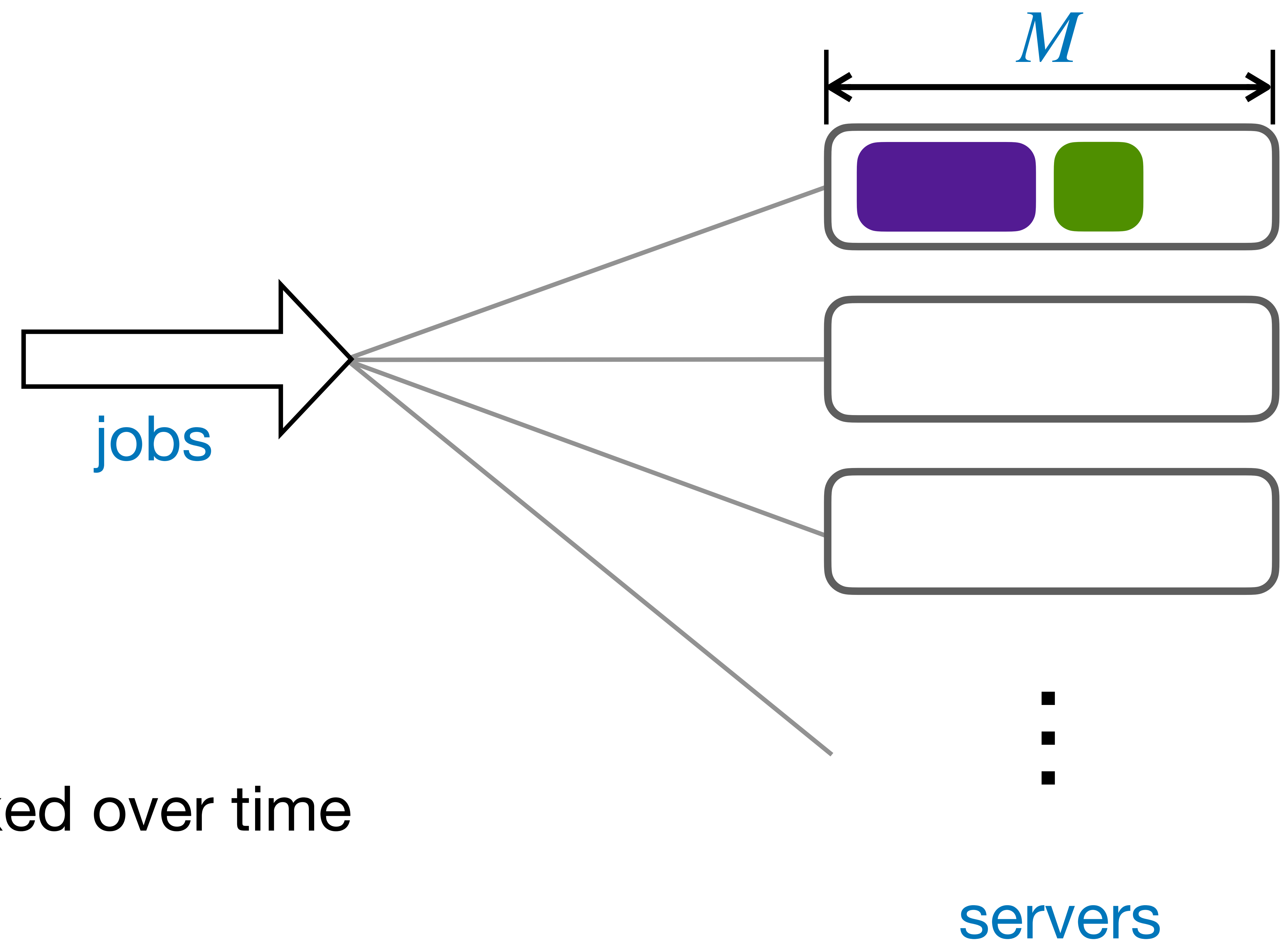

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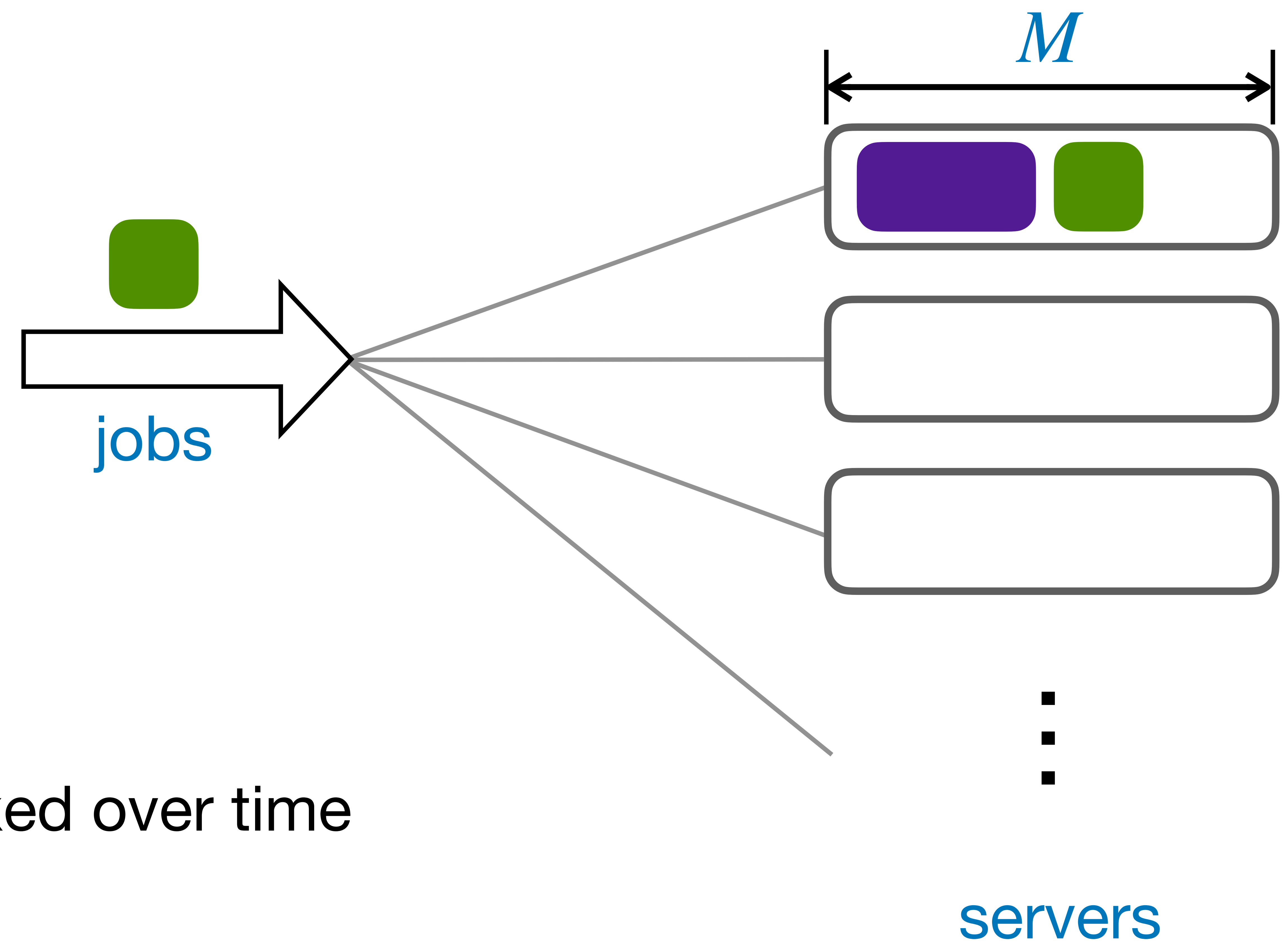

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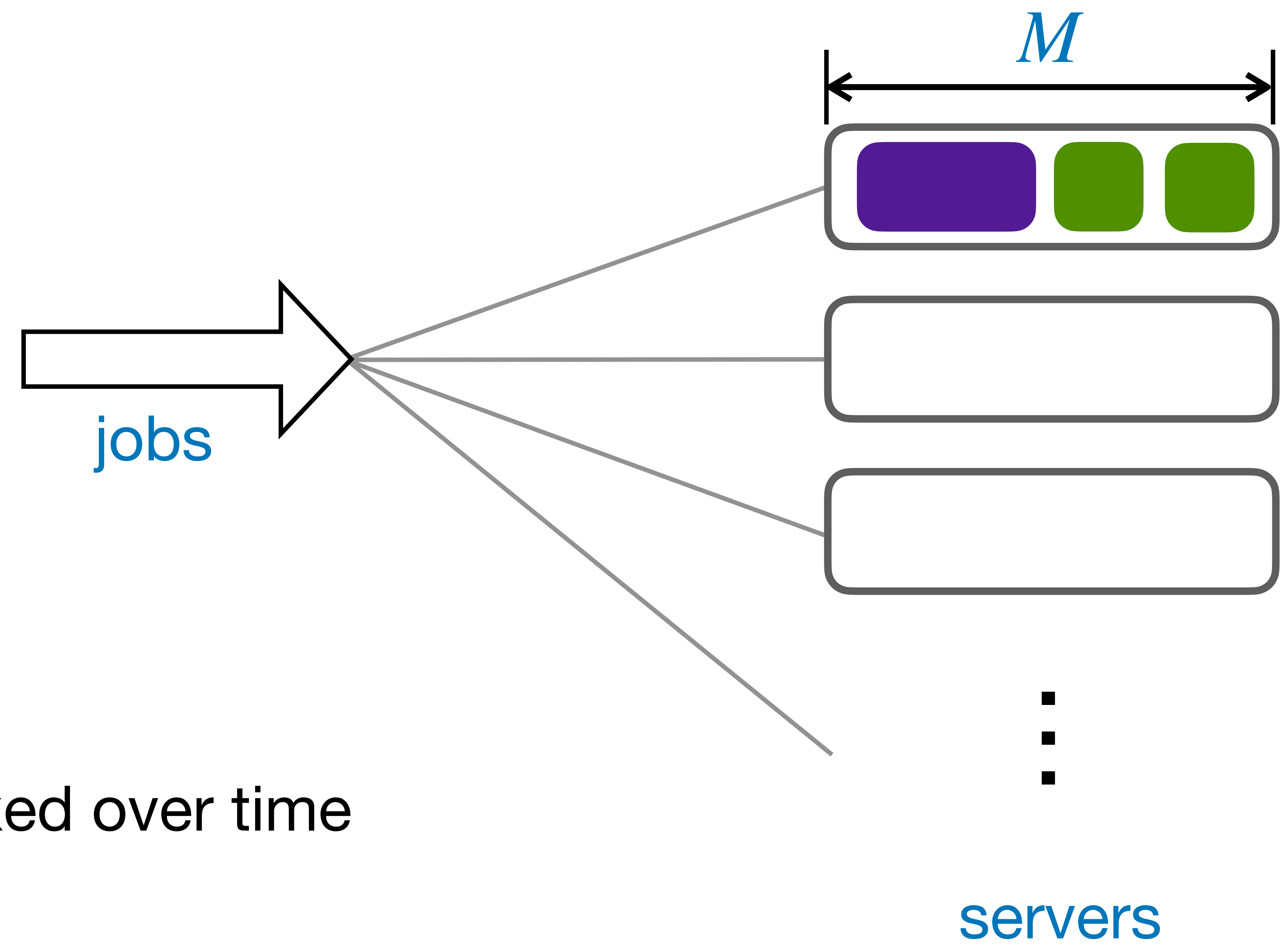

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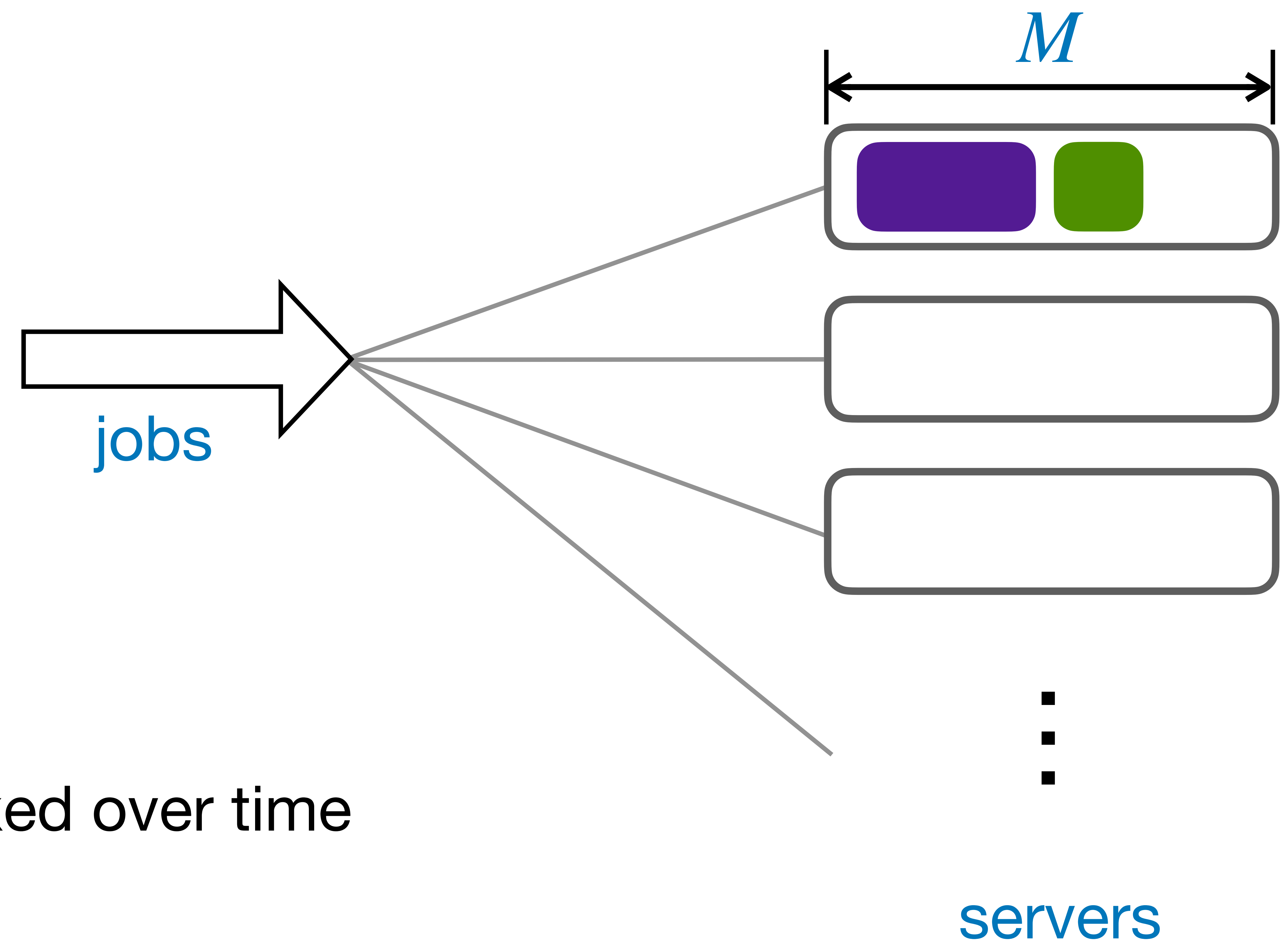

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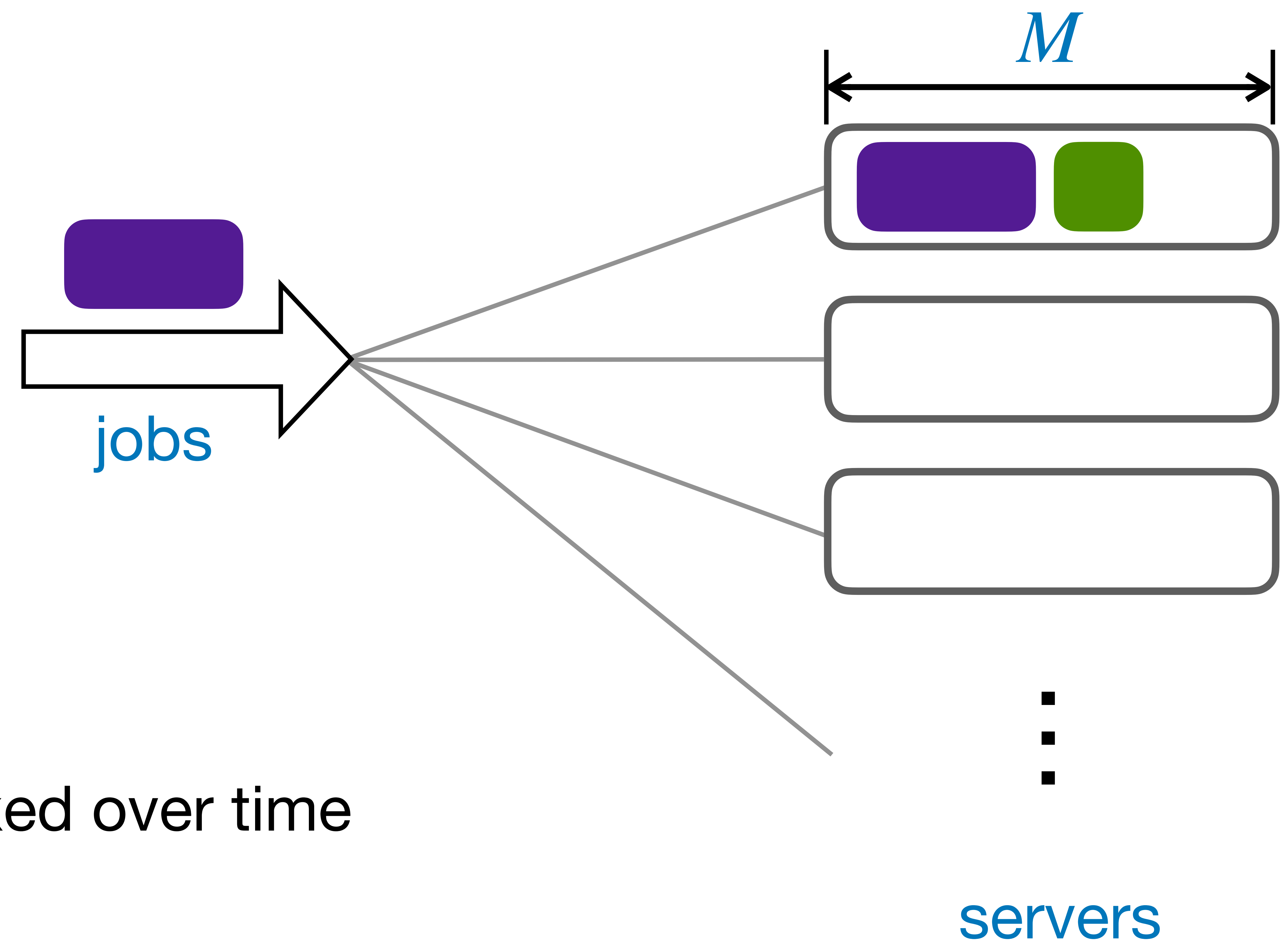

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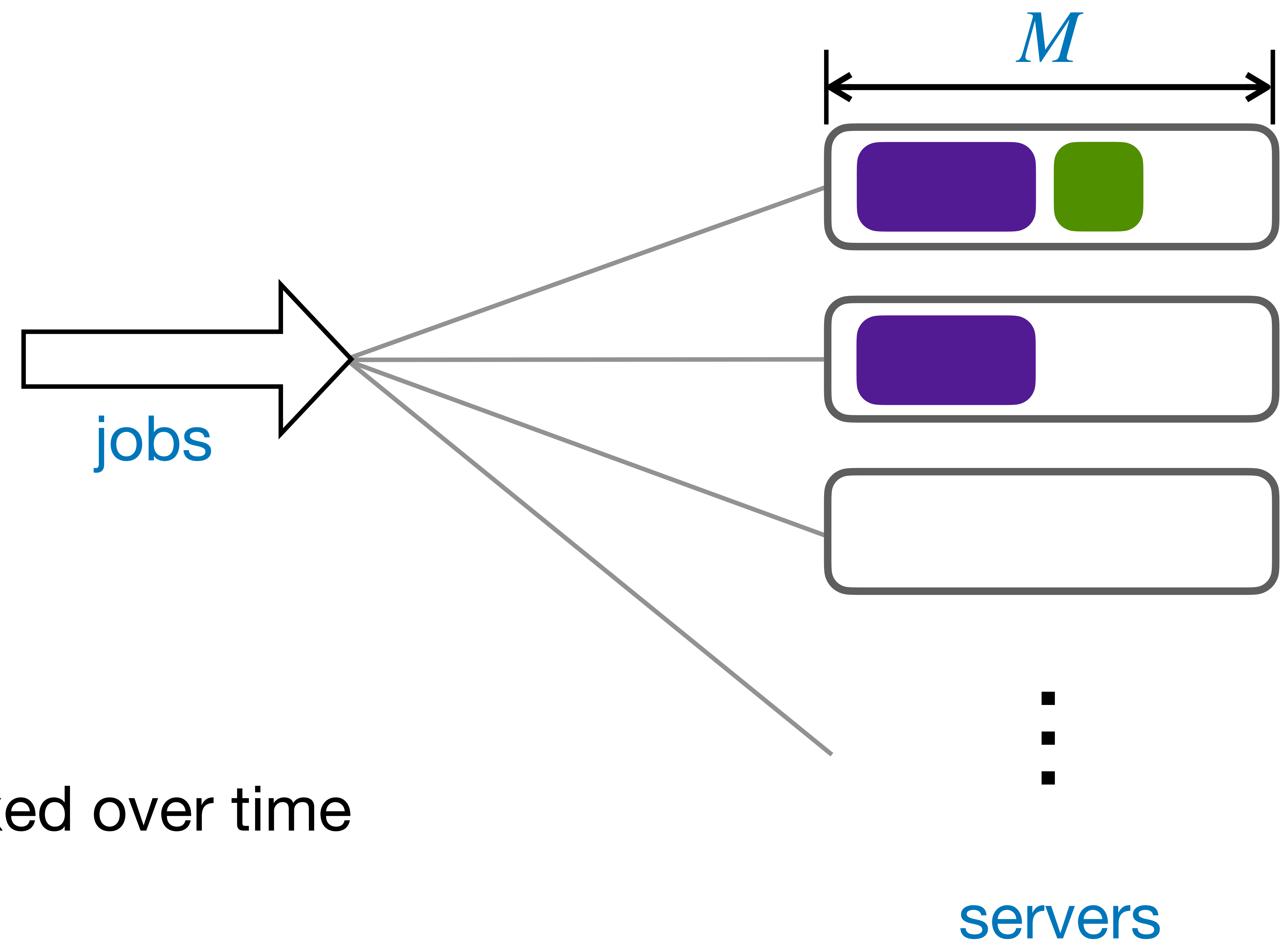

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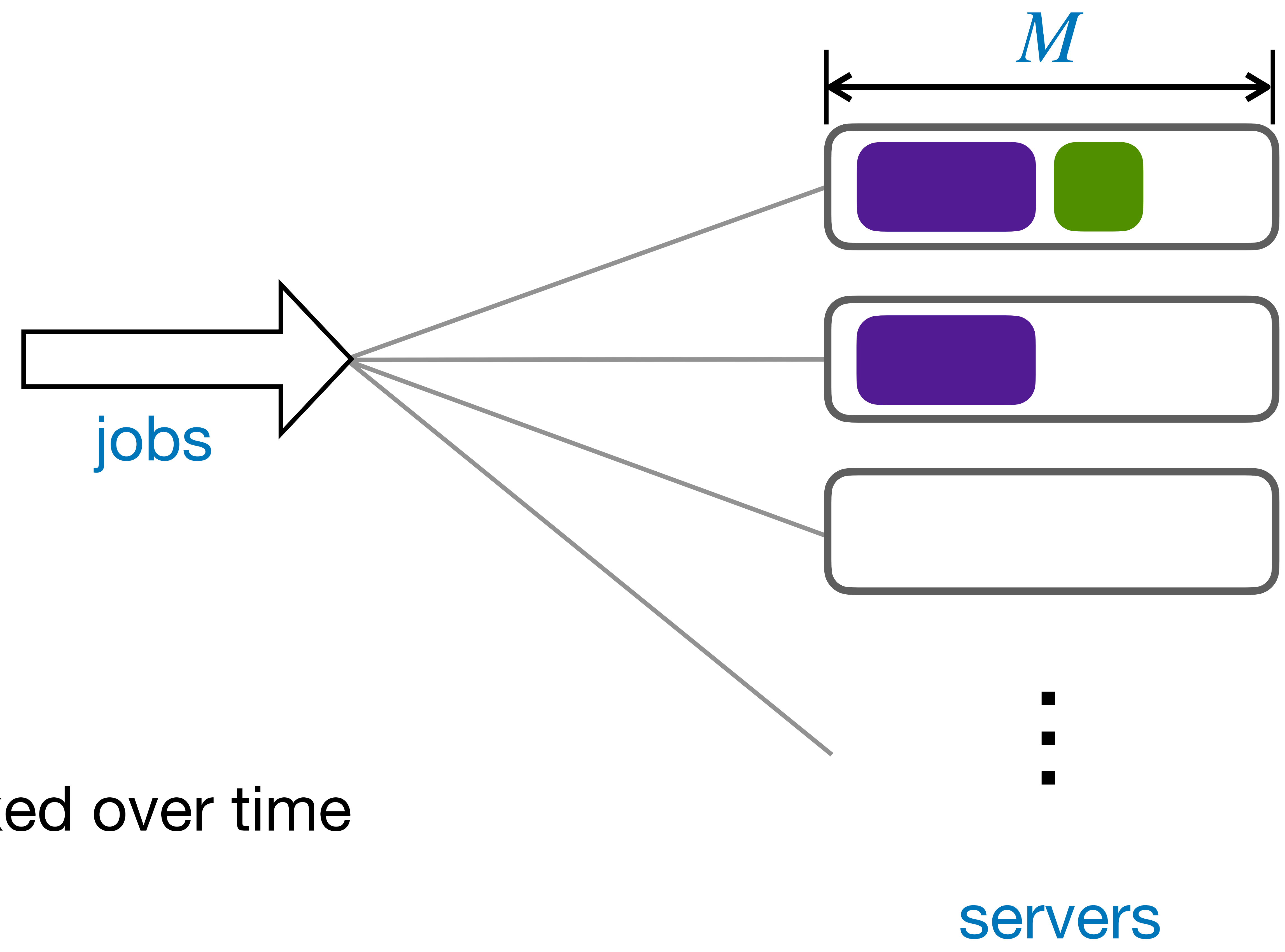

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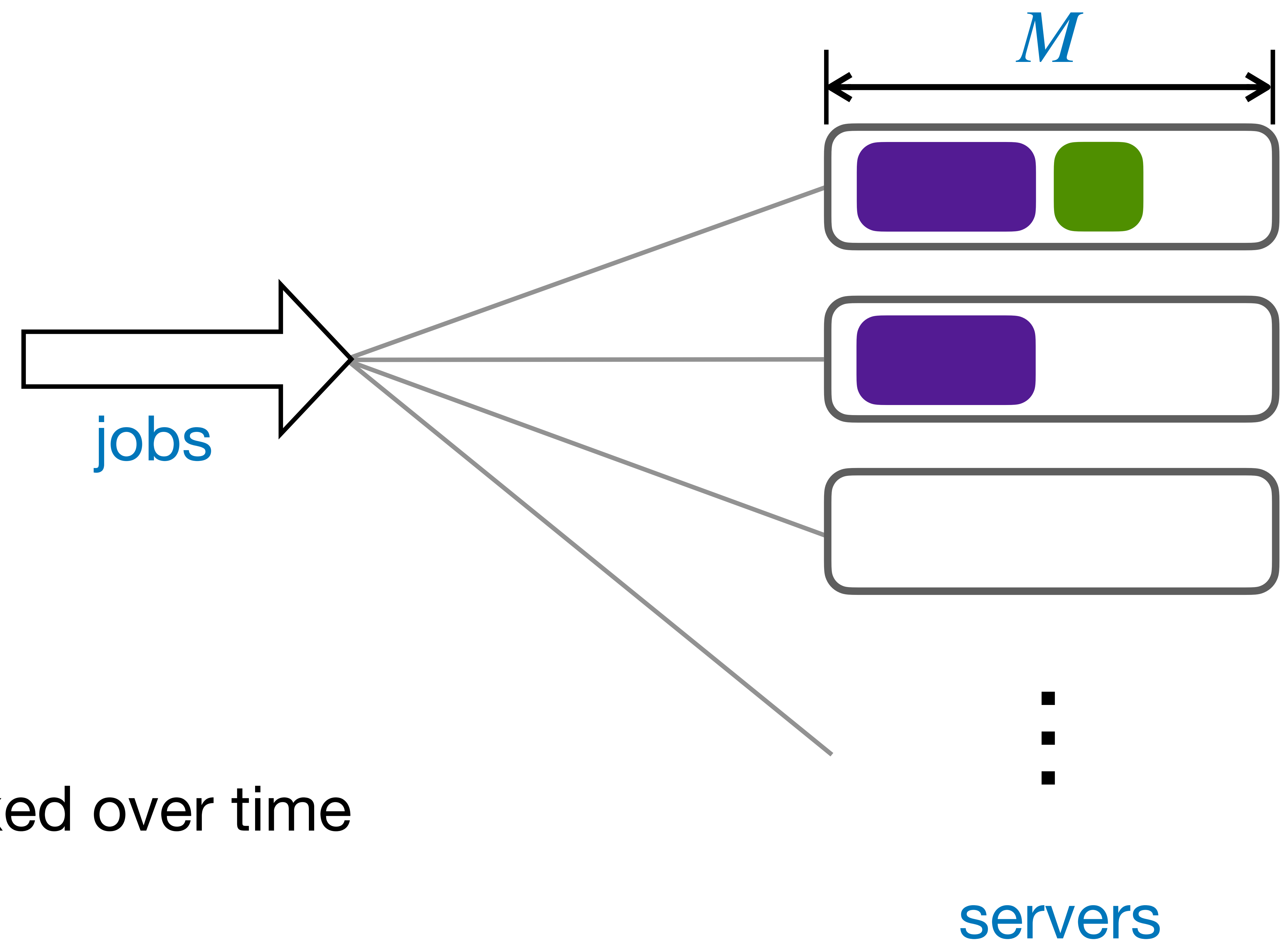

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"Stochastic bin-packing in service systems"

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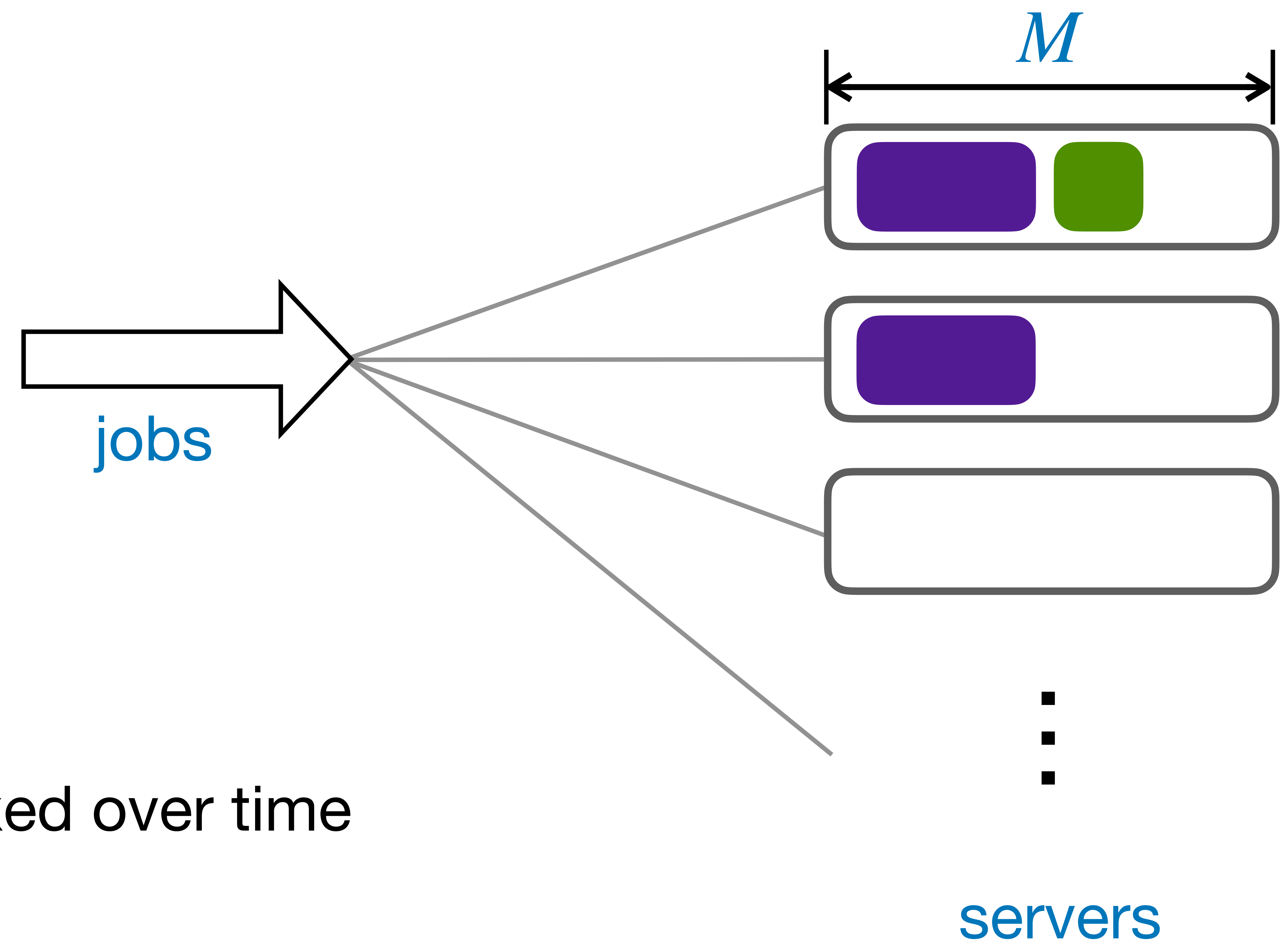

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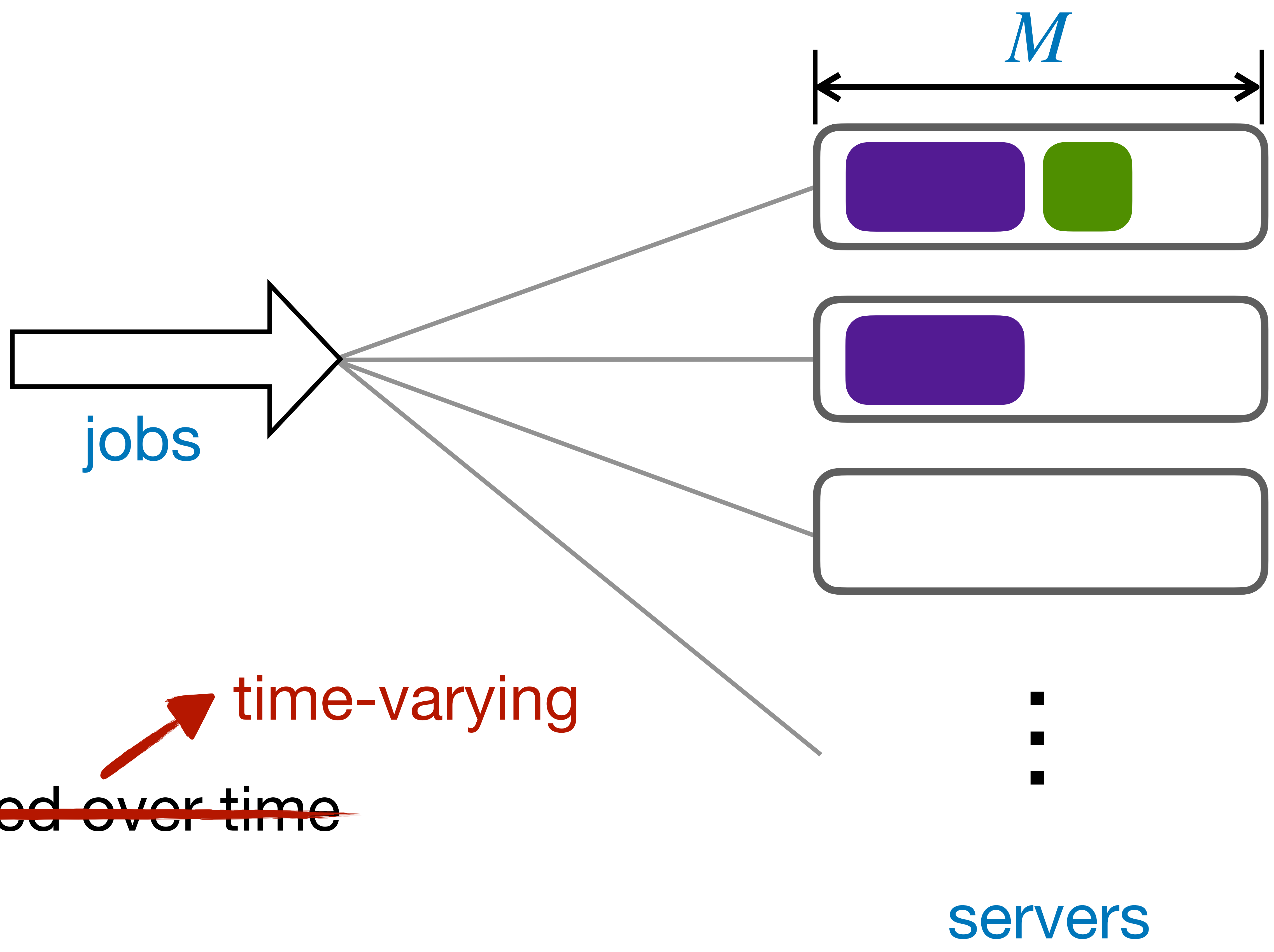

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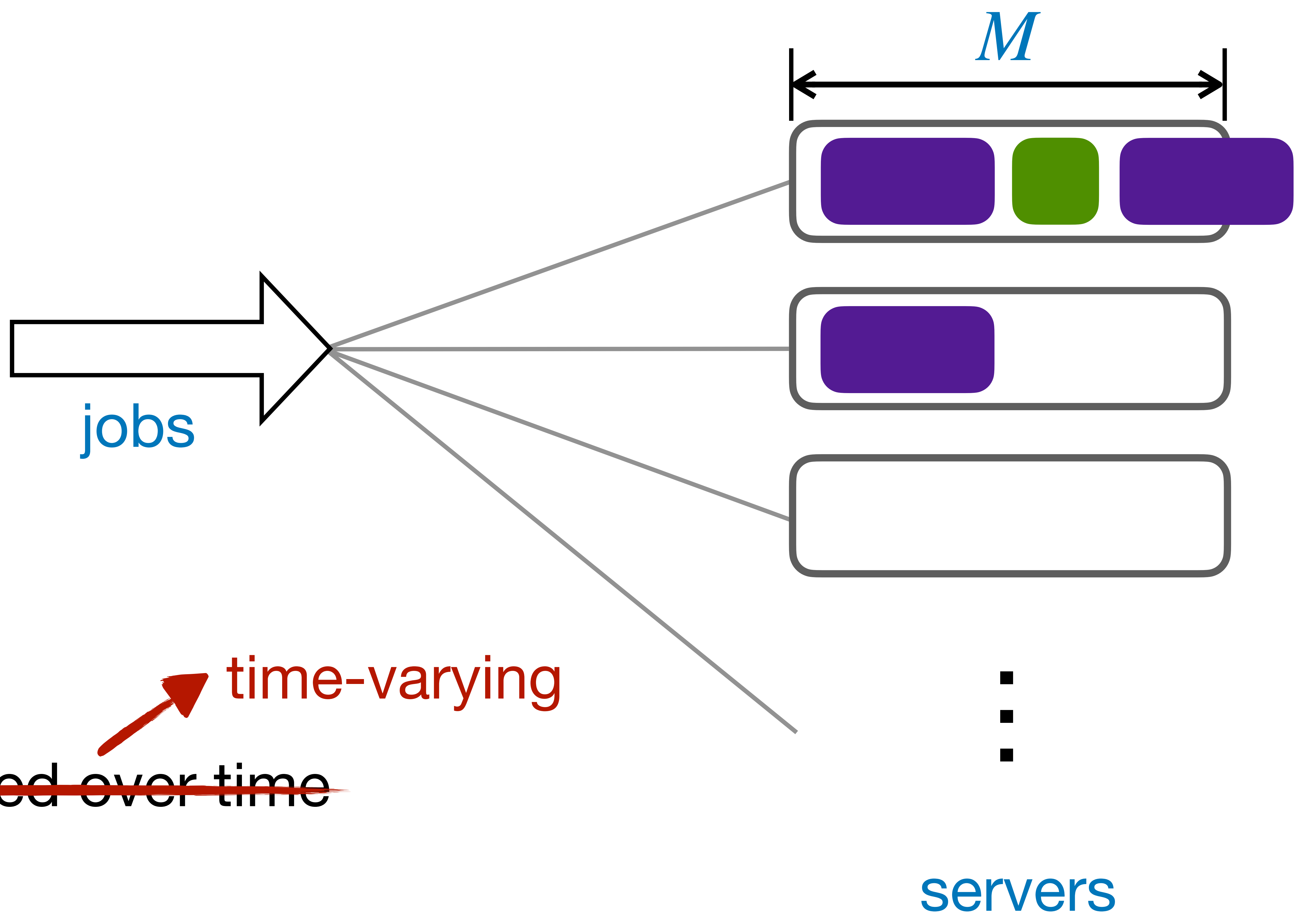

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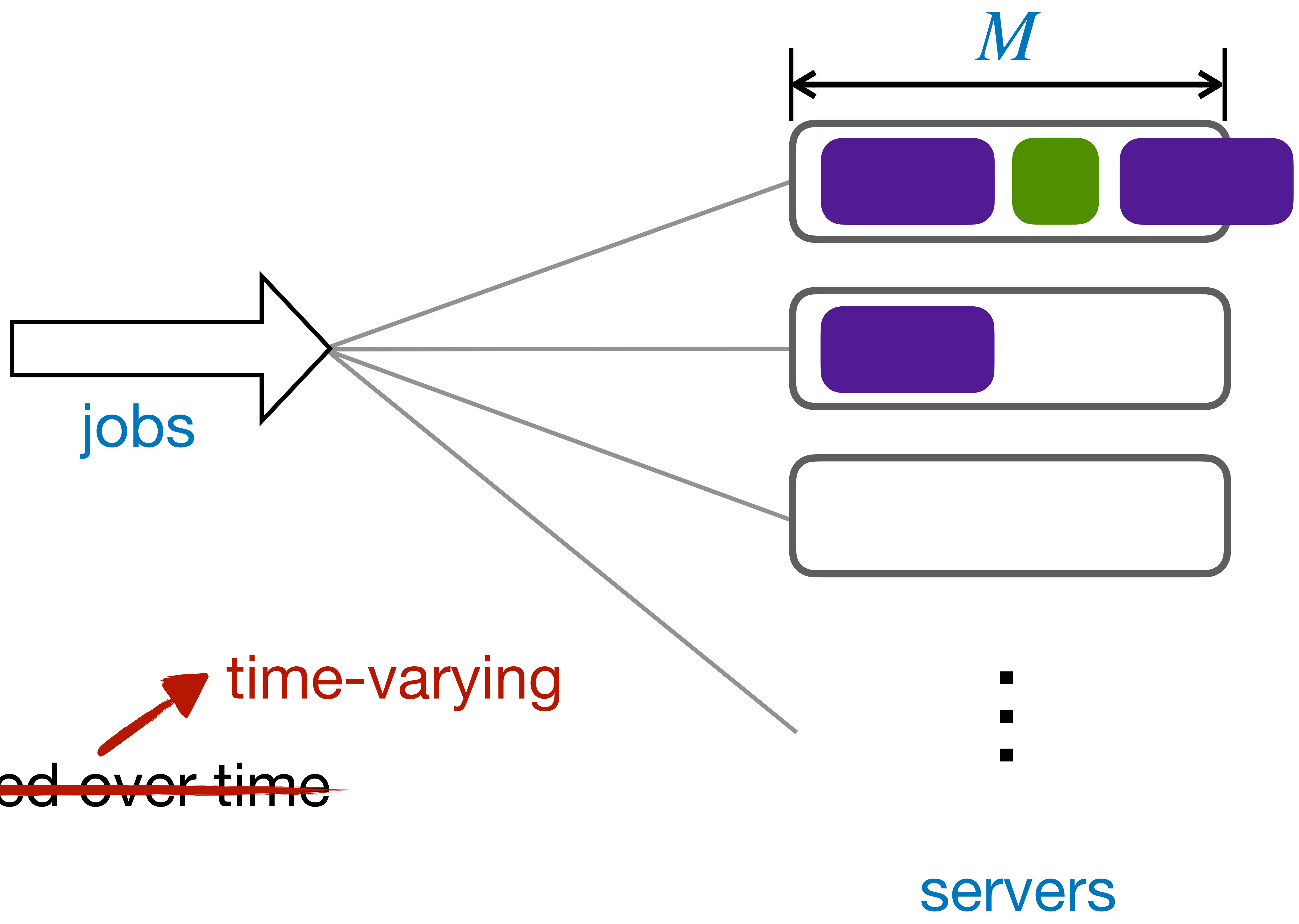

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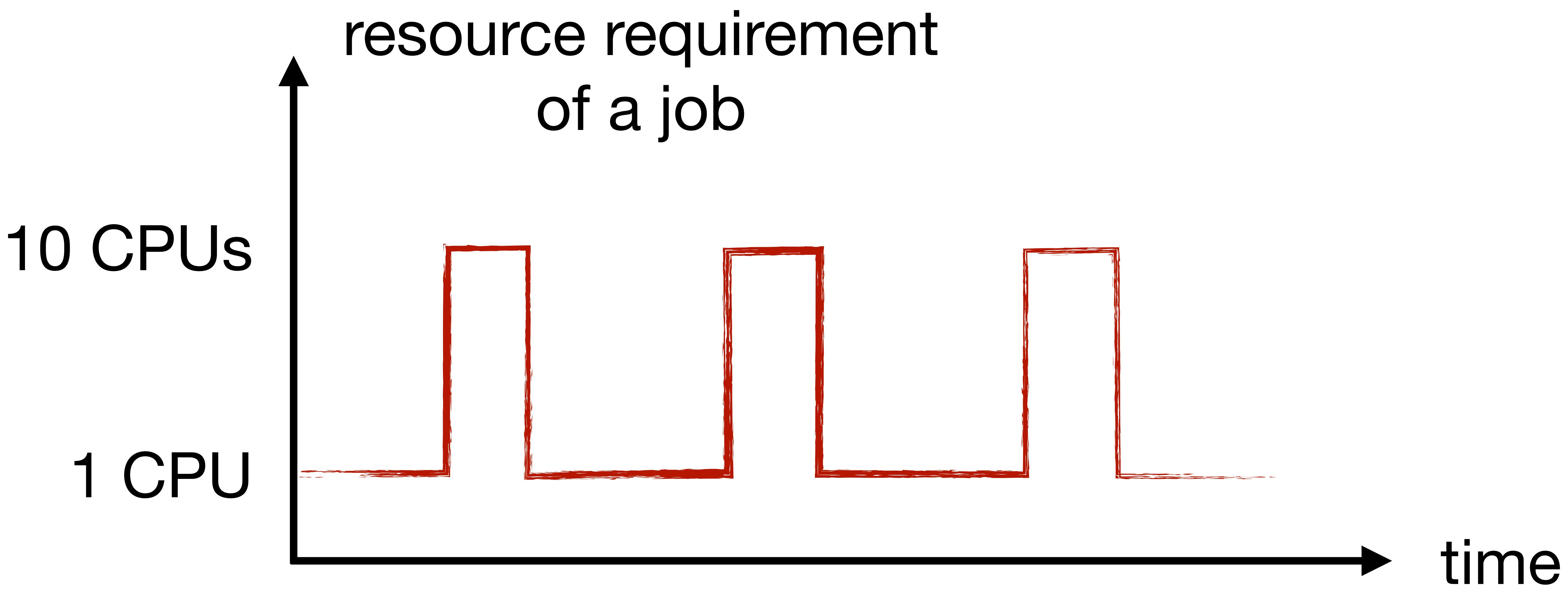
subject to **cost** (resource contention) ≤ budget

"Stochastic bin-packing in service systems"

Why time-varying?



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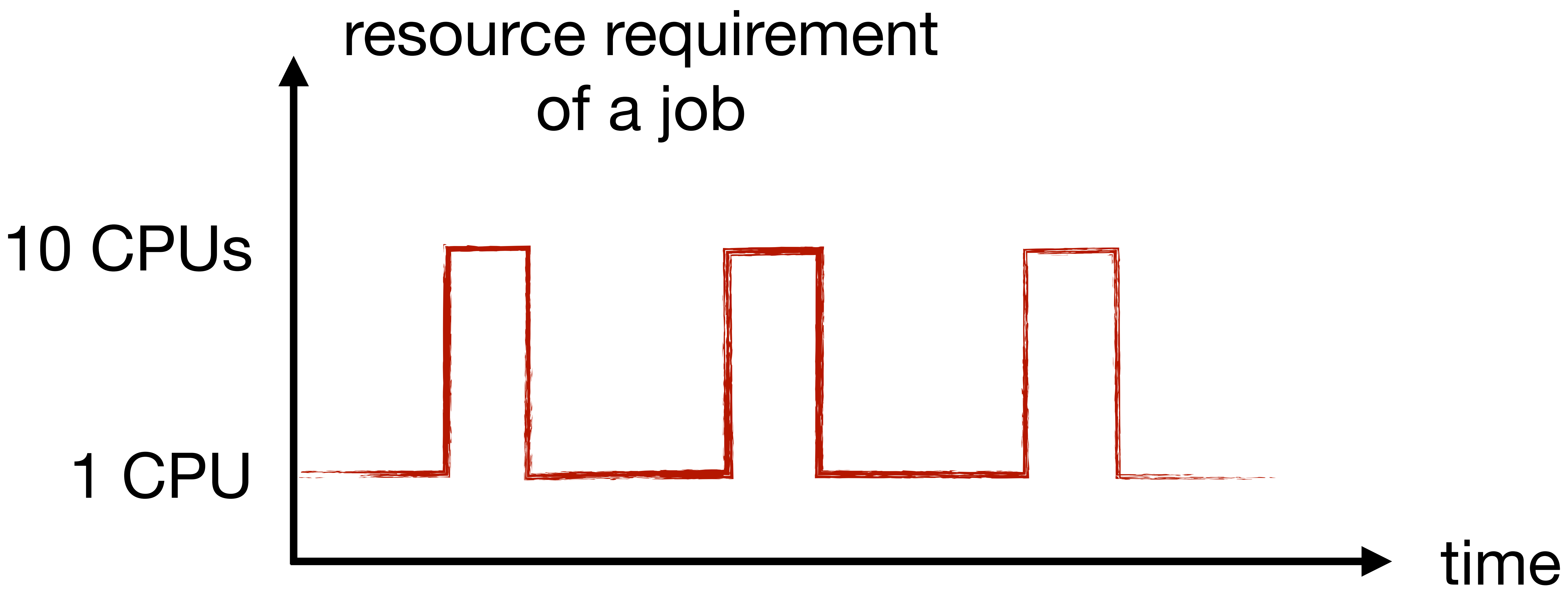

- Reserve resources based on peak requirement

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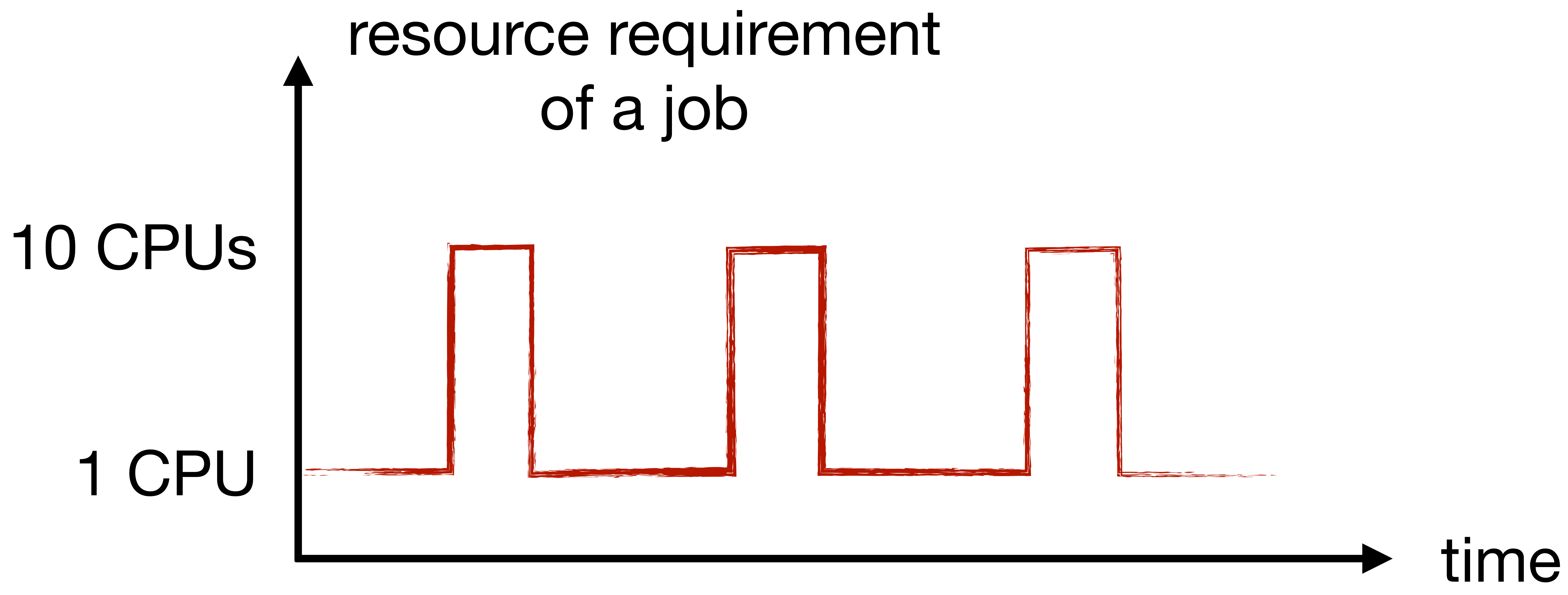

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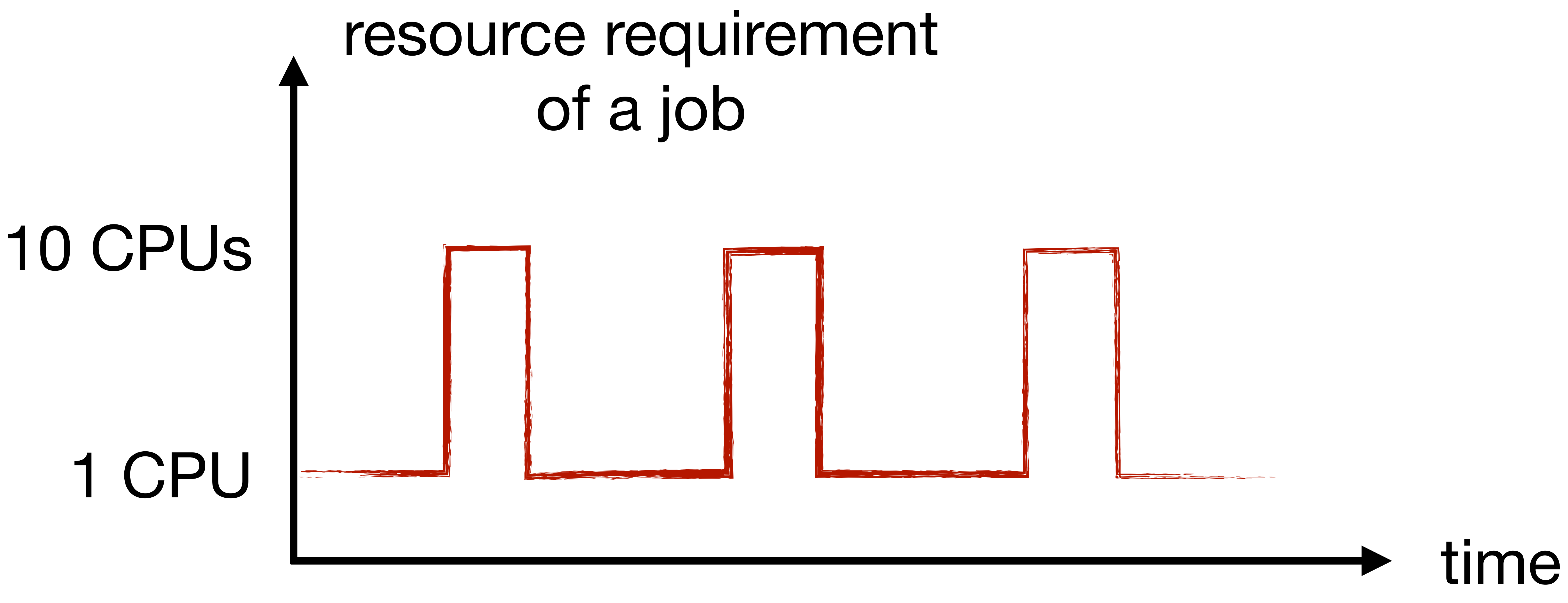

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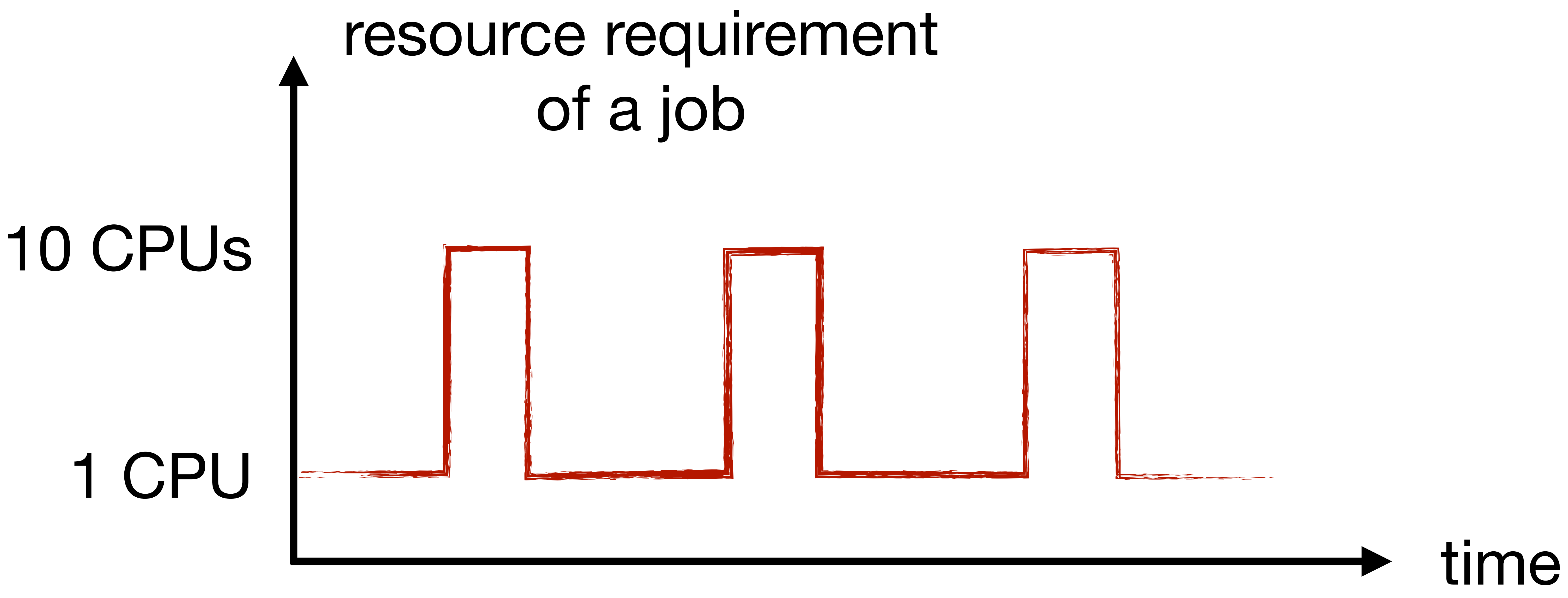

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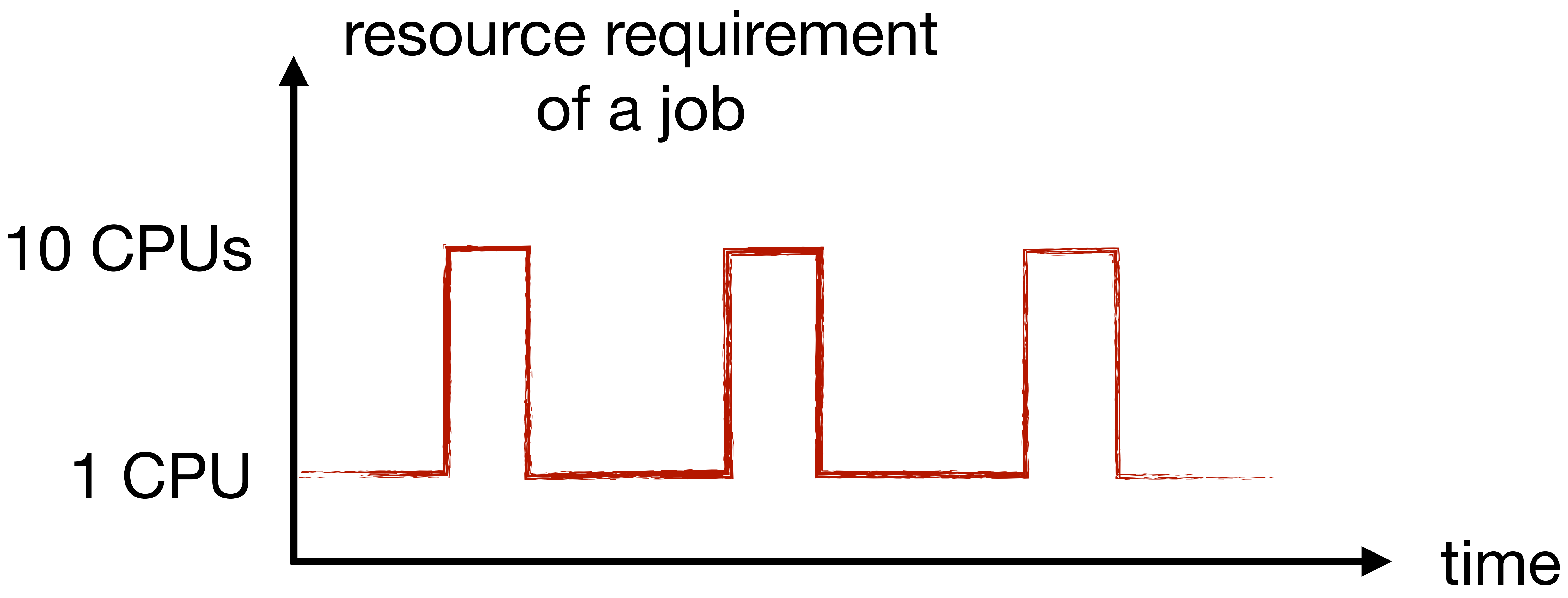

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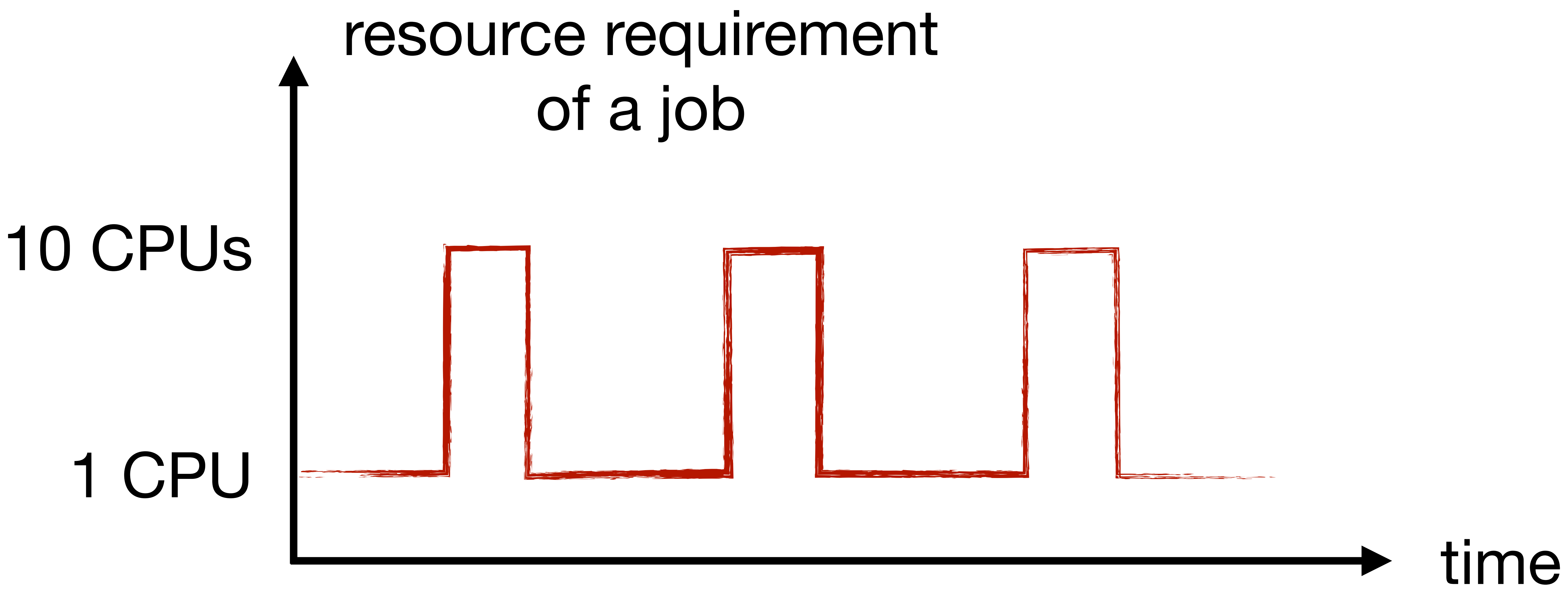

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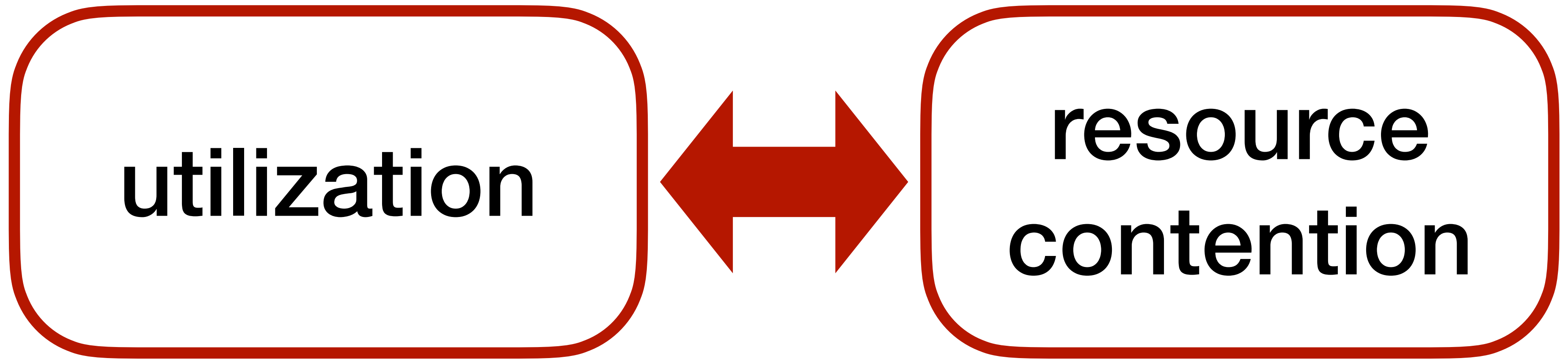
Our formulation captures:



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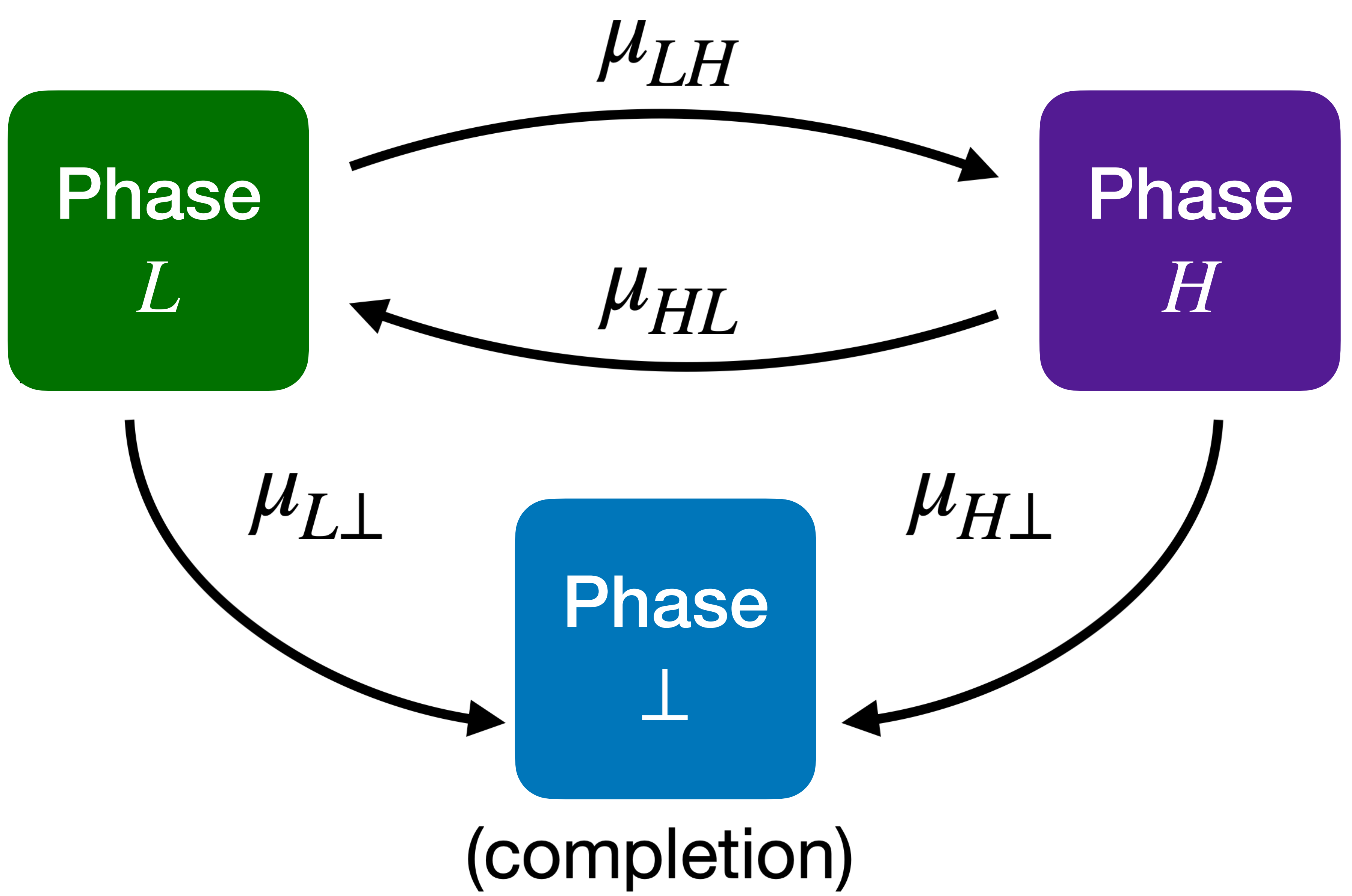
- **Classical bin-packing: Vast literature**

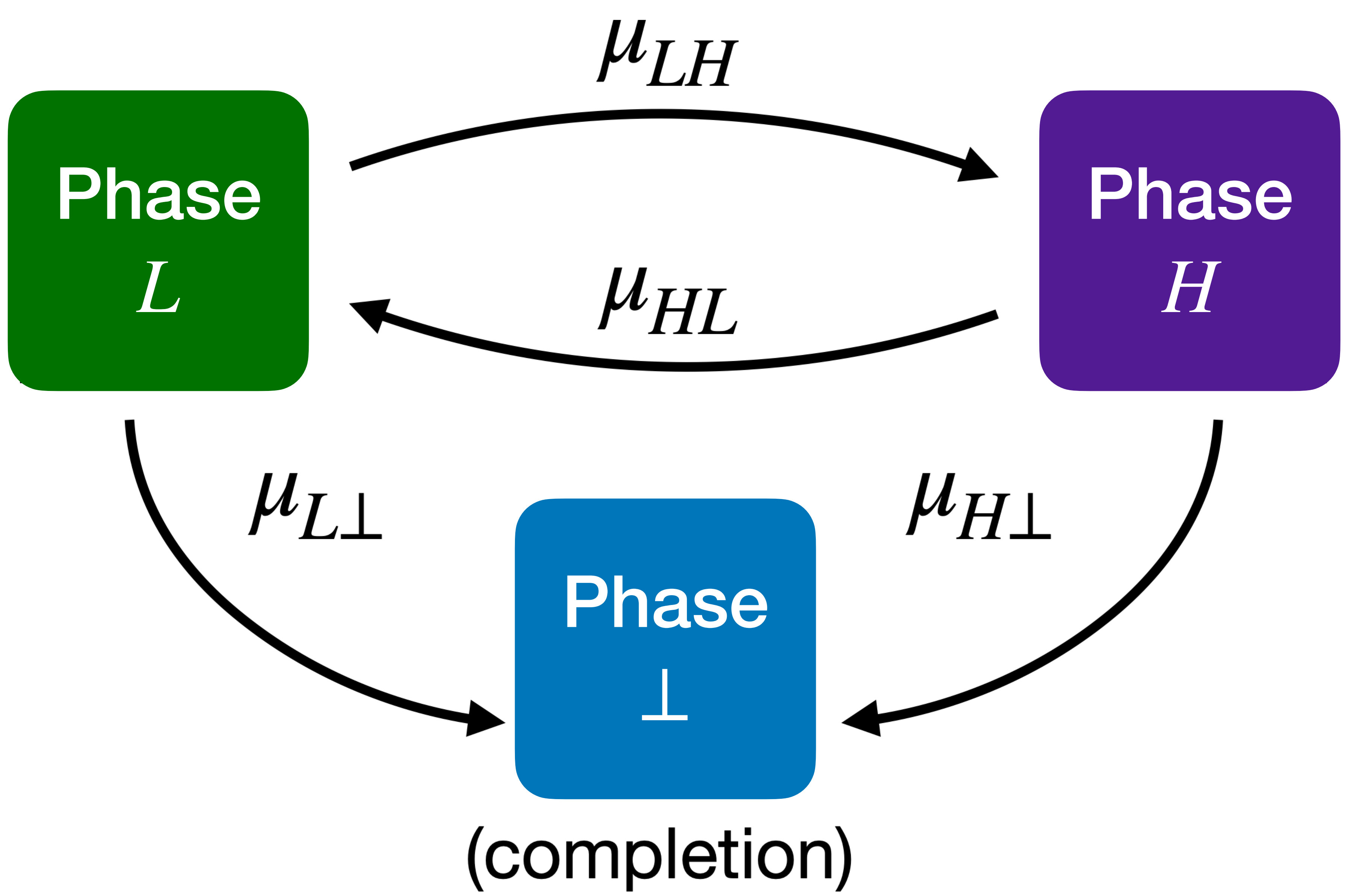
More details on the job model



Example MC

More details on the job model

- Resource requirement of a job evolves over time following a Markov chain



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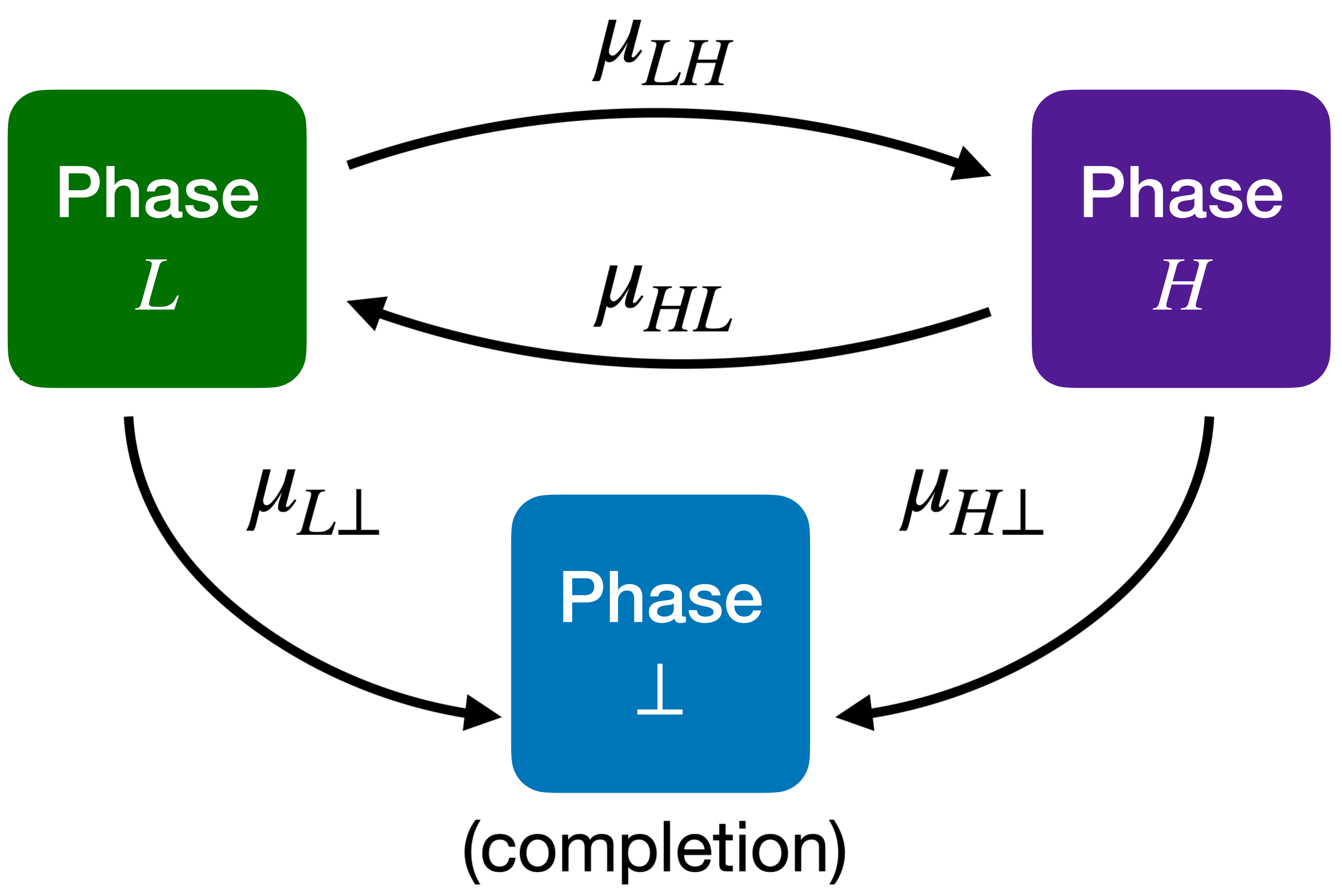

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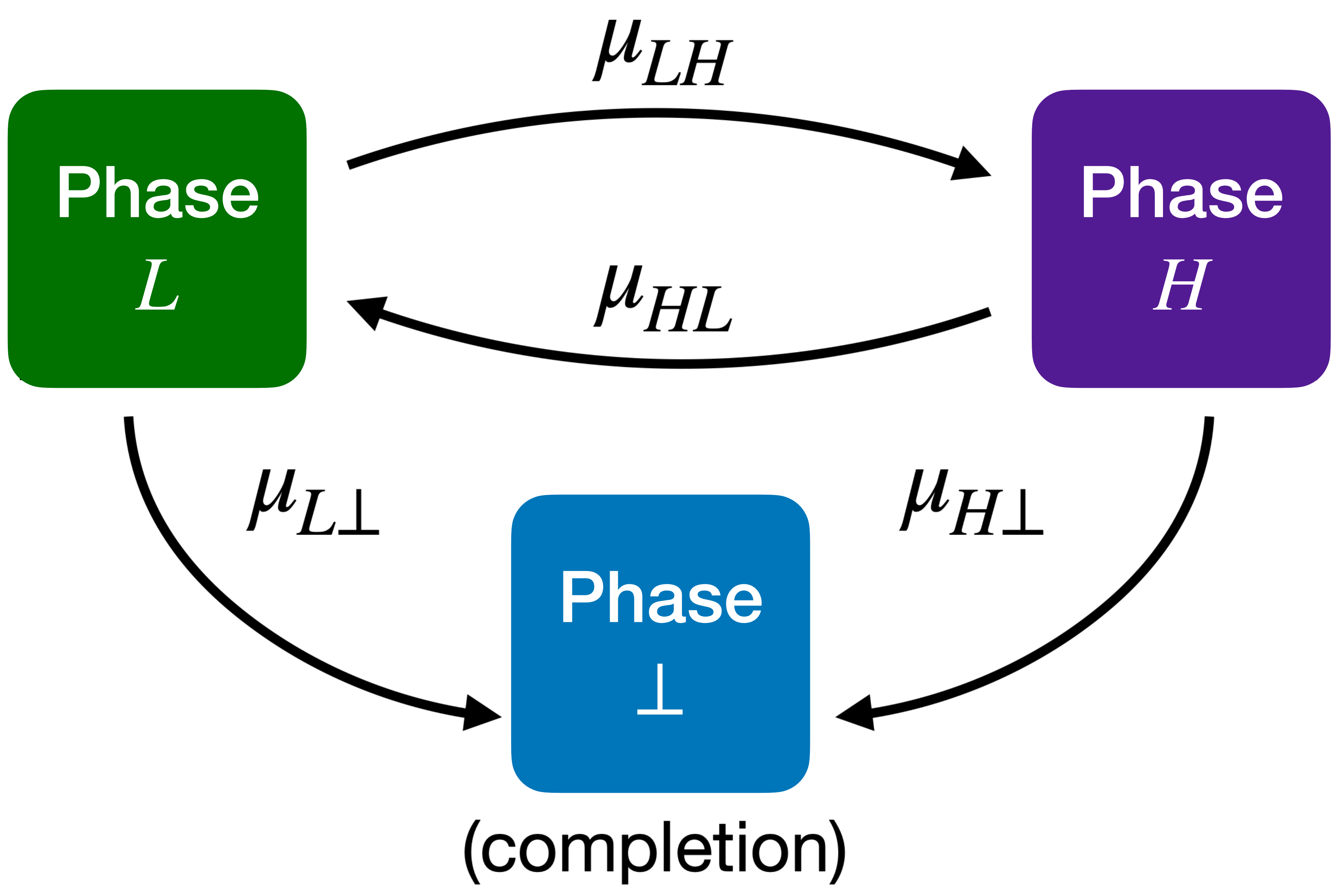

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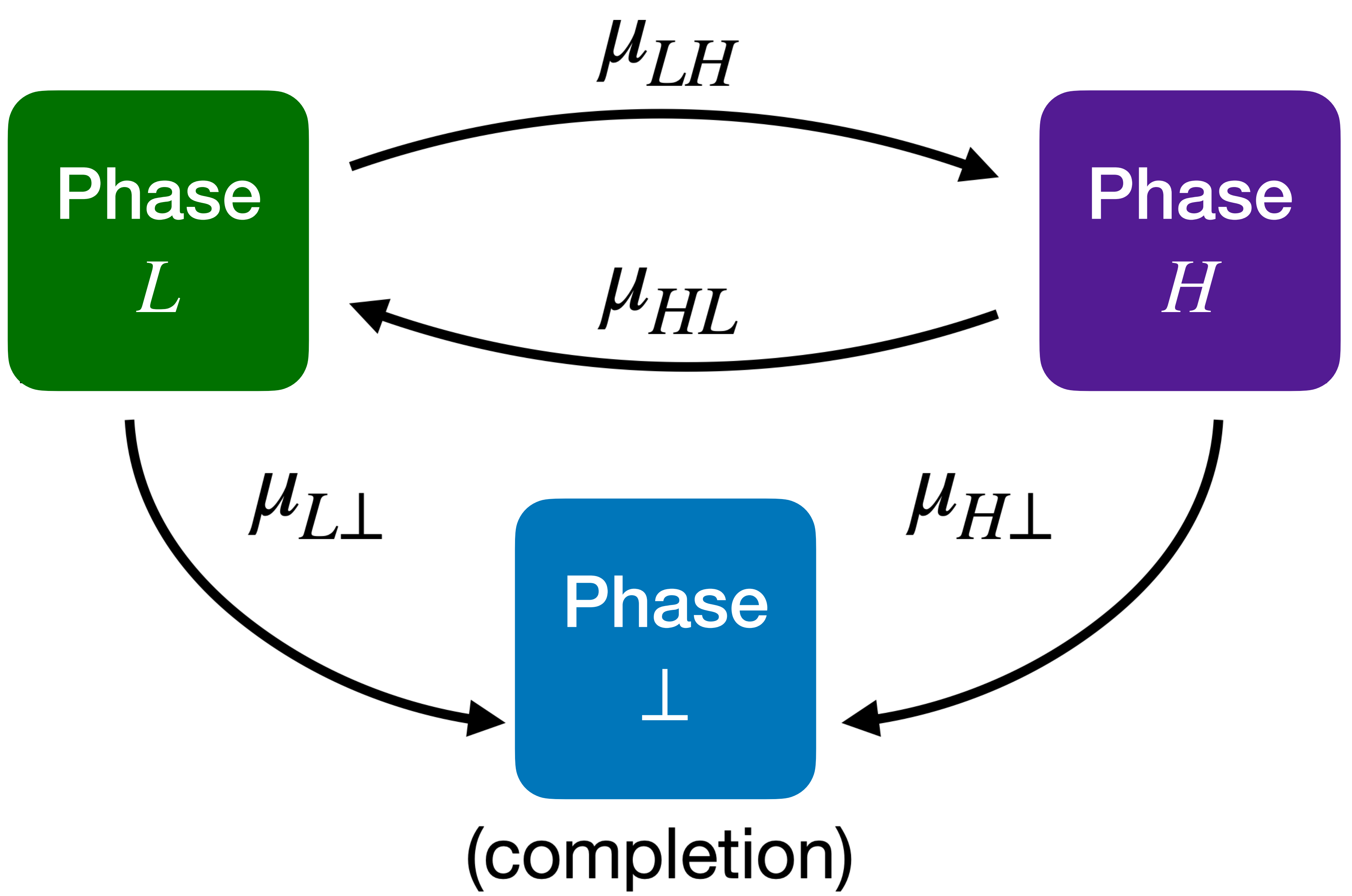
- Jobs arrive following a Poisson process



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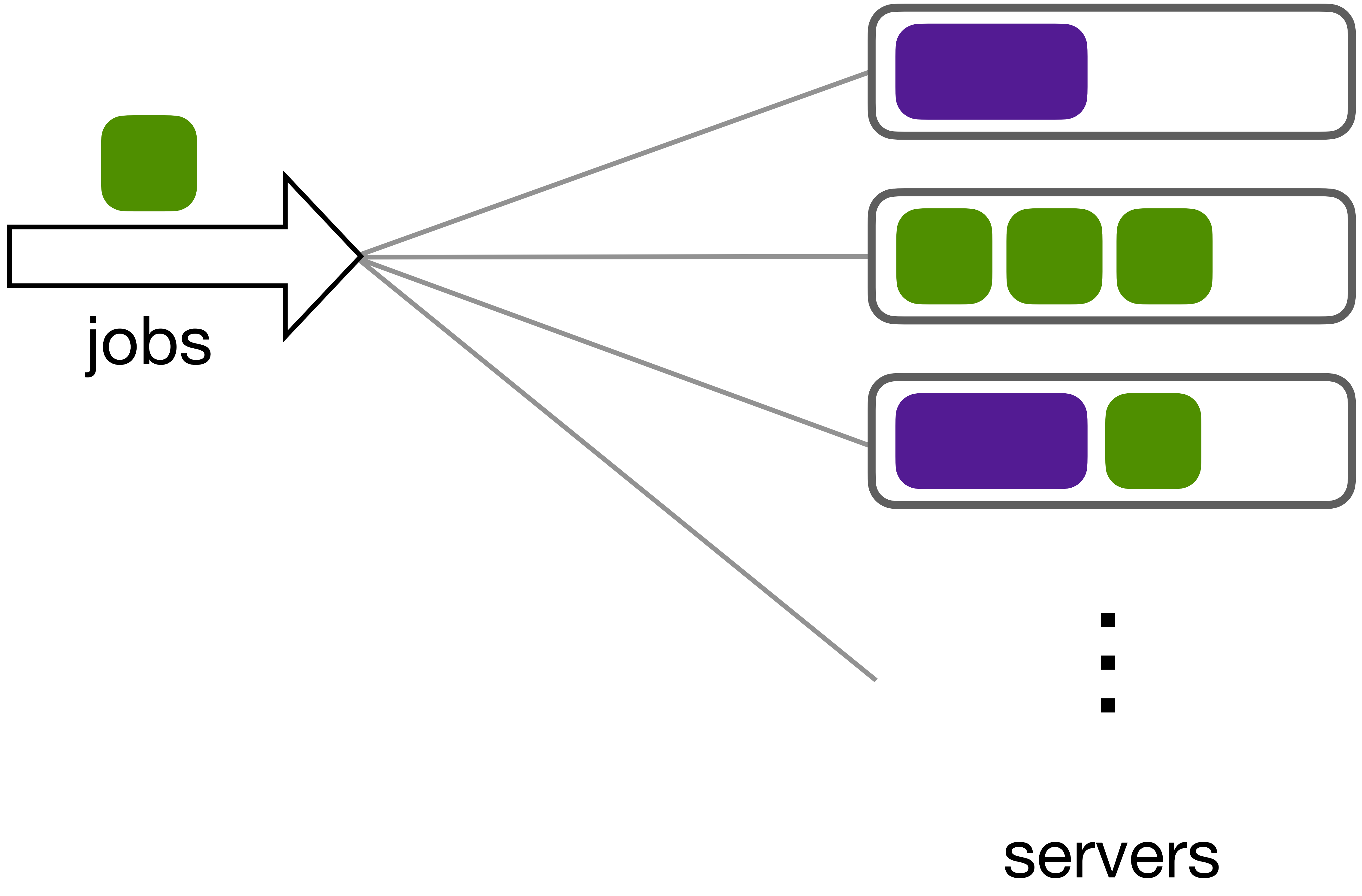
jobs
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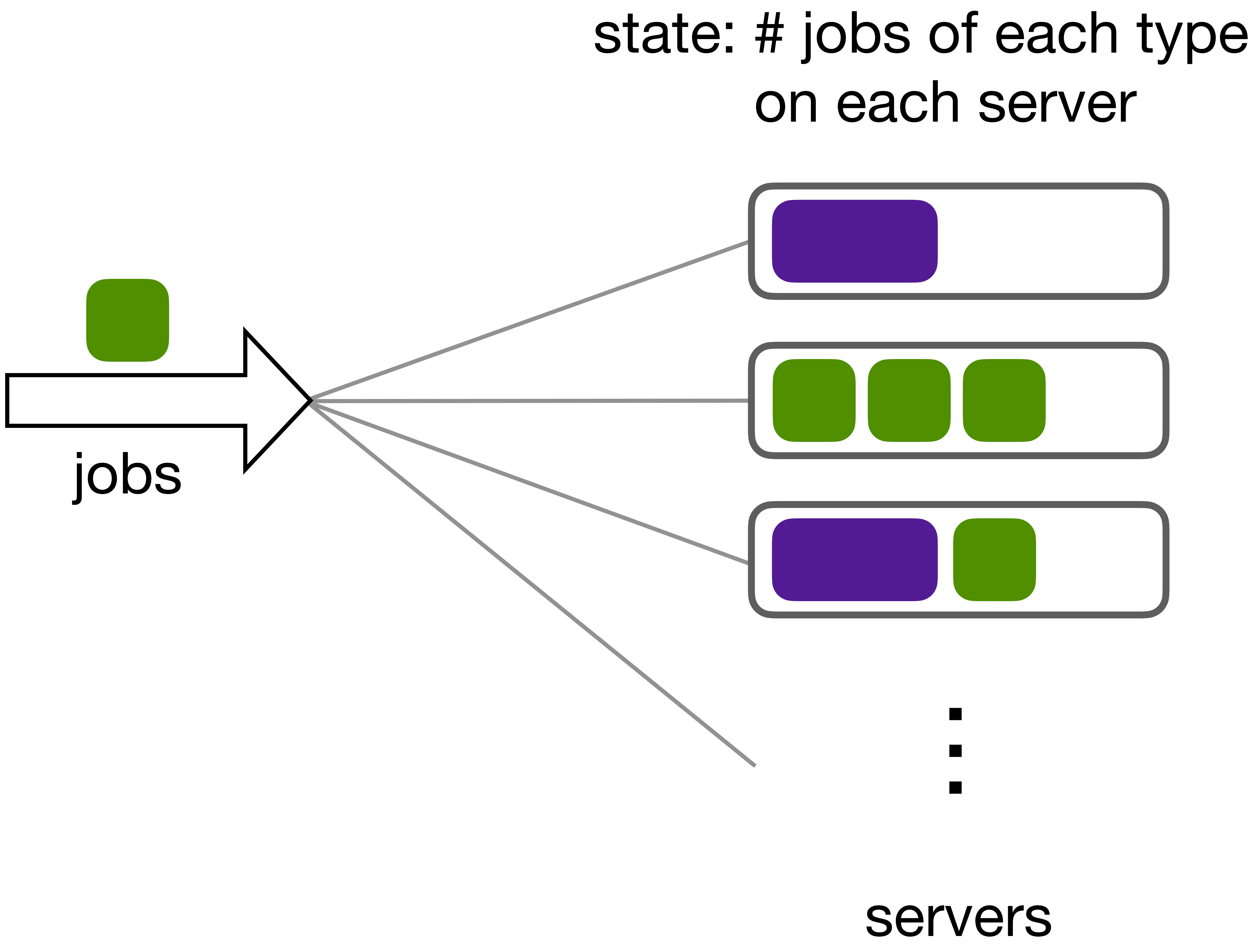
state: # jobs of each type
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state space is large!

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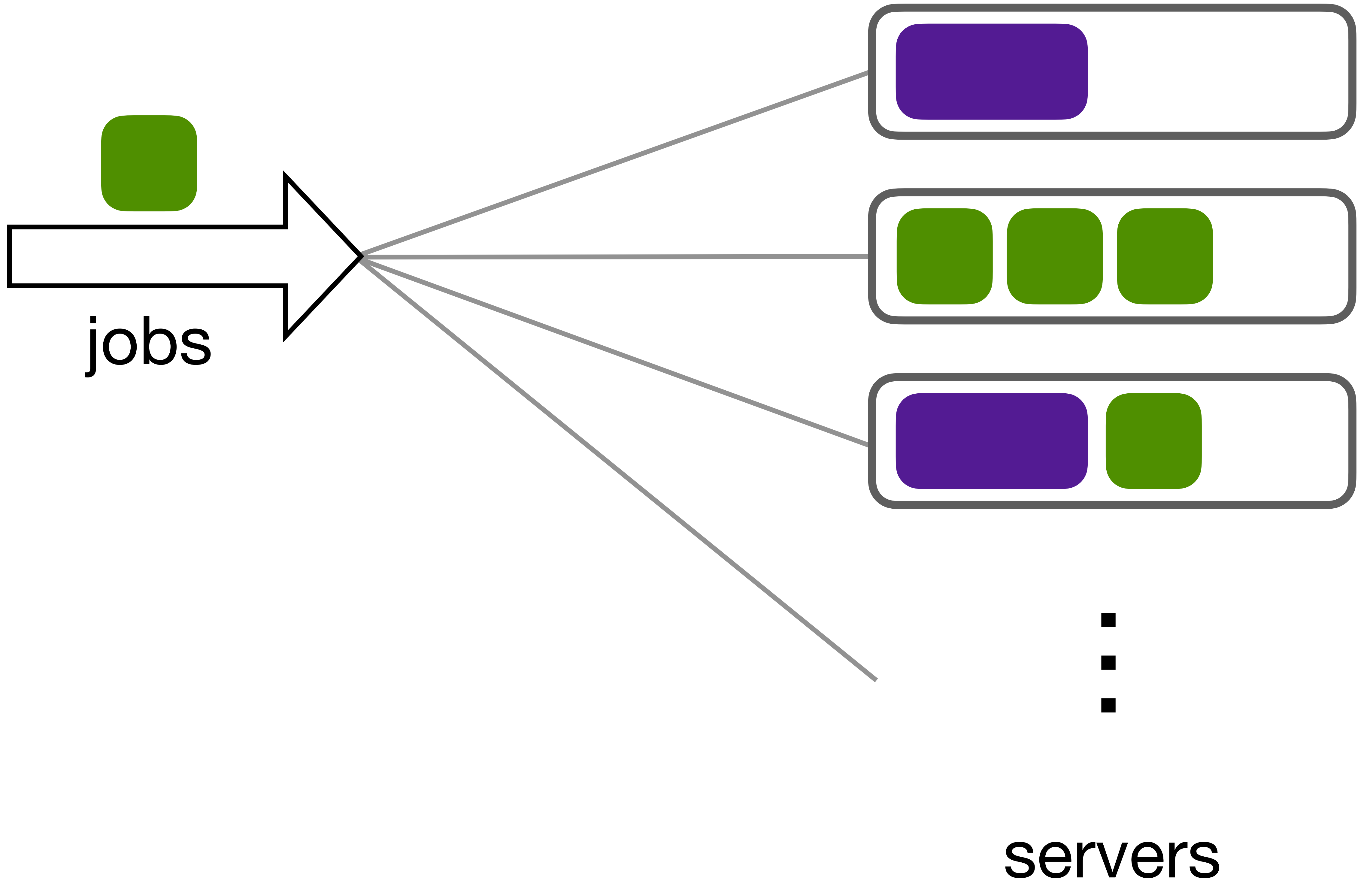

Reducing dimensionality

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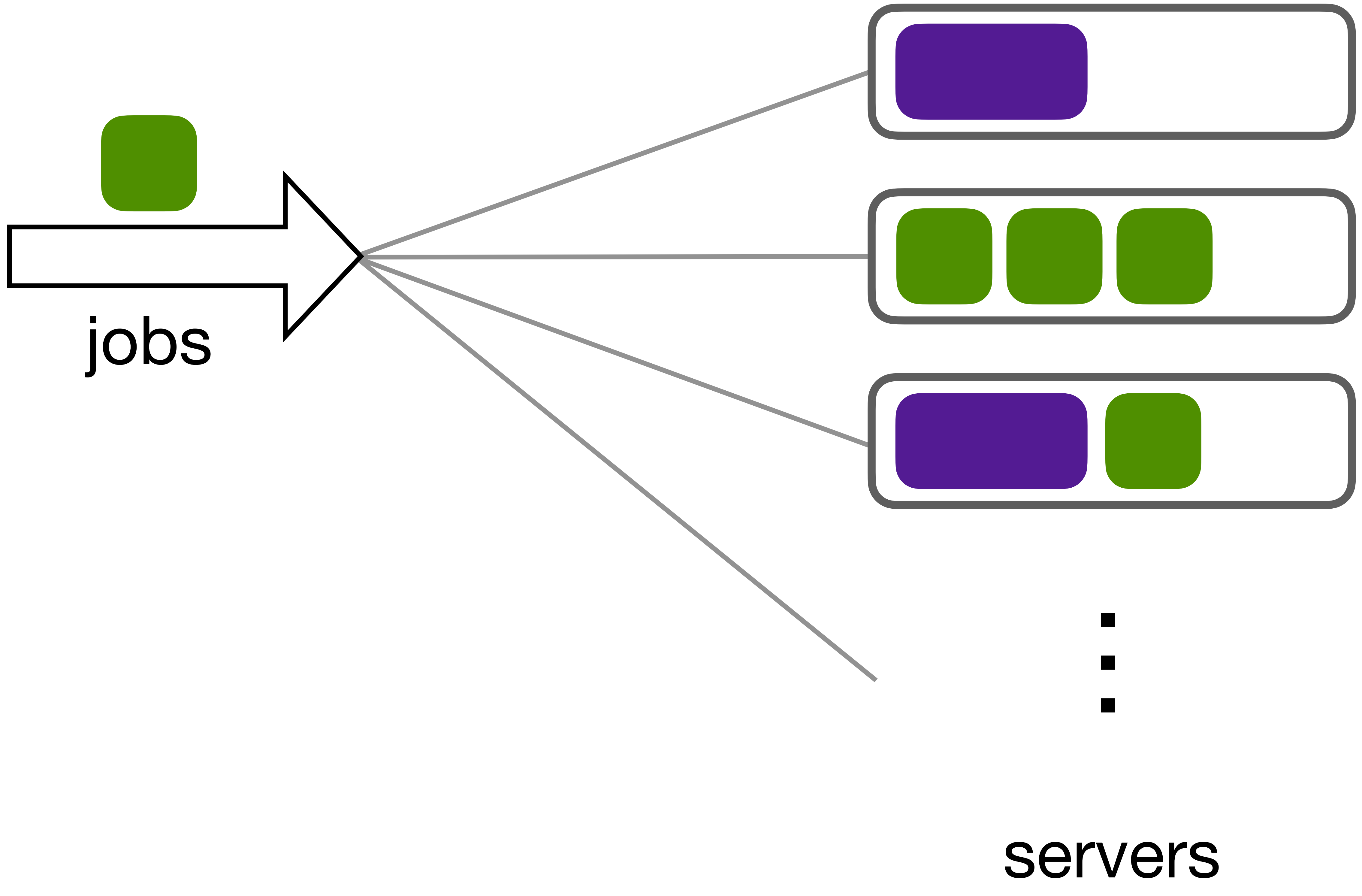

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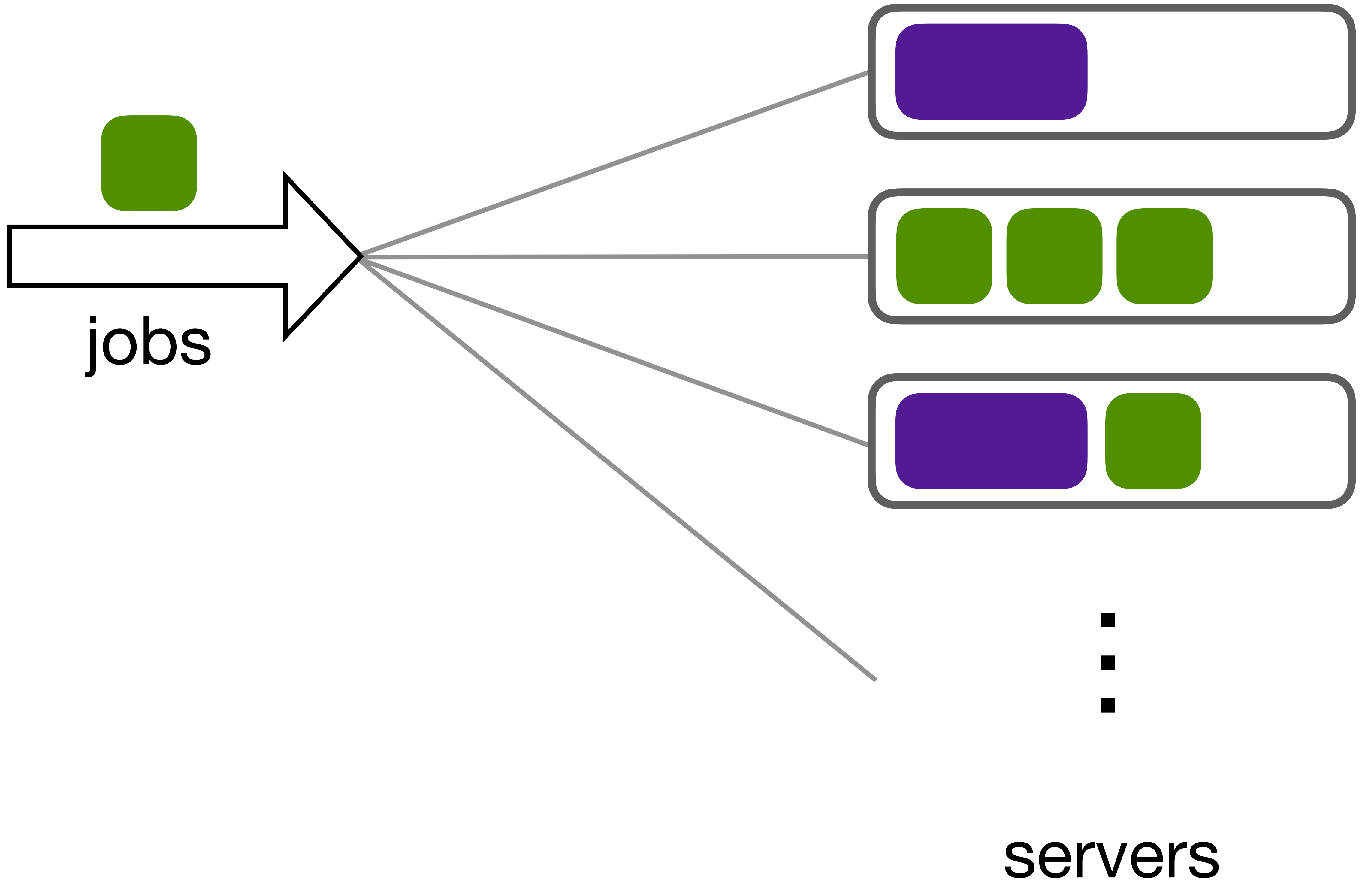
- Server-by-server evaluation:



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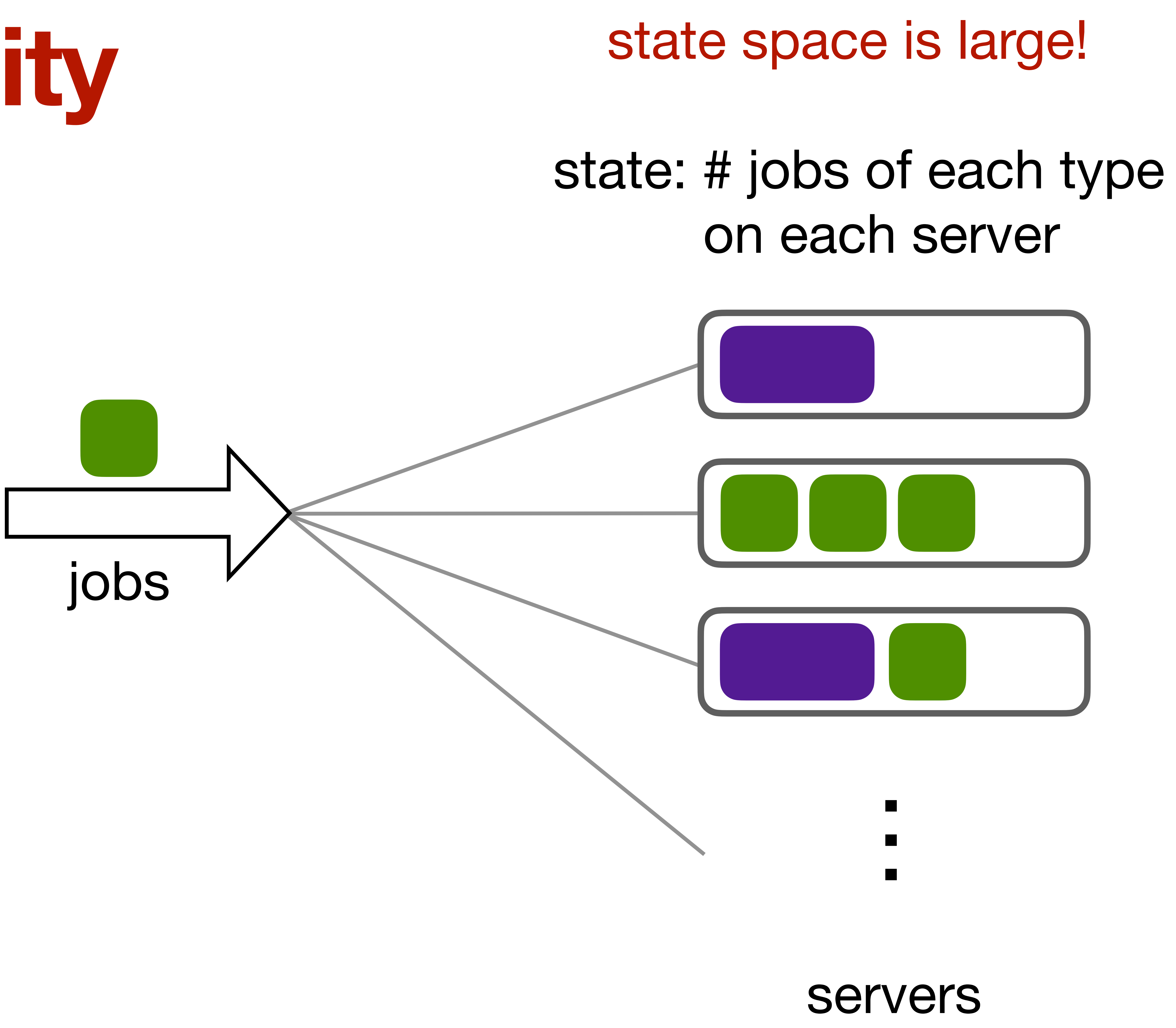
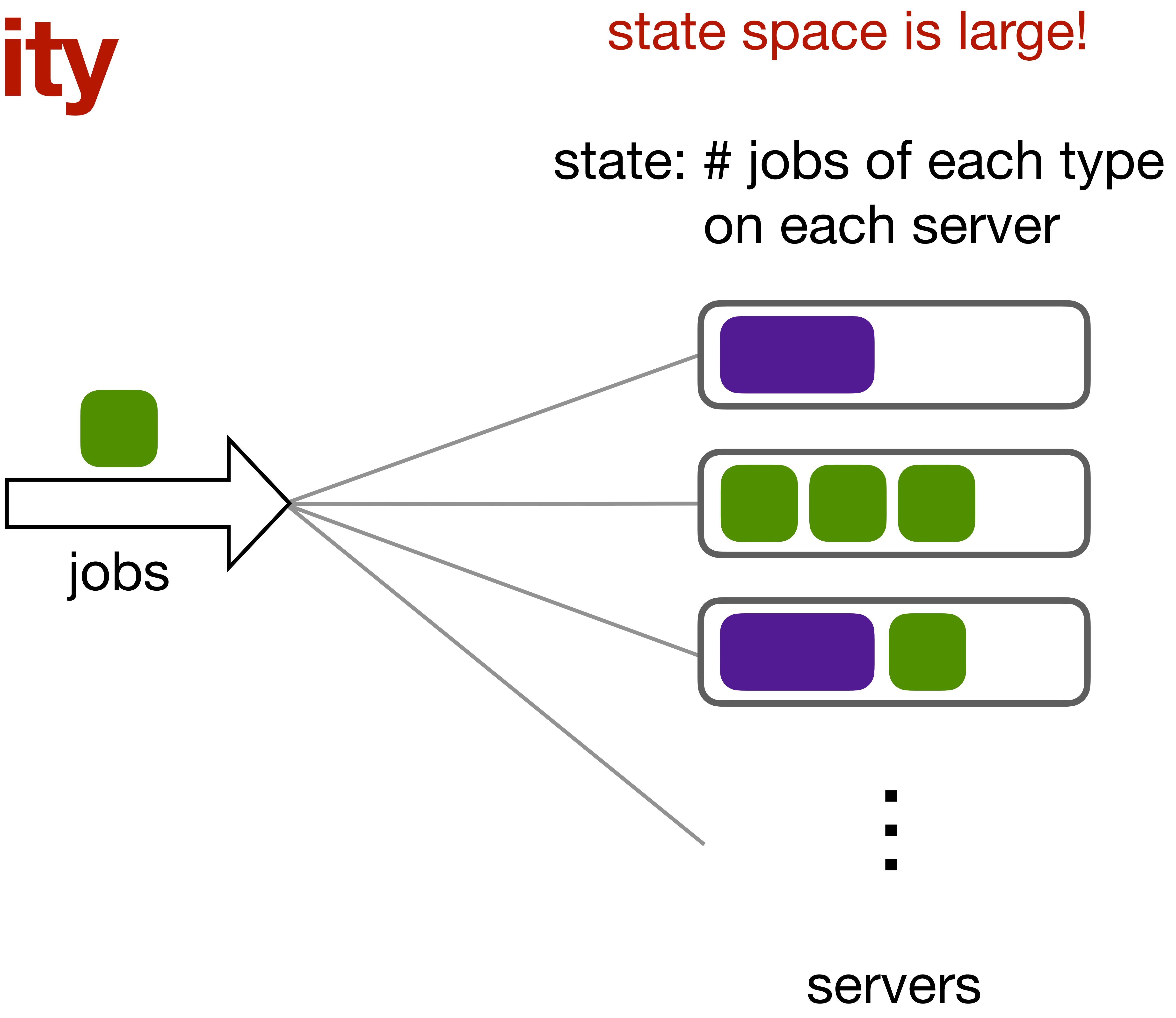
  - How to evaluate each server?



Reducing dimensionality

- Server-by-server evaluation:
  - How to evaluate each server?
  - How to relate to E[# active servers]?



A policy-conversion framework

Policies in the ∞-server system ⟷ Policies in a single-server system

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Single-server system

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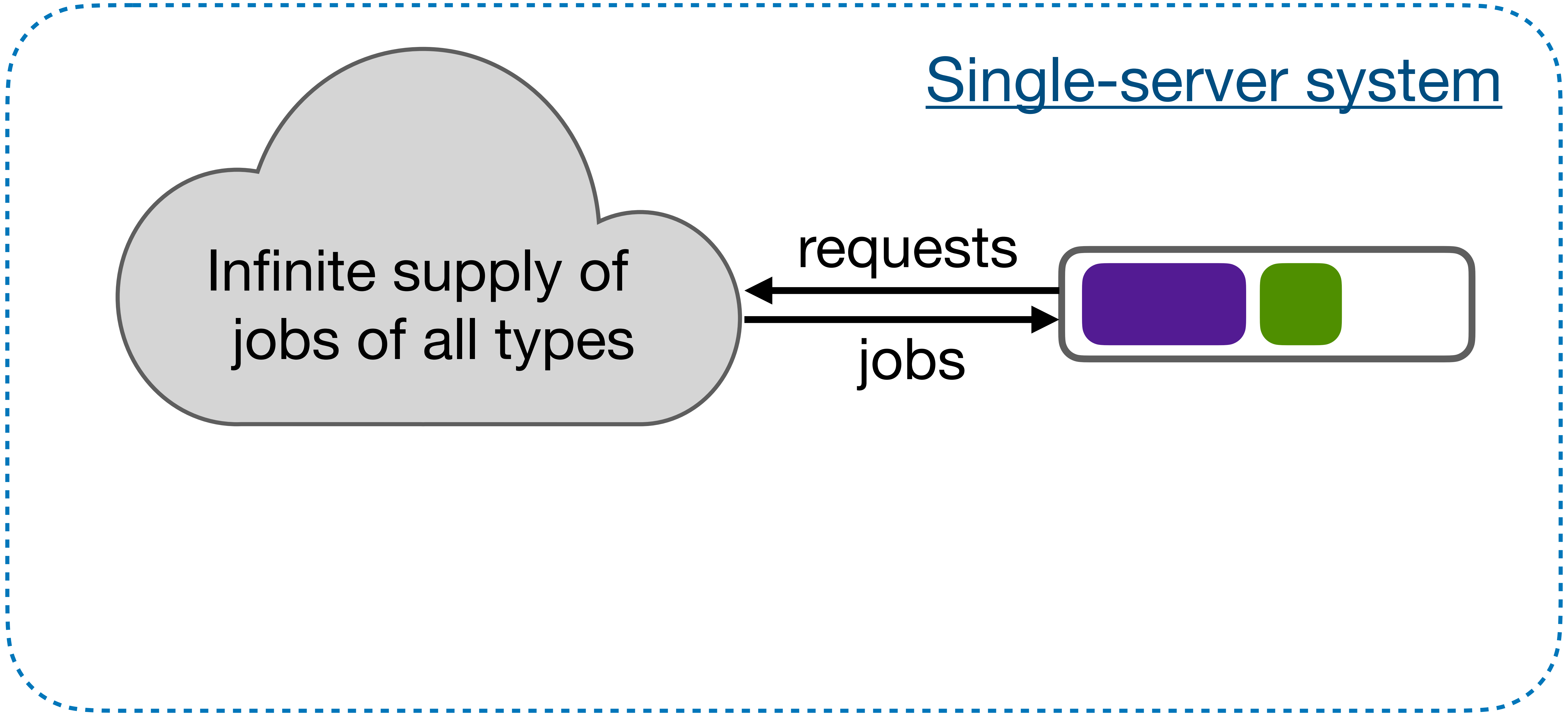

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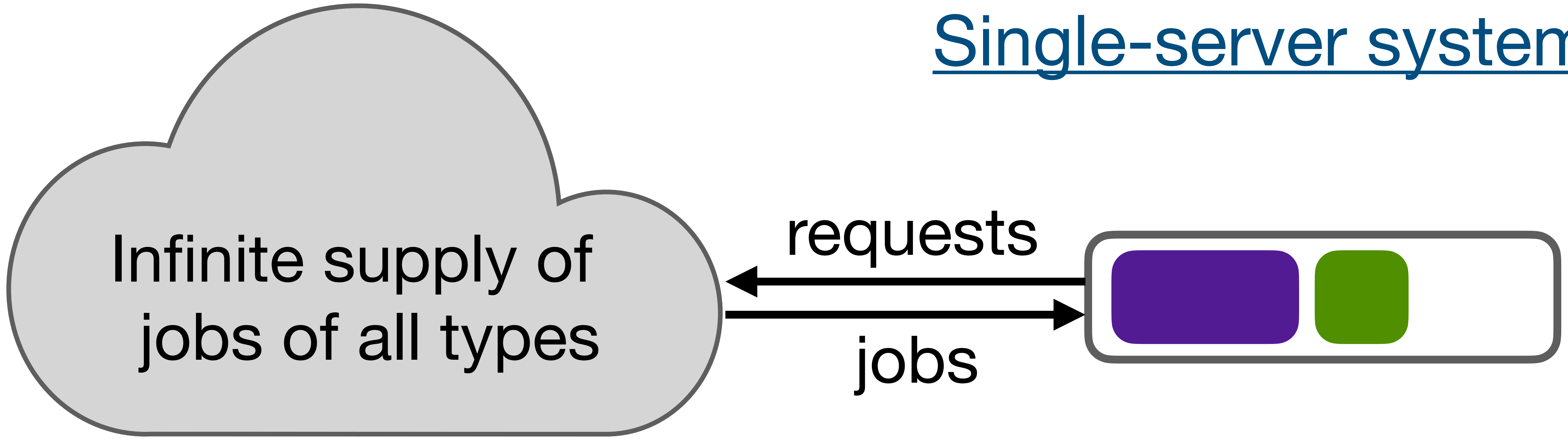

A policy $\overline{\sigma}$ decides when to request what types of jobs to:

maximize throughput

subject to **cost** (resource contention) ≤ budget

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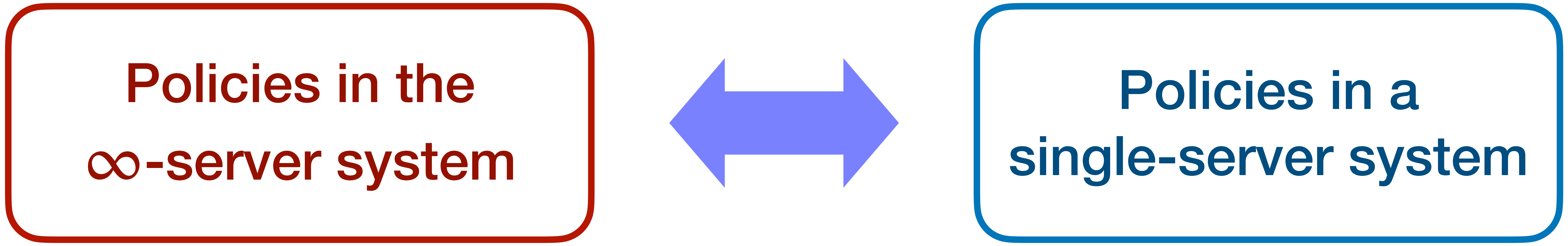

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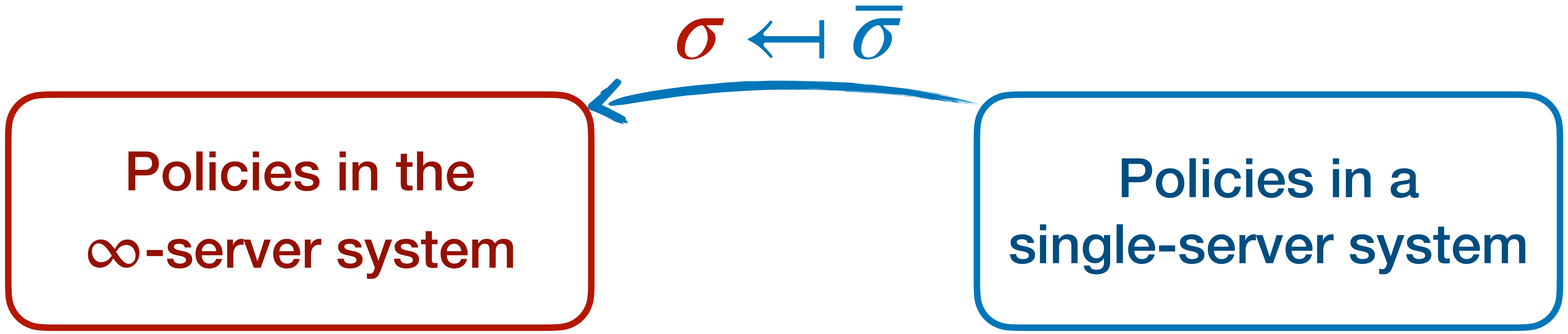

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- Use $\overline{\sigma}$ to tell how to evaluate each server
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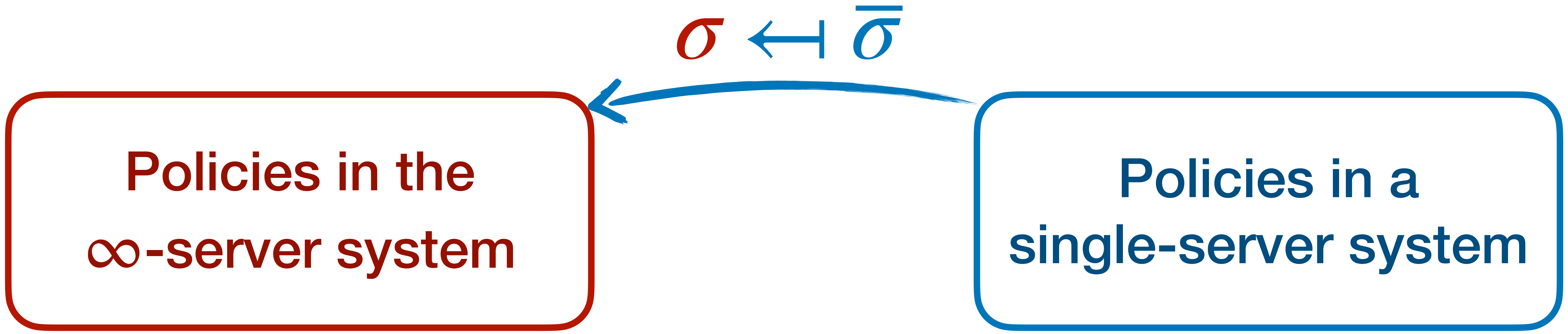

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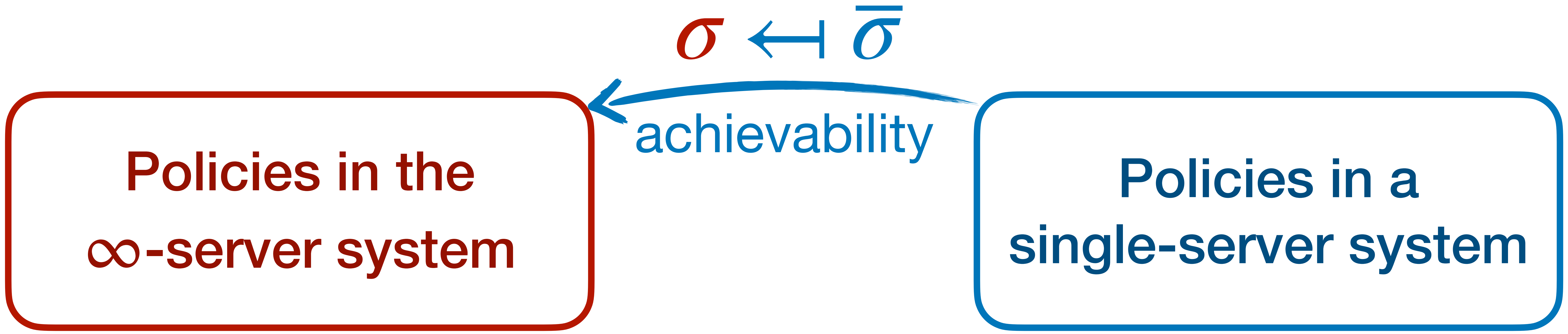

A policy-conversion framework

- Use $\overline{\sigma}$ to tell how to evaluate each server
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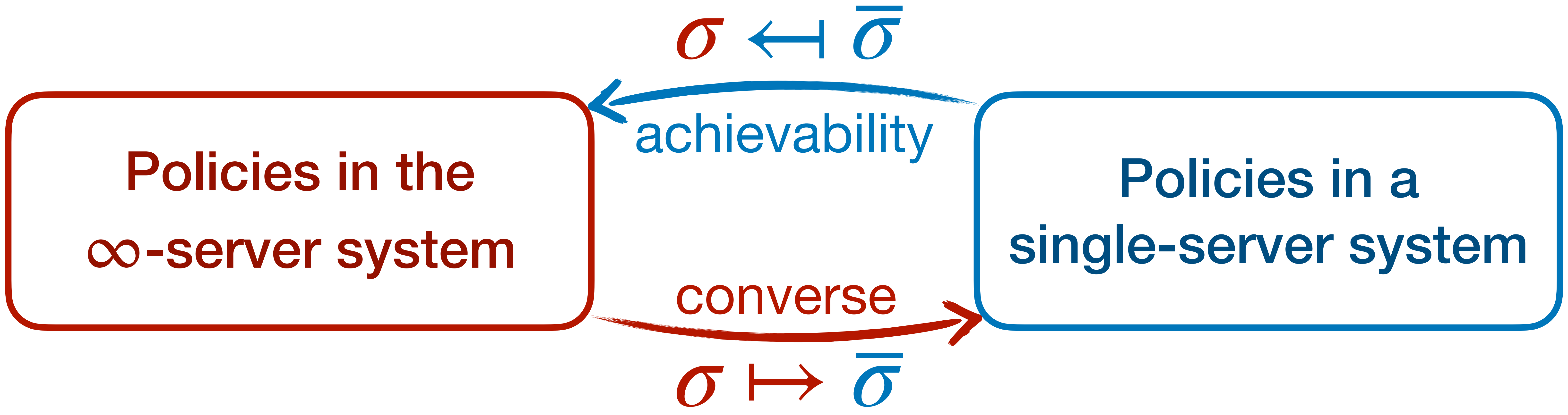

A policy-conversion framework

- Use $\overline{\sigma}$ to tell how to evaluate each server
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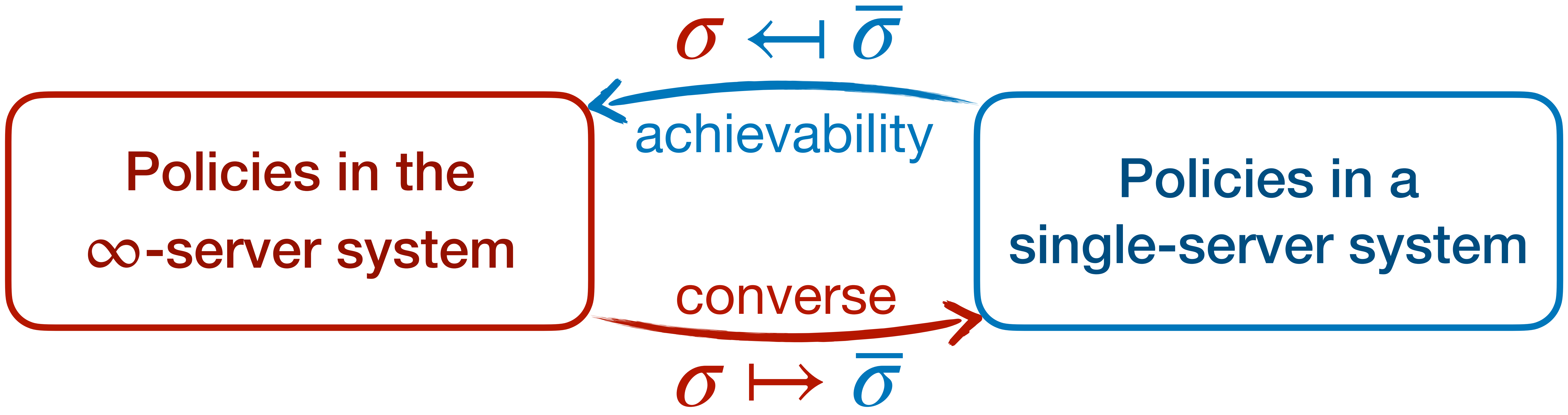


- Allows us to obtain lower bound on E[# active servers]

Policies in the ∞-server system

$\sigma \leftarrow\!\dashv \bar{\sigma}$

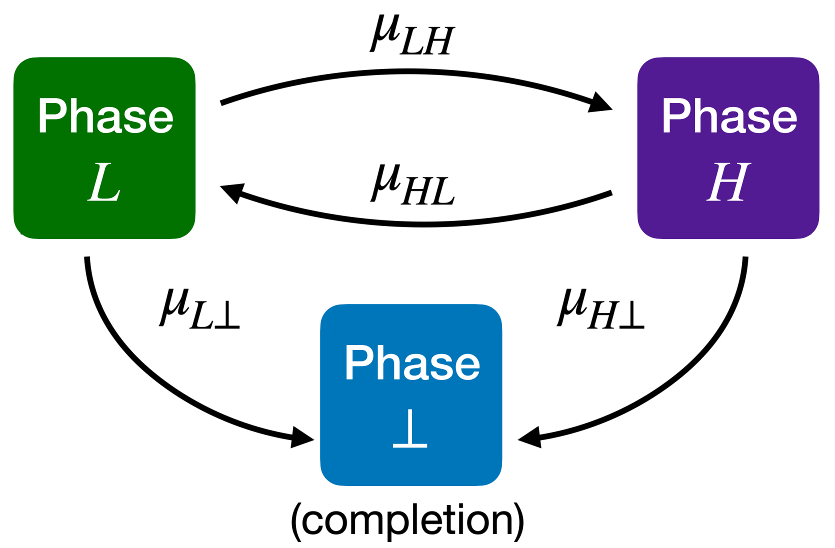
Policies in a single-server system

Policies in the ∞-server system

$\sigma \hookleftarrow \overline{\sigma}$

Policies in a single-server system

- Arrival rates: $r \cdot (\lambda_L, \lambda_H)$



Policies in the ∞-server system

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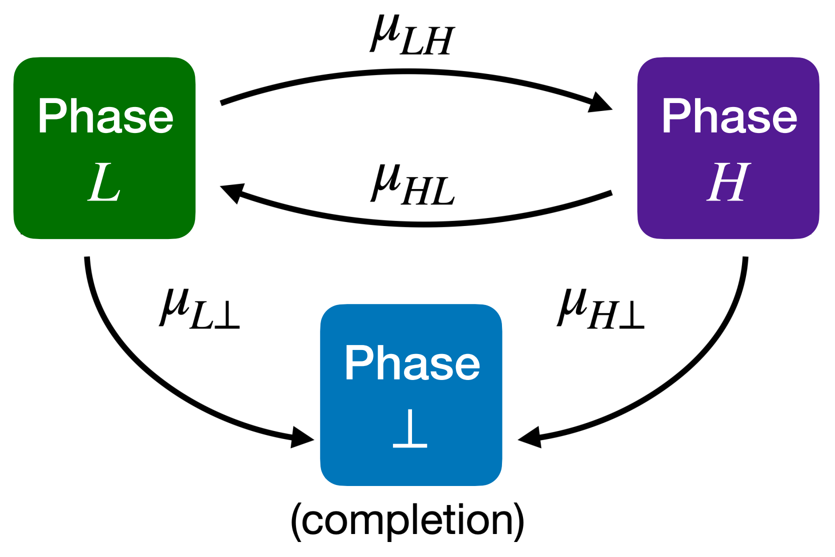
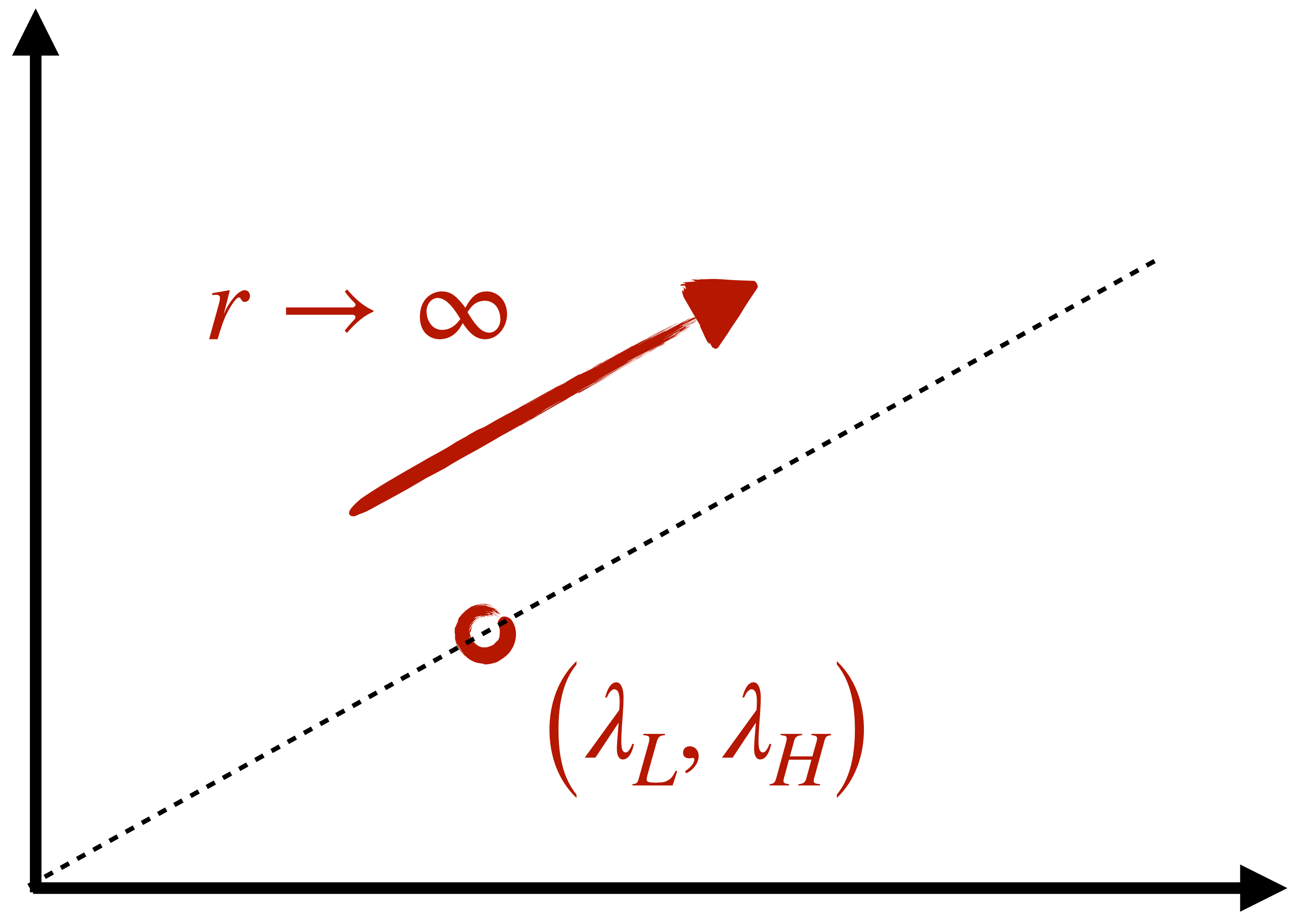
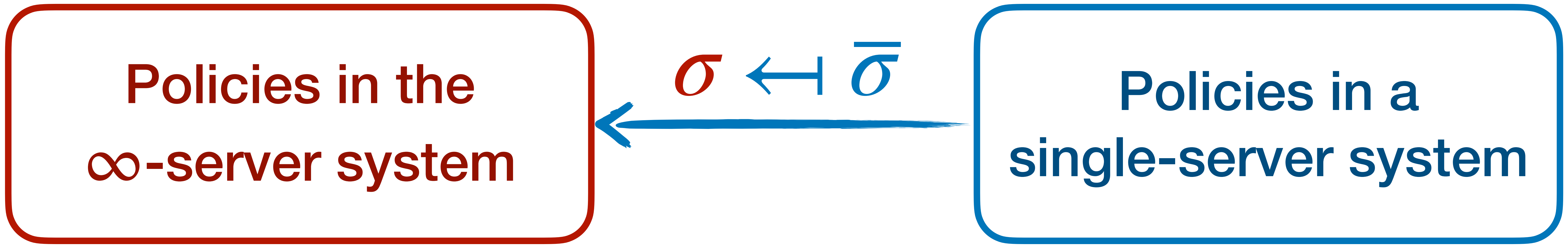
Policies in a single-server system

- Arrival rates: $r \cdot (\lambda_L, \lambda_H)$
- Asymptotic regime: $r \to +\infty$







- Arrival rates: $r \cdot (\lambda_L, \lambda_H)$
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Policies in the ∞-server system

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Policies in a single-server system

- Arrival rates: $r \cdot (\lambda_L, \lambda_H)$
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Policy $\bar{\sigma}$

$\bar{N} \cdot \left(\textbf{throughput}_L, \textbf{throughput}_R\right) = r \cdot (\lambda_L, \lambda_H)$

cost (resource contention) ≤ budget

Policies in the ∞-server system

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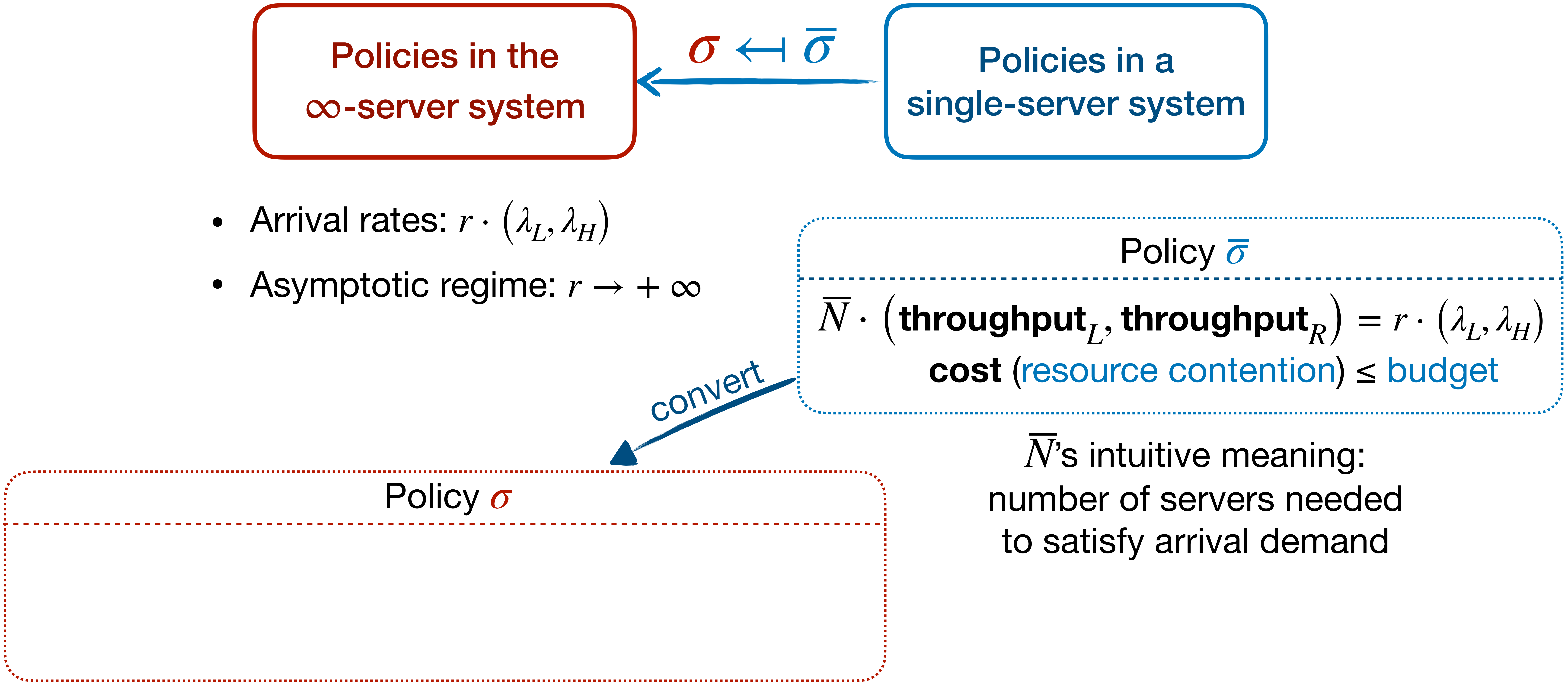
$\bar{N}$'s intuitive meaning:
number of servers needed
to satisfy arrival demand



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cost (resource contention) ≤ budget

convert

Policy σ

E [# active servers] $\leq \left(1 + O\left(r^{-0.5}\right)\right) \cdot \bar{N}$

$\bar{N}$'s intuitive meaning: number of servers needed to satisfy arrival demand

Policies in the ∞-server system

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Policies in the ∞-server system $\xleftarrow{\sigma \leftarrow \overline{\sigma}}$ Policies in a single-server system

- Arrival rates: $r \cdot (\lambda_L, \lambda_H)$
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Policy $\overline{\sigma}^*$

$\overline{N}^* \cdot \left(\textbf{throughput}_L, \textbf{throughput}_R\right) = r \cdot (\lambda_L, \lambda_H)$

cost (resource contention) ≤ budget

convert

Policy σ^*

$\mathbf{E}$ [# active servers] $\leq \left(1 + O\left(r^{-0.5}\right)\right) \cdot \overline{N}^*$

cost (resource contention) $\leq \left(1 + O\left(r^{-0.5}\right)\right) \cdot$ budget

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Policies in a single-server system

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cost (resource contention) ≤ budget

$\overline{\sigma}^*$ maximizes the throughput and thus achieves the minimum $\overline{N}^*$

convert

Policy σ^*

E [# active servers] $\leq \left(1 + O\left(r^{-0.5}\right)\right) \cdot \overline{N}^*$

cost (resource contention) $\leq \left(1 + O\left(r^{-0.5}\right)\right) \cdot$ budget

Policies in the
∞-server system

$\sigma \leftarrow \overline{\sigma}$

Policies in a
single-server system

Main Results:

Policy $\overline{\sigma}^*$

$\overline{N}^* \cdot \left(\textbf{throughput}_L, \textbf{throughput}_R\right) = r \cdot (\lambda_L, \lambda_H)$

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Policy σ^*

$$\textbf{E}\,[\text{\# active servers}] \le \left(1 + O\left(r^{-0.5}\right)\right) \cdot \overline{N}^*$$

$$\textbf{cost}\ (\text{resource contention}) \le \left(1 + O\left(r^{-0.5}\right)\right) \cdot \text{budget}$$

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Policies in a single-server system

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Policies in the ∞-server system

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Policies in a single-server system

Main Results:

- E[# active servers] $\geq \overline{N}^*$ under any policy
- The policy σ^* converted from $\overline{\sigma}^*$ is *asymptotically optimal*

Policy $\overline{\sigma}^*$

$$\overline{N}^* \cdot \left(\textbf{throughput}_L, \textbf{throughput}_R\right) = r \cdot (\lambda_L, \lambda_H)$$

cost (resource contention) ≤ budget

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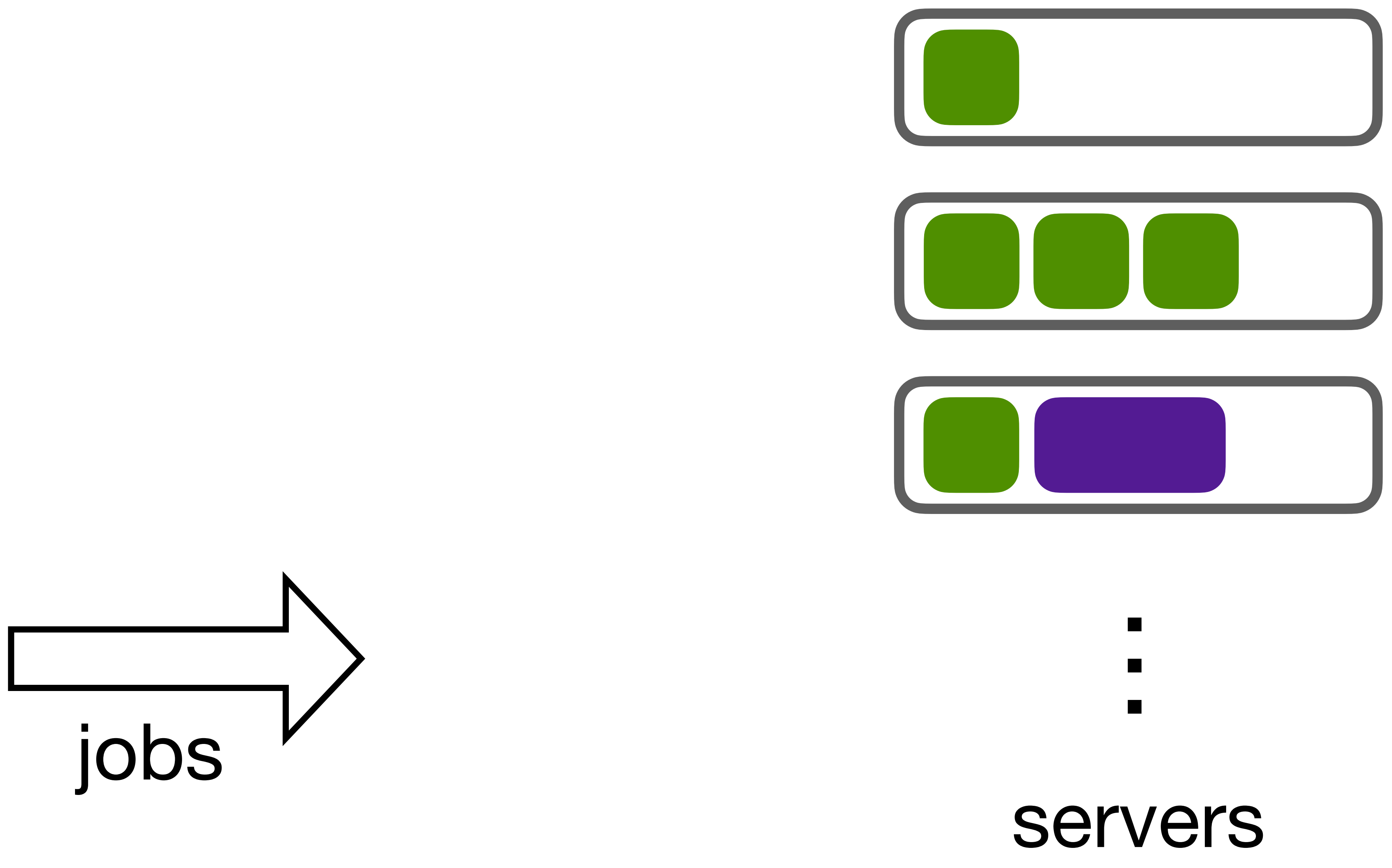
Policy conversion: single-server to ∞-server

Meta-algorithm: JOIN-REQUESTING-SERVER ($\vec{\sigma}$)



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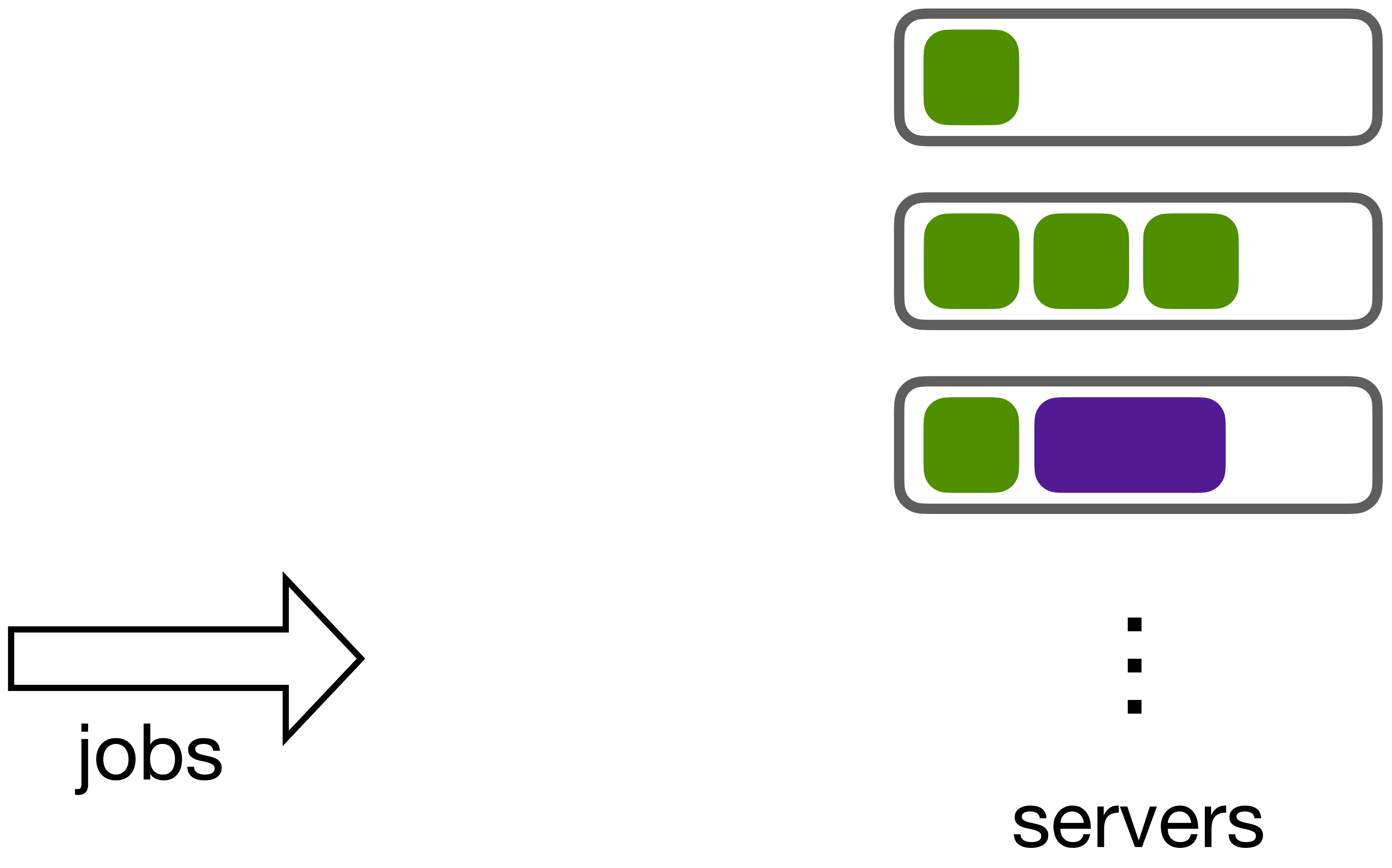
- For each server, run a single-server policy $\overline{\sigma}$



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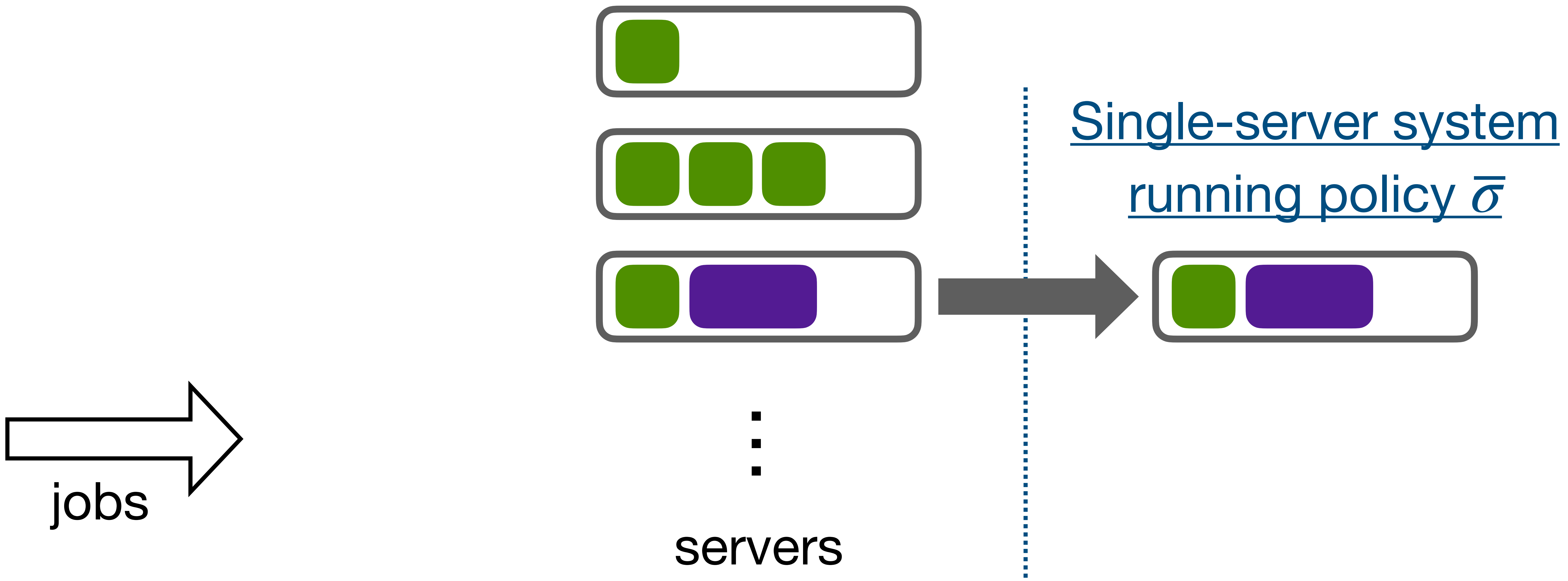

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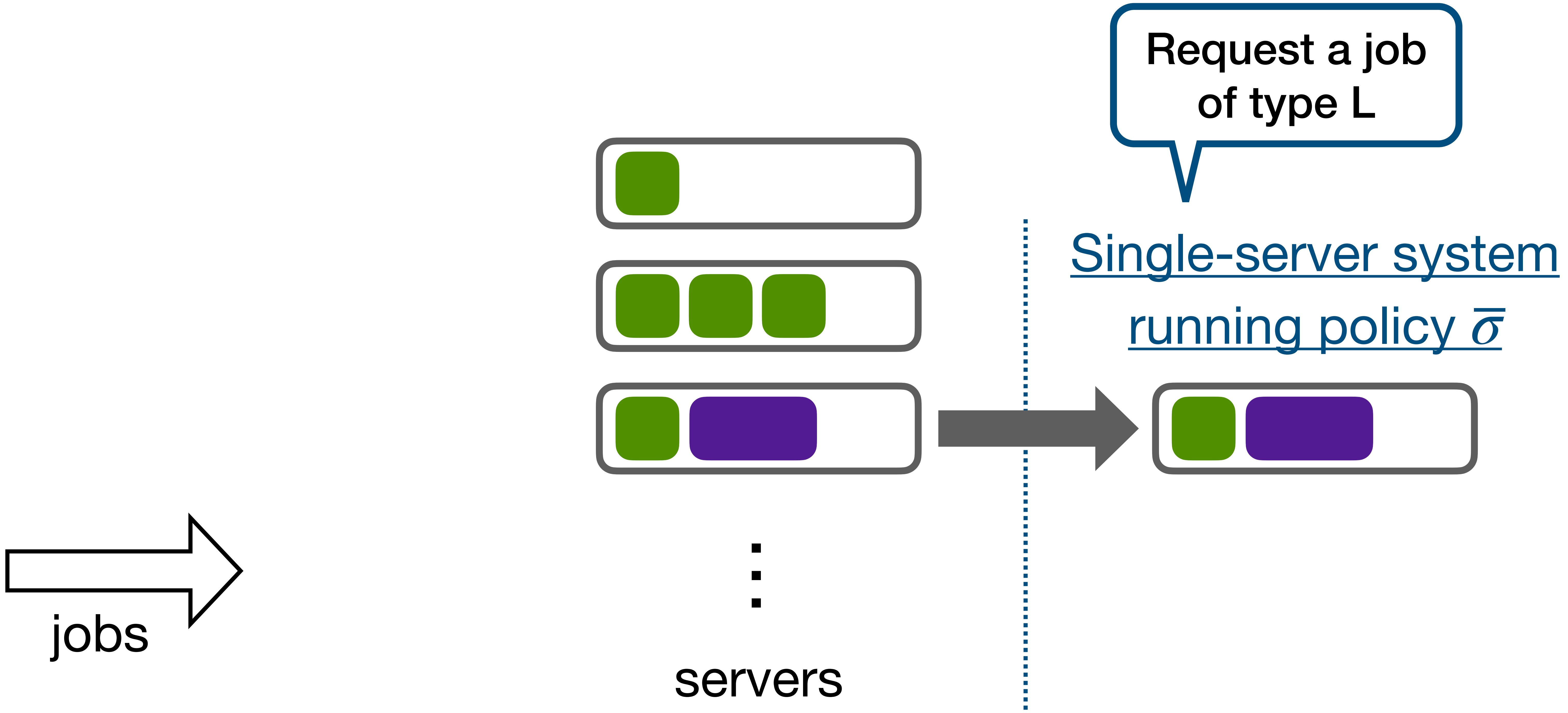
- If $\overline{\sigma}$ requests a job of type i, generate a token of type i



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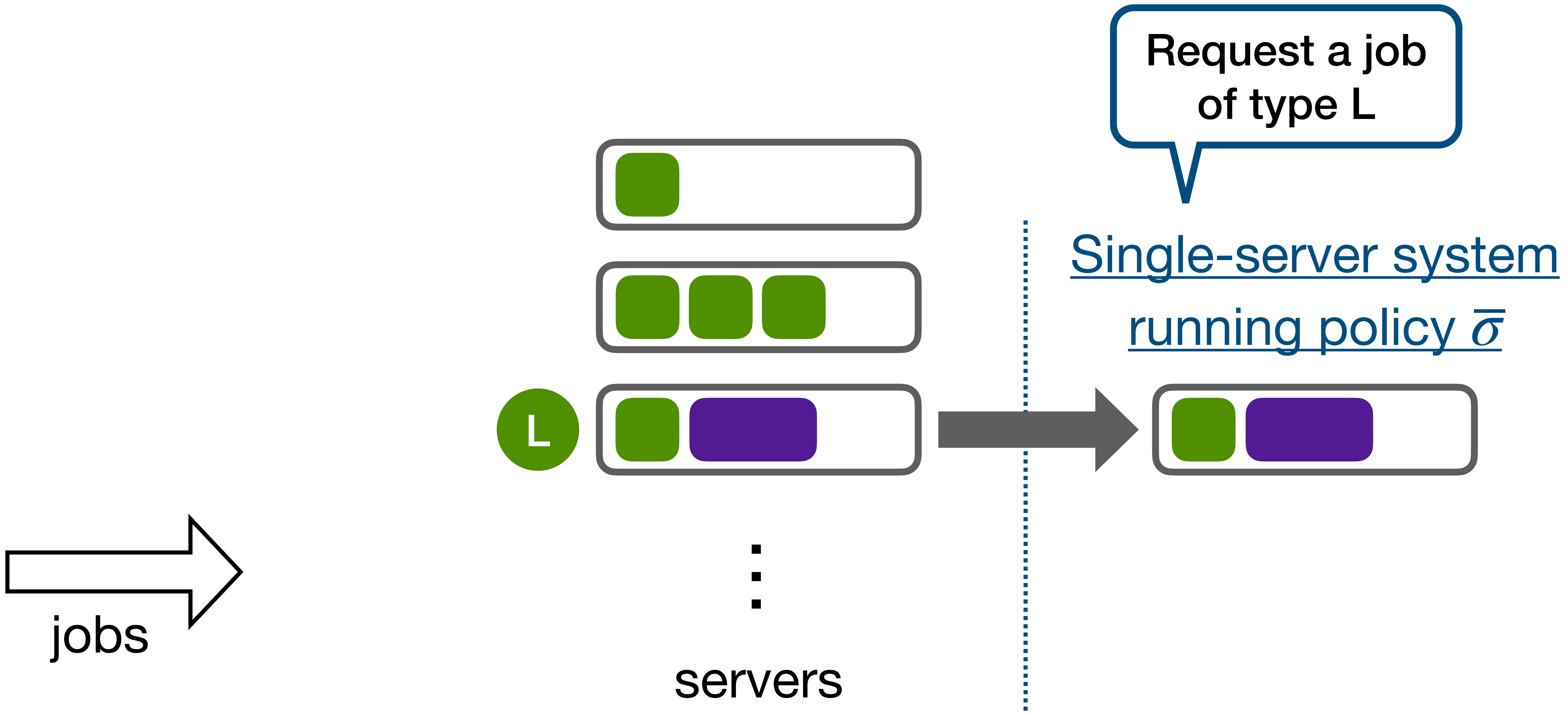

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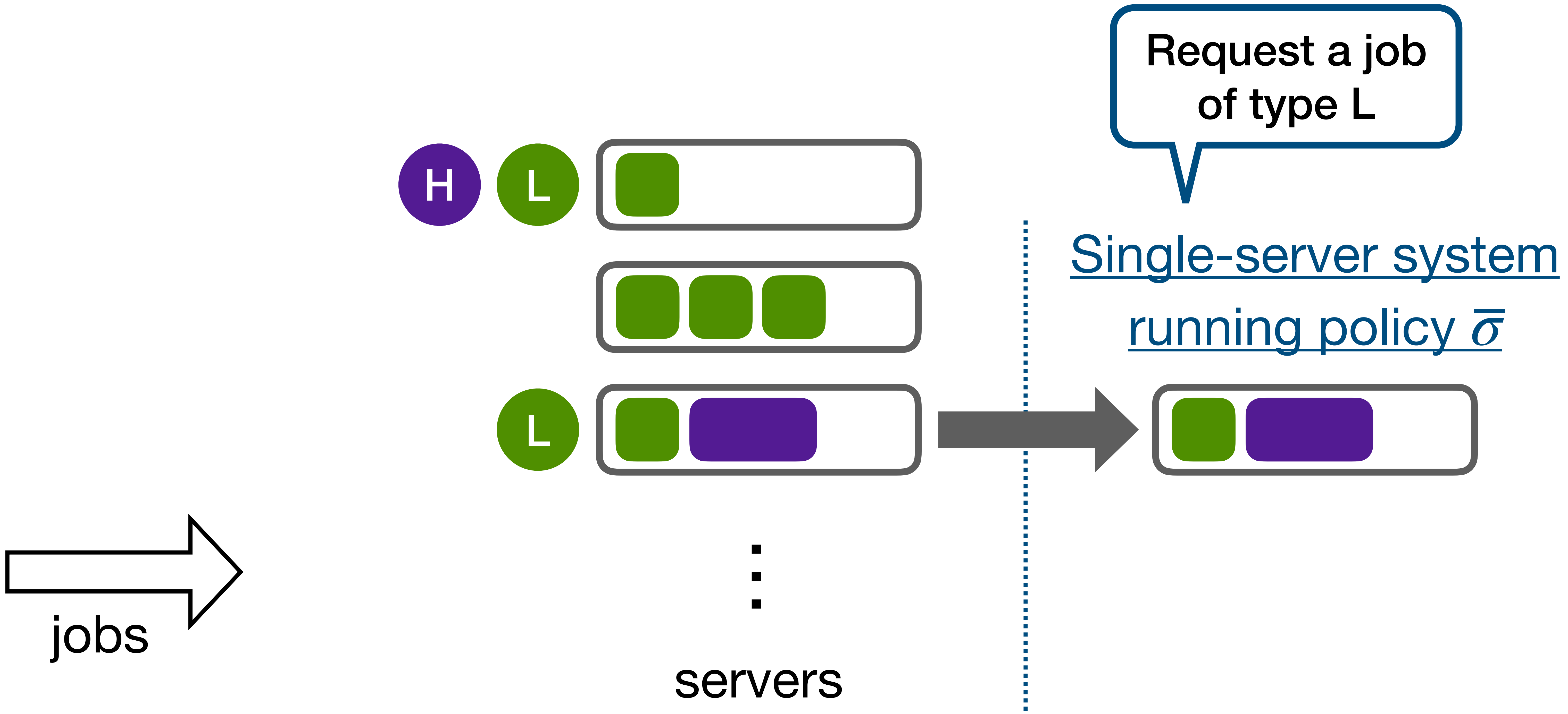

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- For each server, run a single-server policy $\overline{\sigma}$
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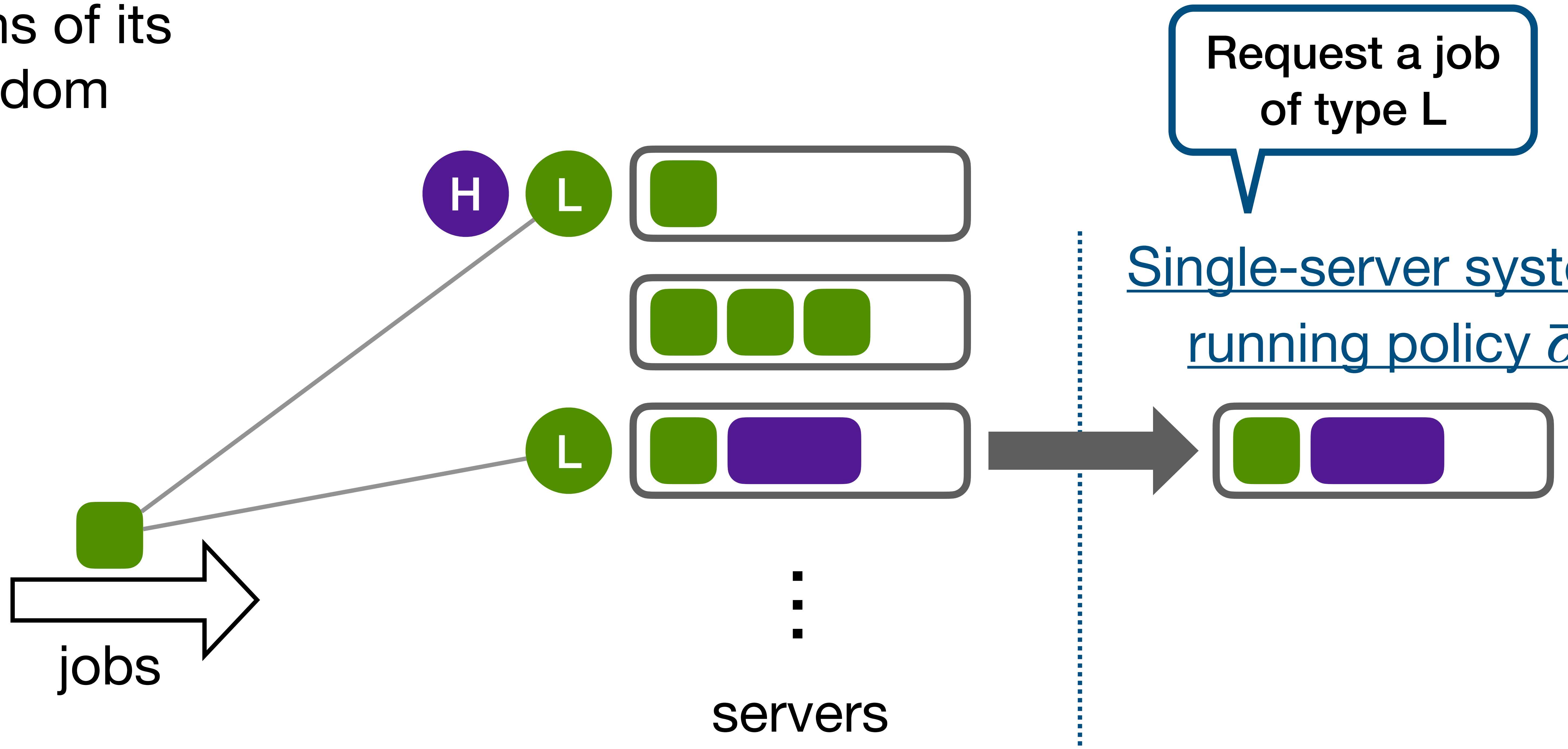

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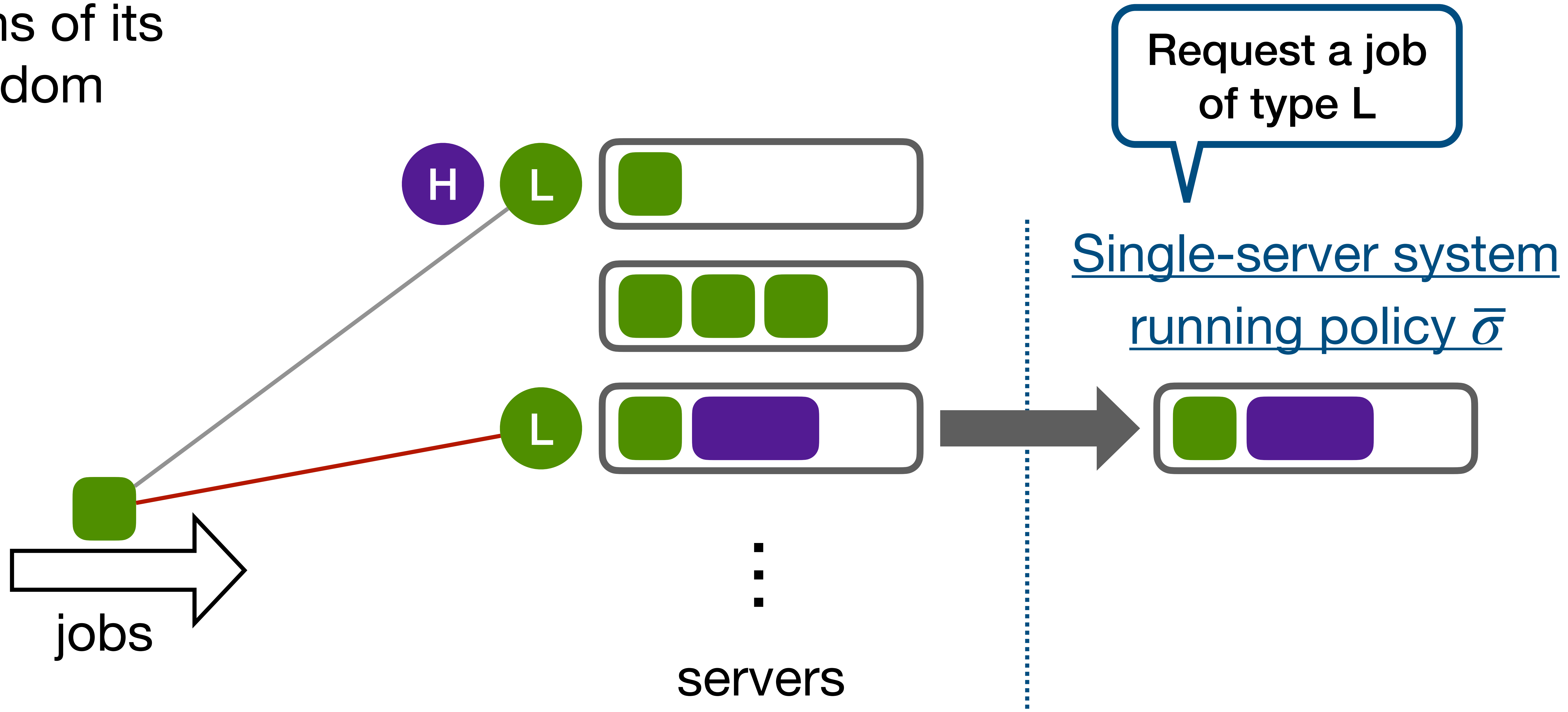

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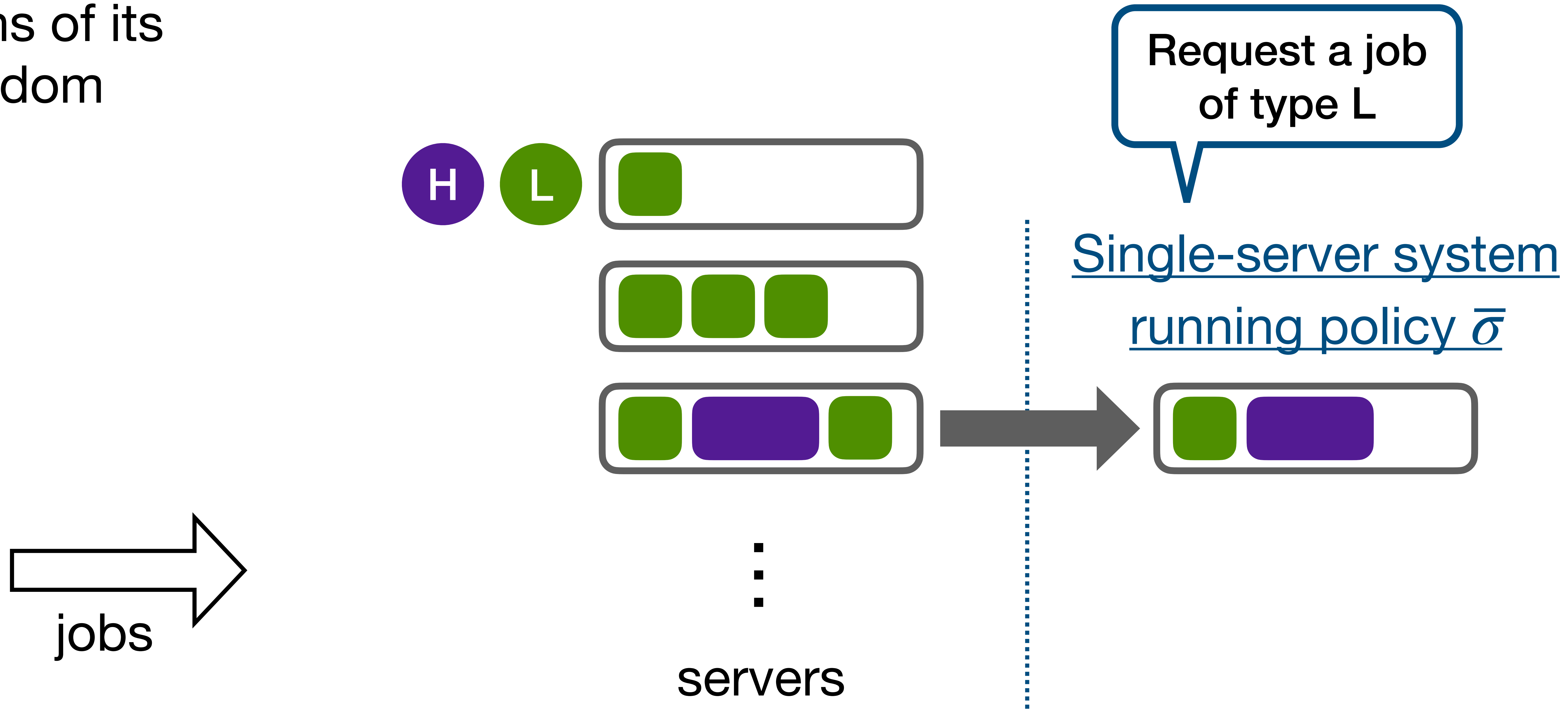

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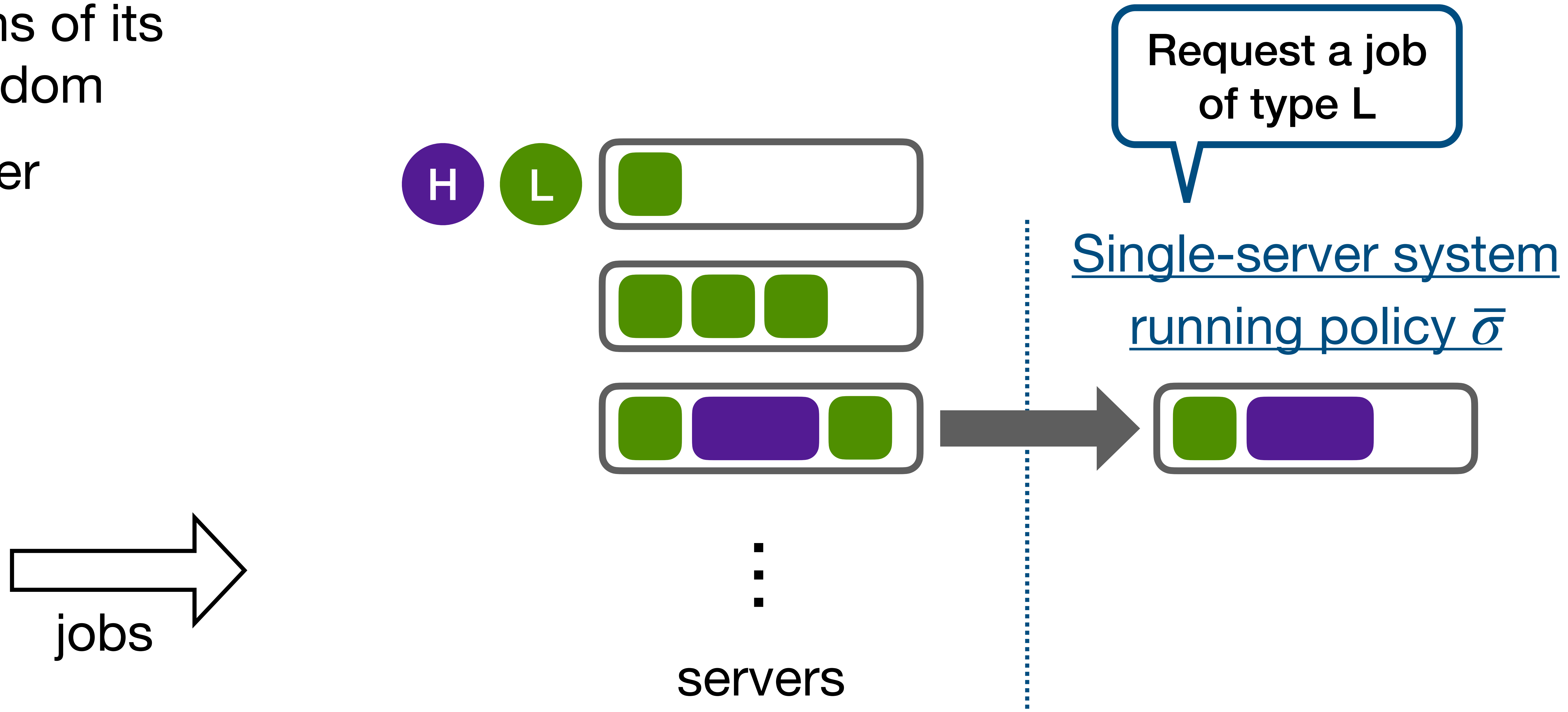
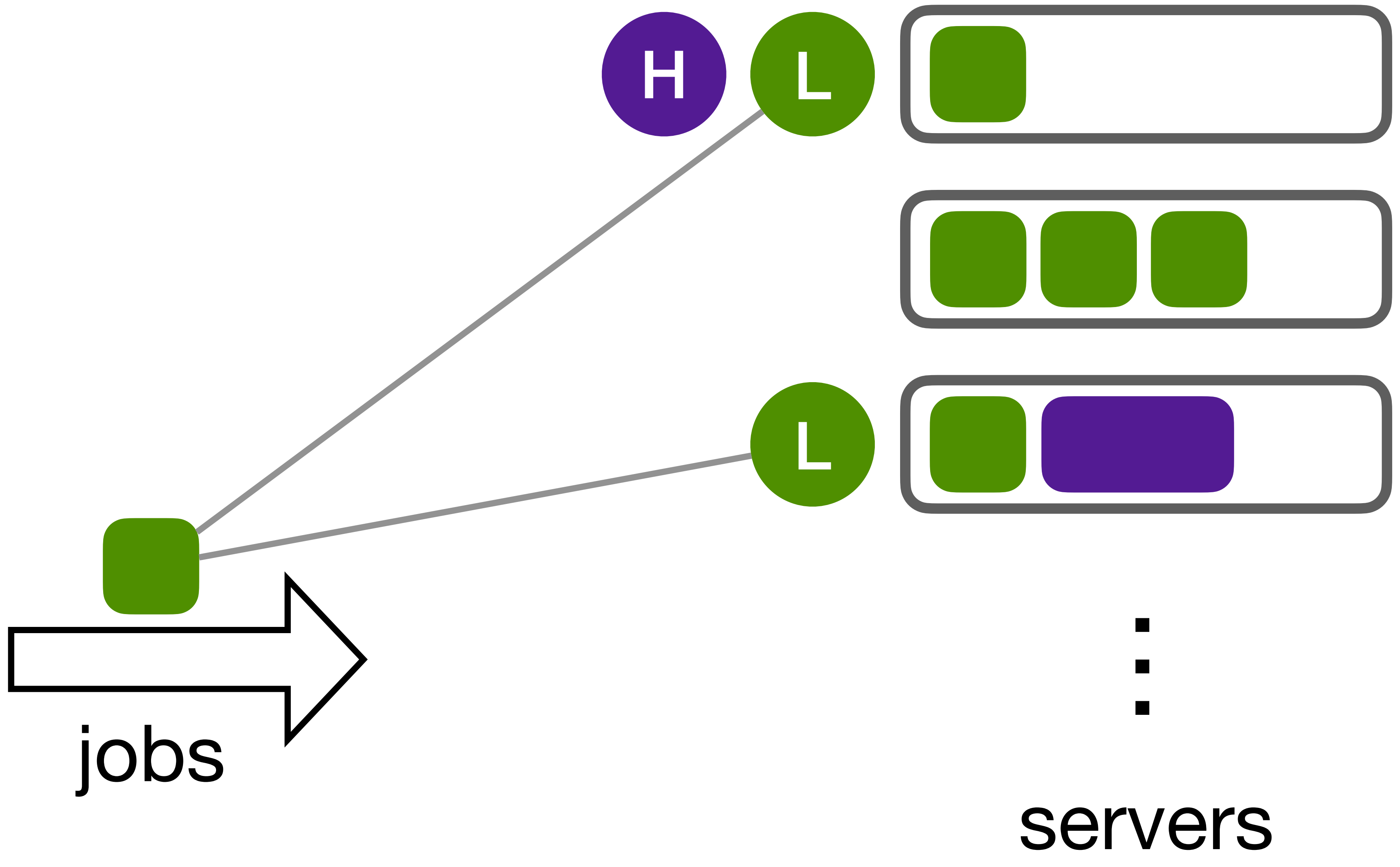
- When a job arrives, it checks tokens of its type and joins one uniformly at random
- If no tokens, go to an inactive server



Policy conversion: more details



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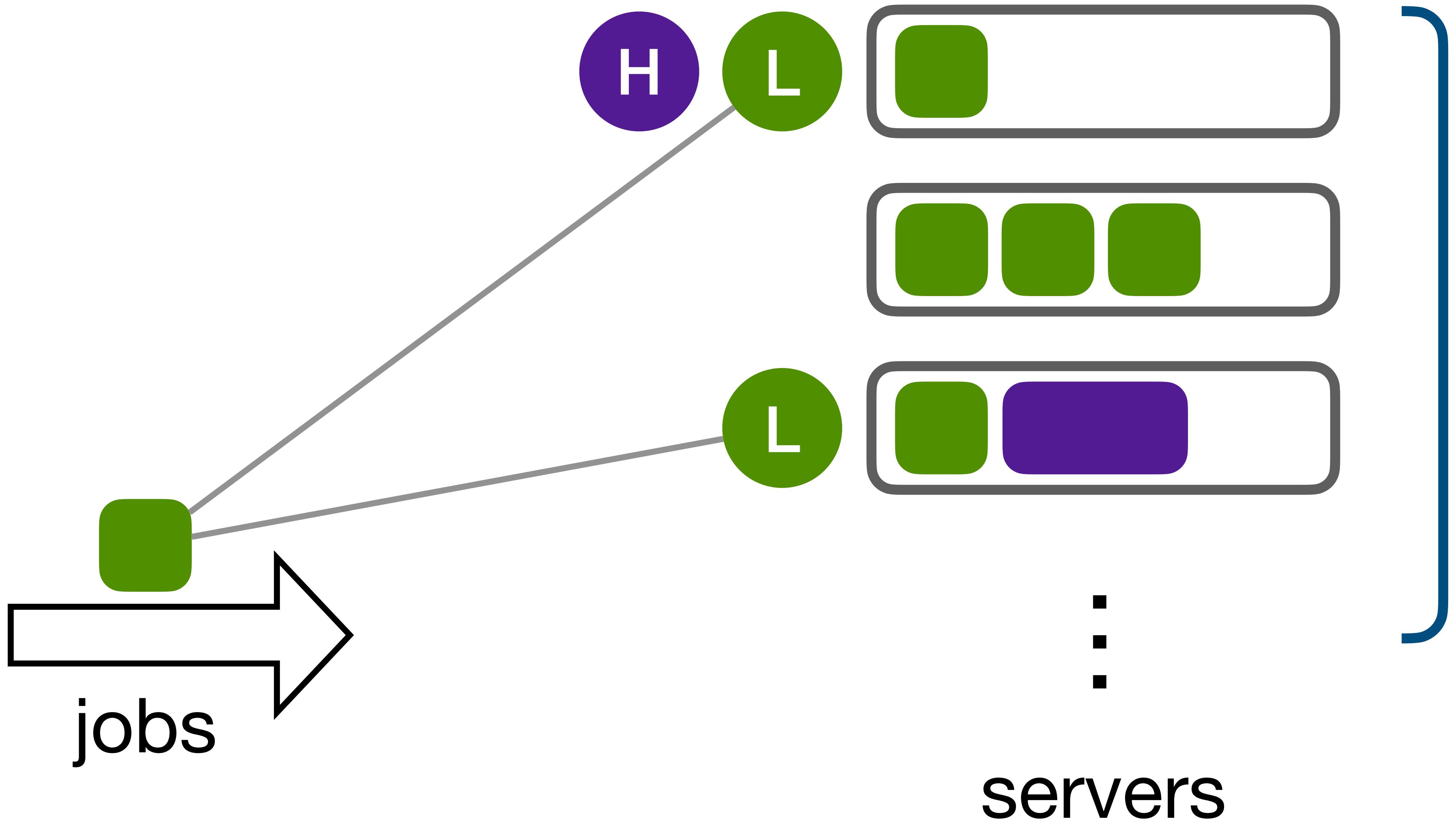
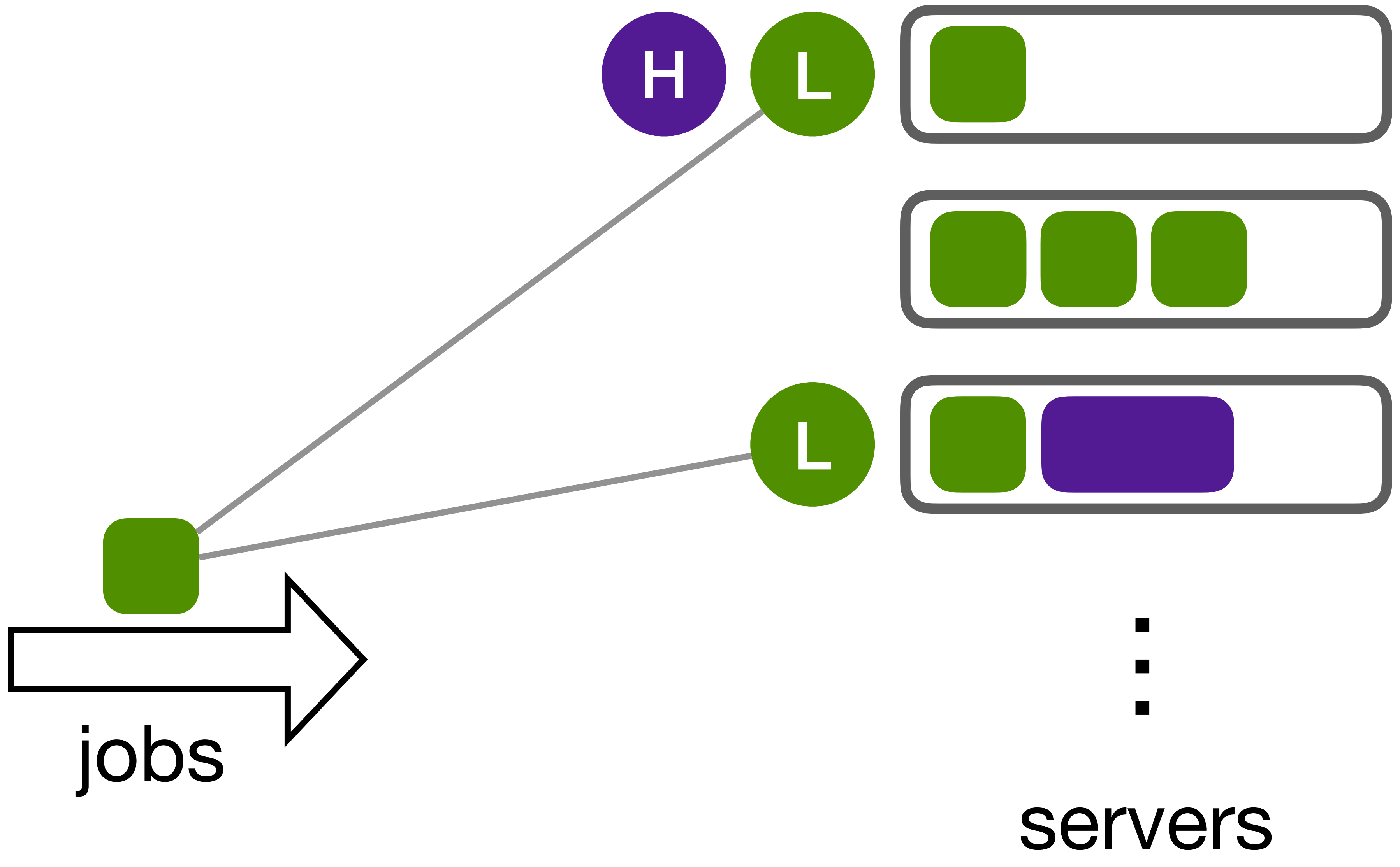


Run single-server policy $\overline{\sigma}$ for only

$$\overline{N} = \frac{\text{arrival rate}}{\text{throughtput}(\overline{\sigma})} \text{ servers}$$

(regular servers)

Policy conversion: more details



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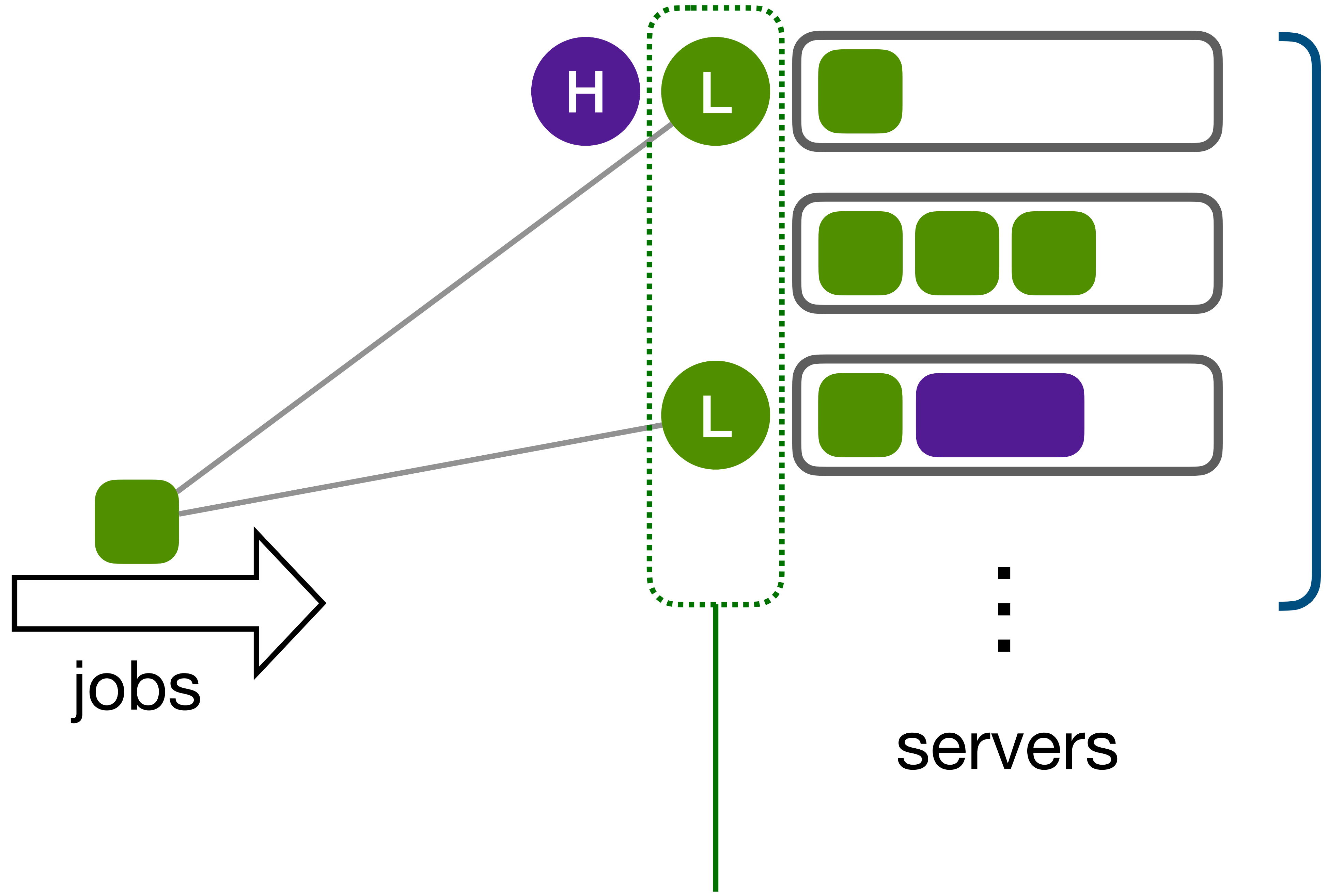
Recall that we aim to show

$$\mathbf{E}\left[\text{\# active servers}\right] \leq \left(1 + O\left(r^{-0.5}\right)\right) \cdot \overline{N}$$

Policy conversion: more details



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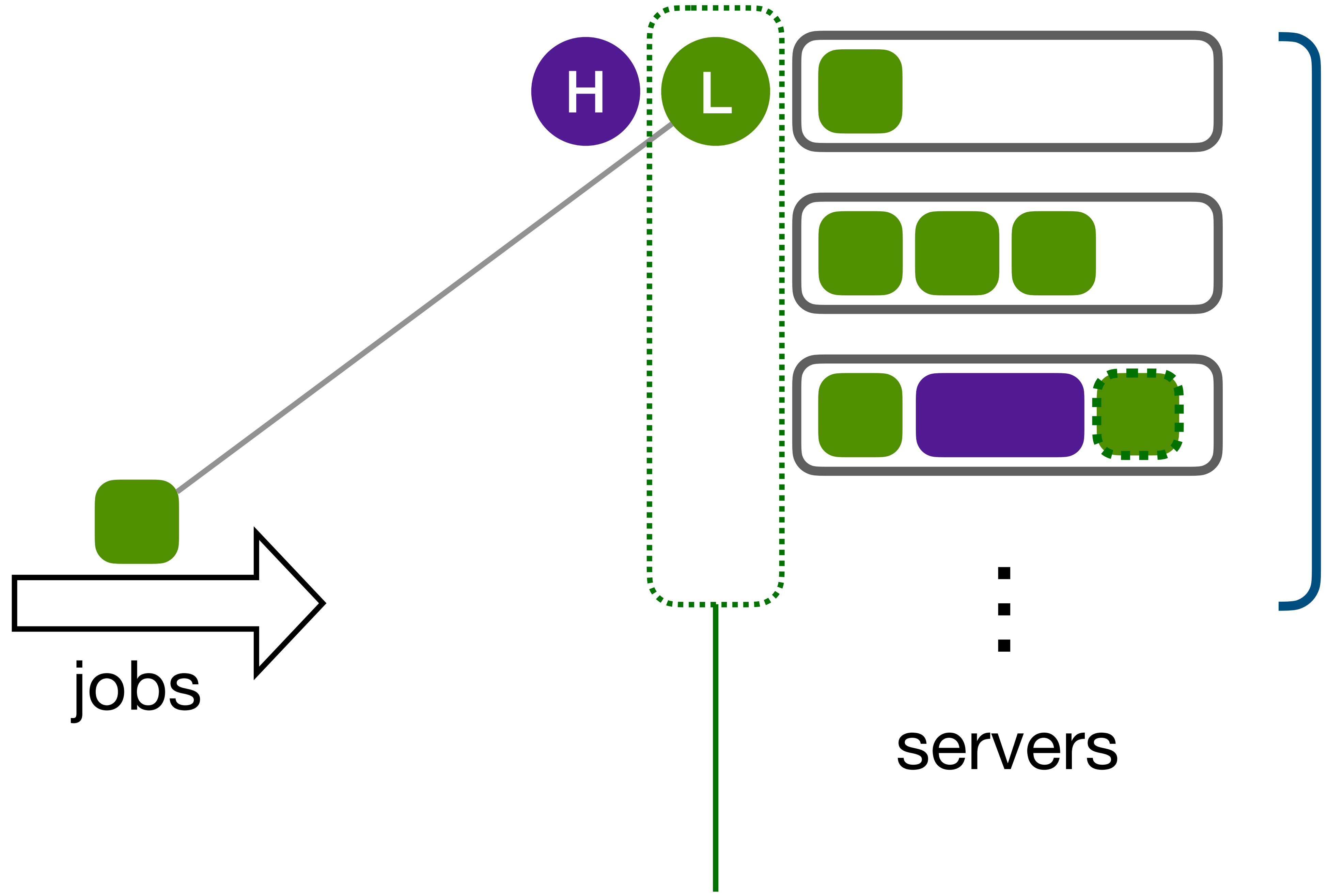
When the # tokens of a type $> \sqrt{r}$, remove the overflow tokens and generate virtual jobs

Policy conversion: more details



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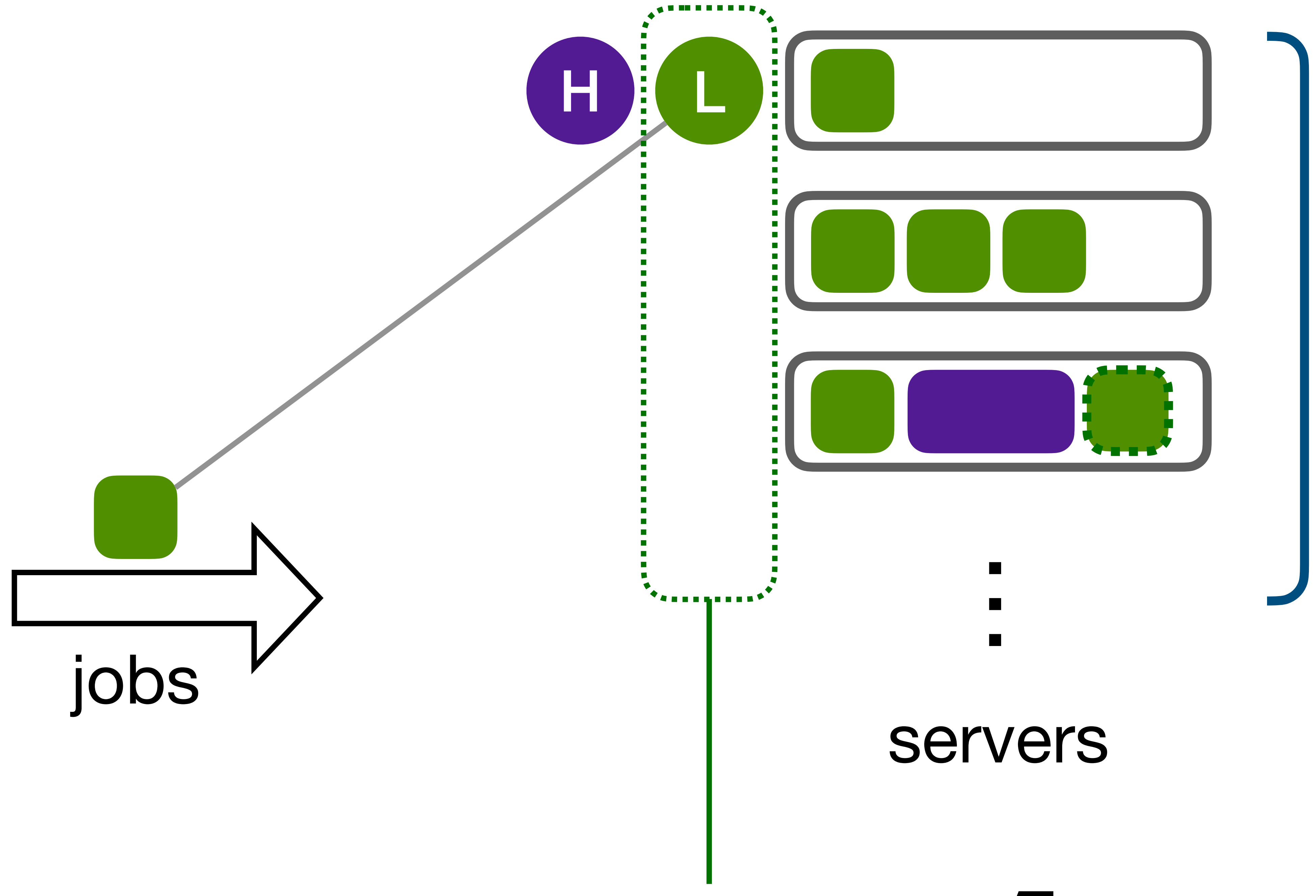

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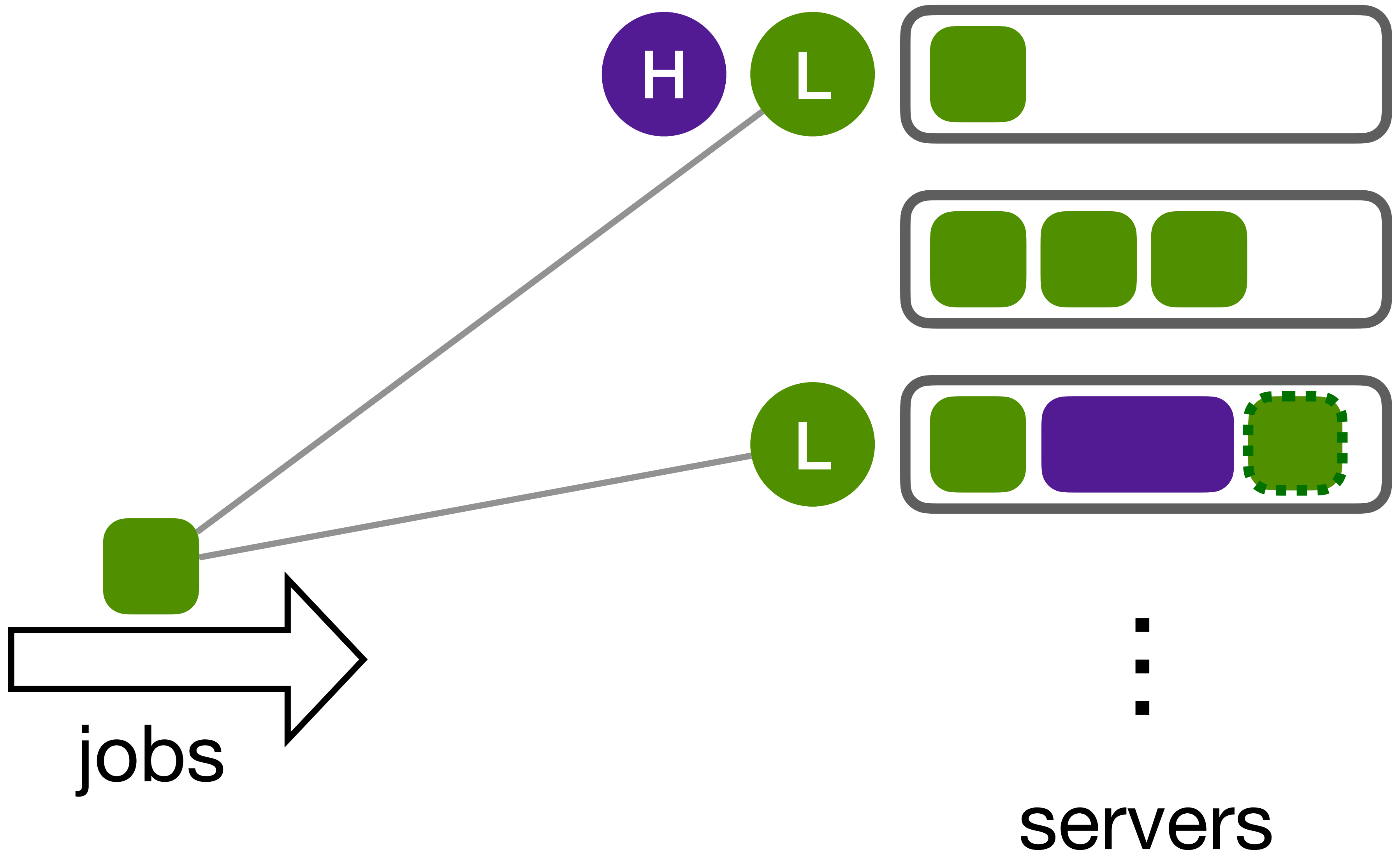
We can prove that $\mathbf{E}\,[\#\text{ virtual jobs}] = O\left(r^{0.5}\right)$

Key proof idea 1



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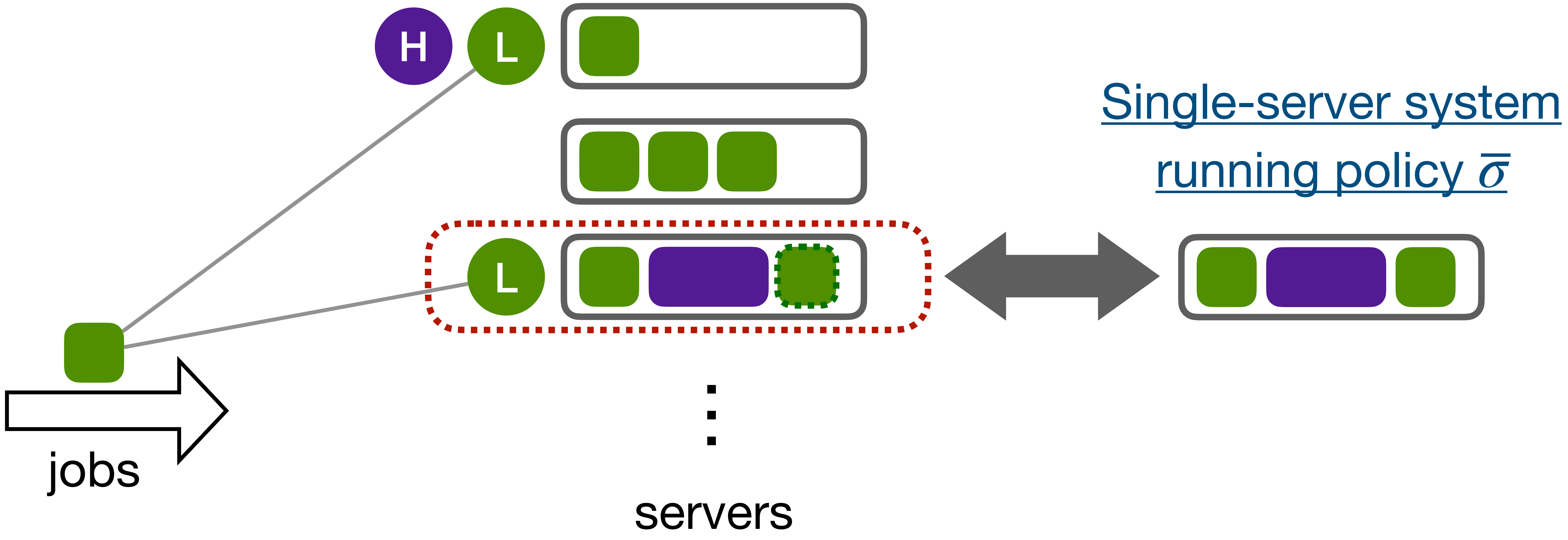
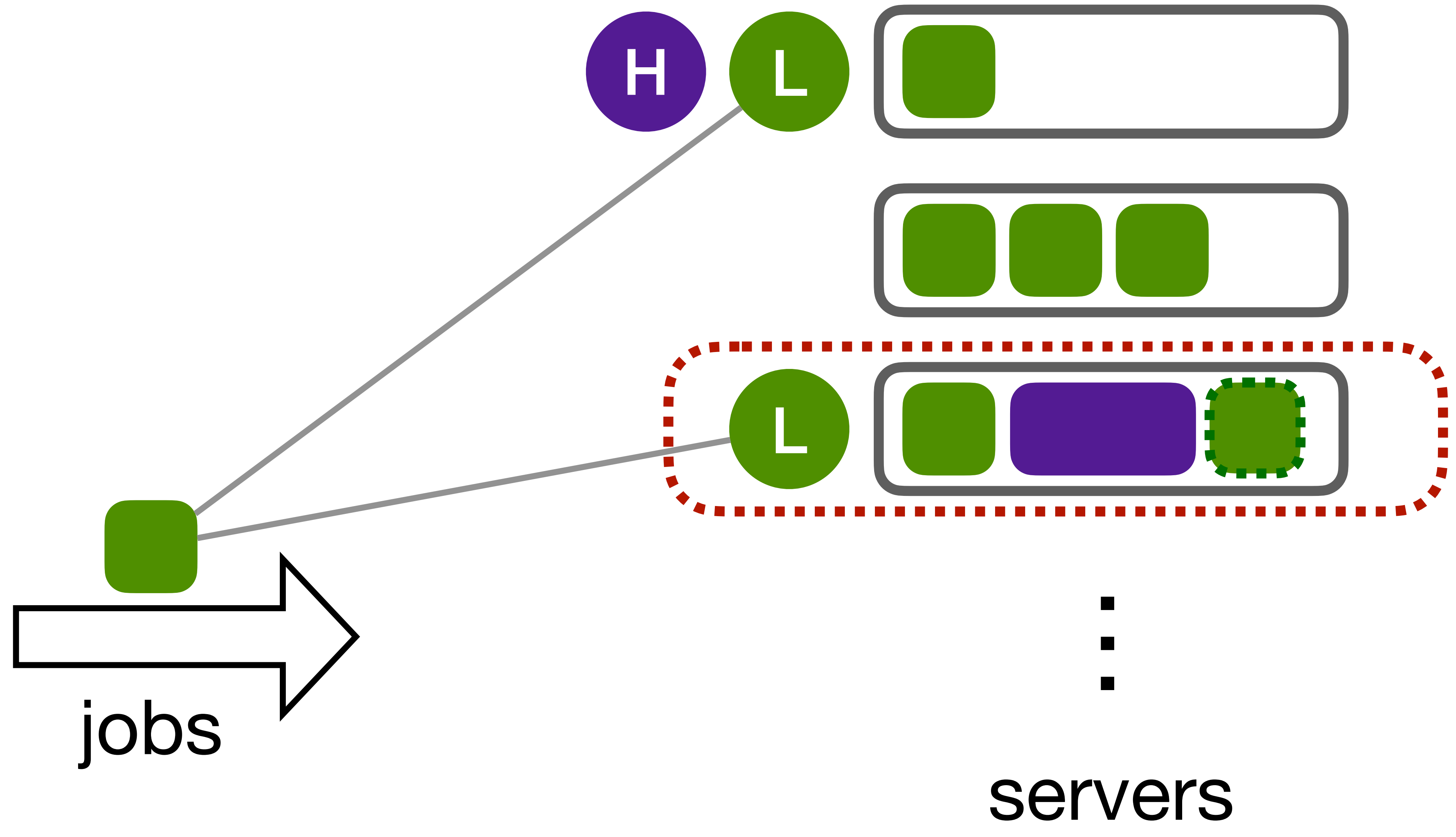
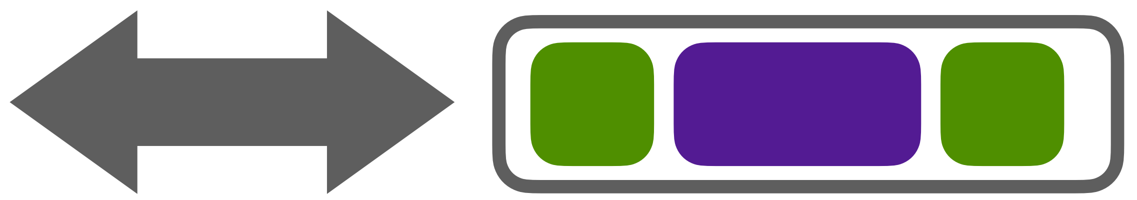


Key proof idea 1

Will show that regular servers in the original system $\approx \overline{N}$ independent single-server systems

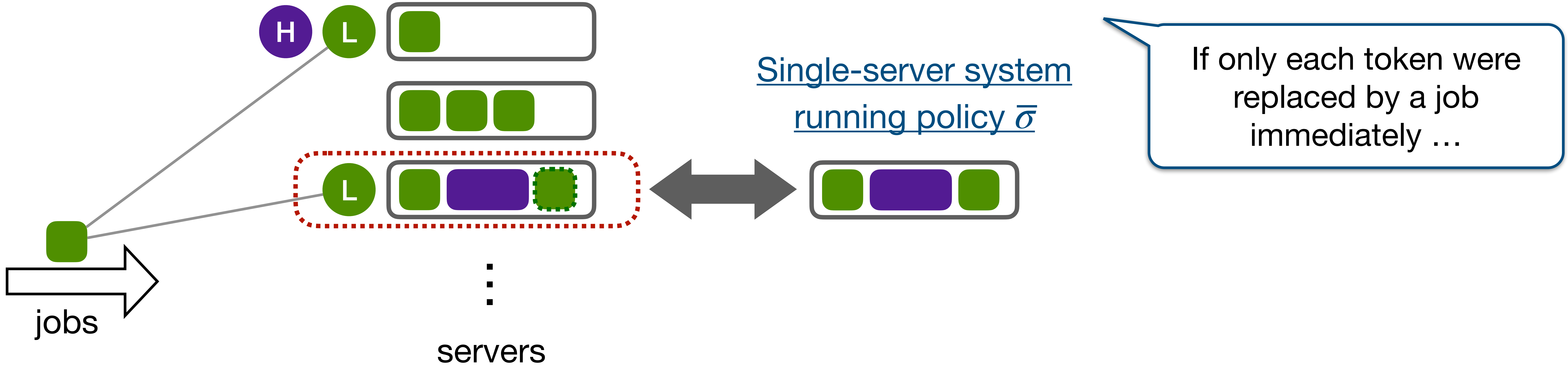


Single-server system running policy $\overline{\sigma}$



Key proof idea 1

Will show that regular servers in the original system $\approx \bar{N}$ independent single-server systems



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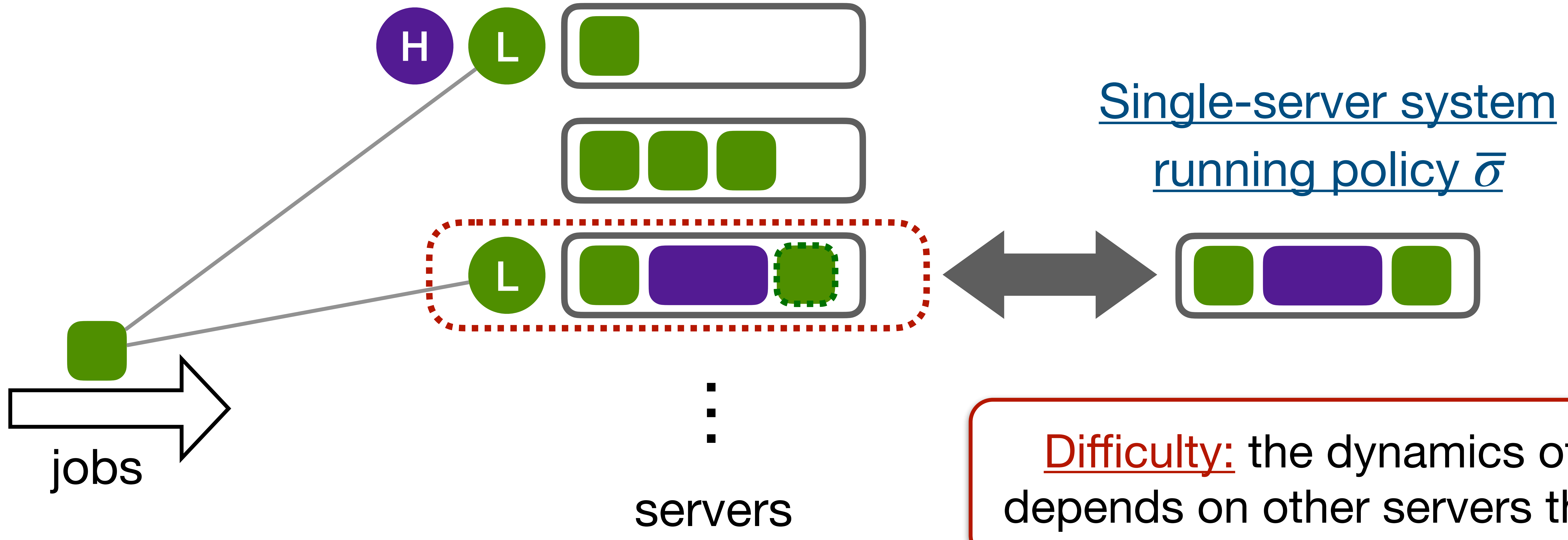

Difficulty: the dynamics of a server in the original system depends on other servers through arrivals & token overflows

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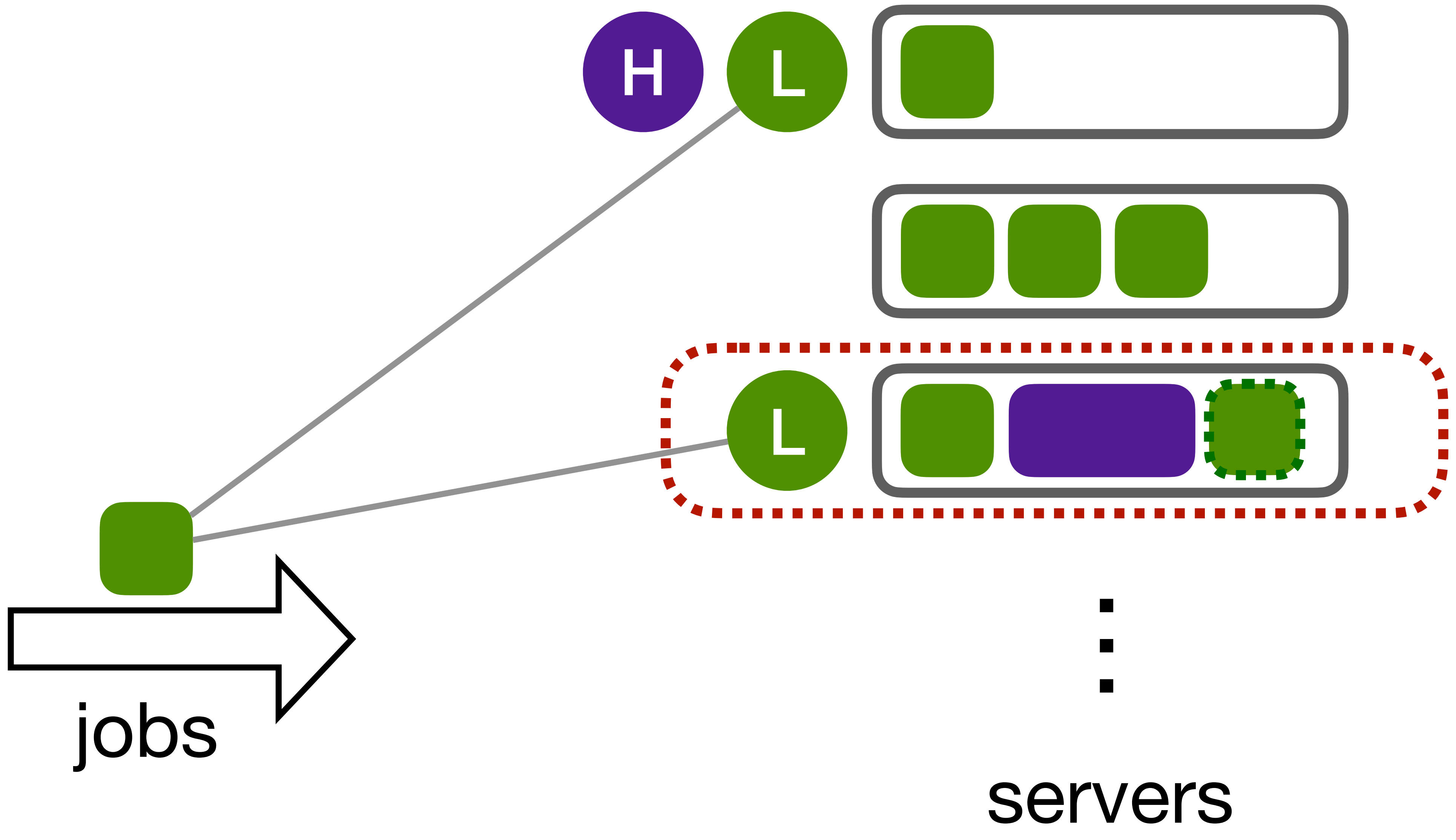
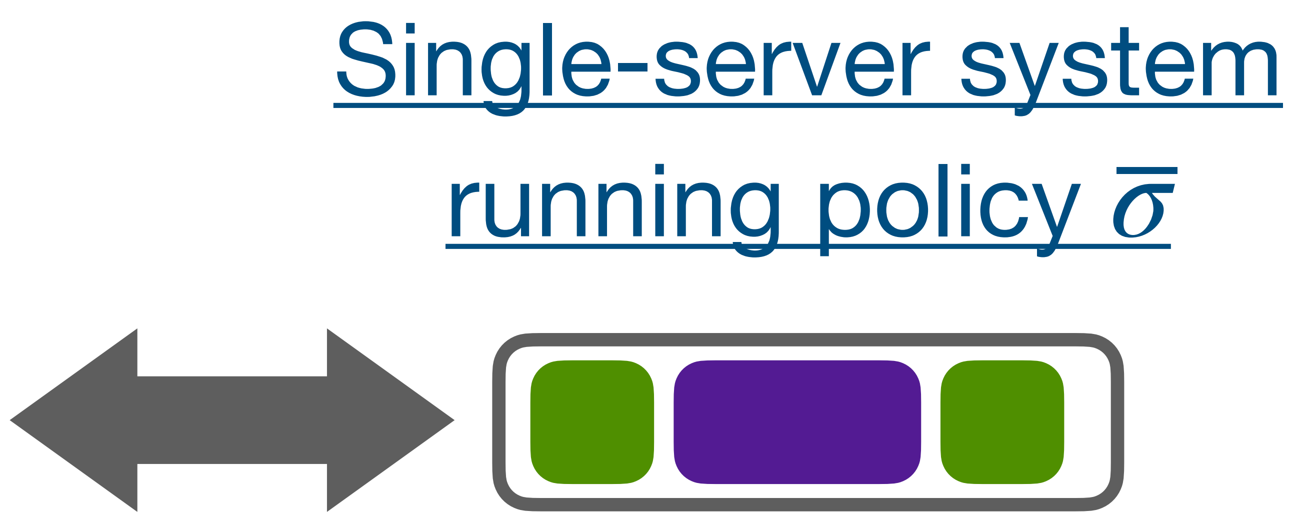




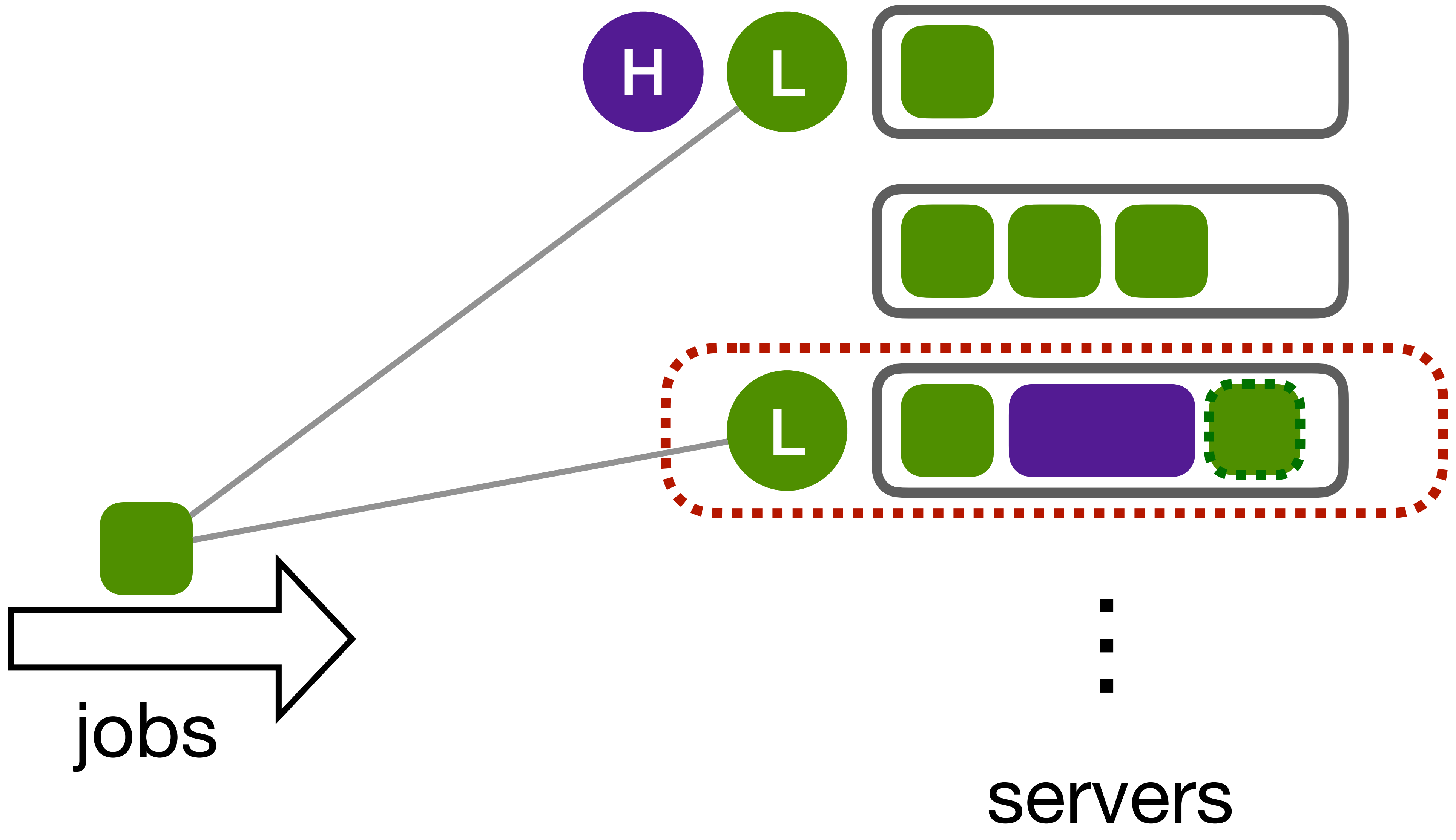
Difficulty: the dynamics of a server in the original system depends on other servers through arrivals & token overflows

Idea: for each type i, consider

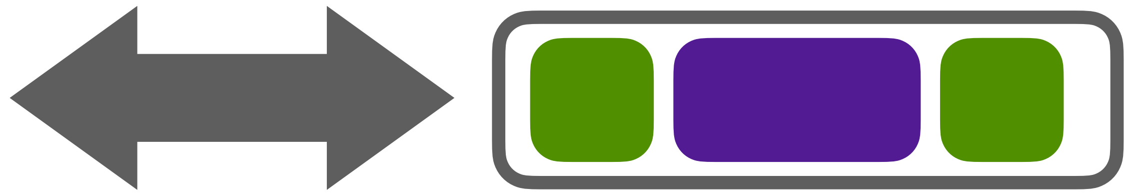
$\widetilde{K}_i =$ # jobs + # virtual jobs + # tokens

Key proof idea 1

Will show that regular servers in the original system $\approx \overline{N}$ independent single-server systems



Single-server system running policy $\overline{\sigma}$



Difficulty: the dynamics of a server in the original system depends on other servers through arrivals & token overflows

Idea: for each type i, consider

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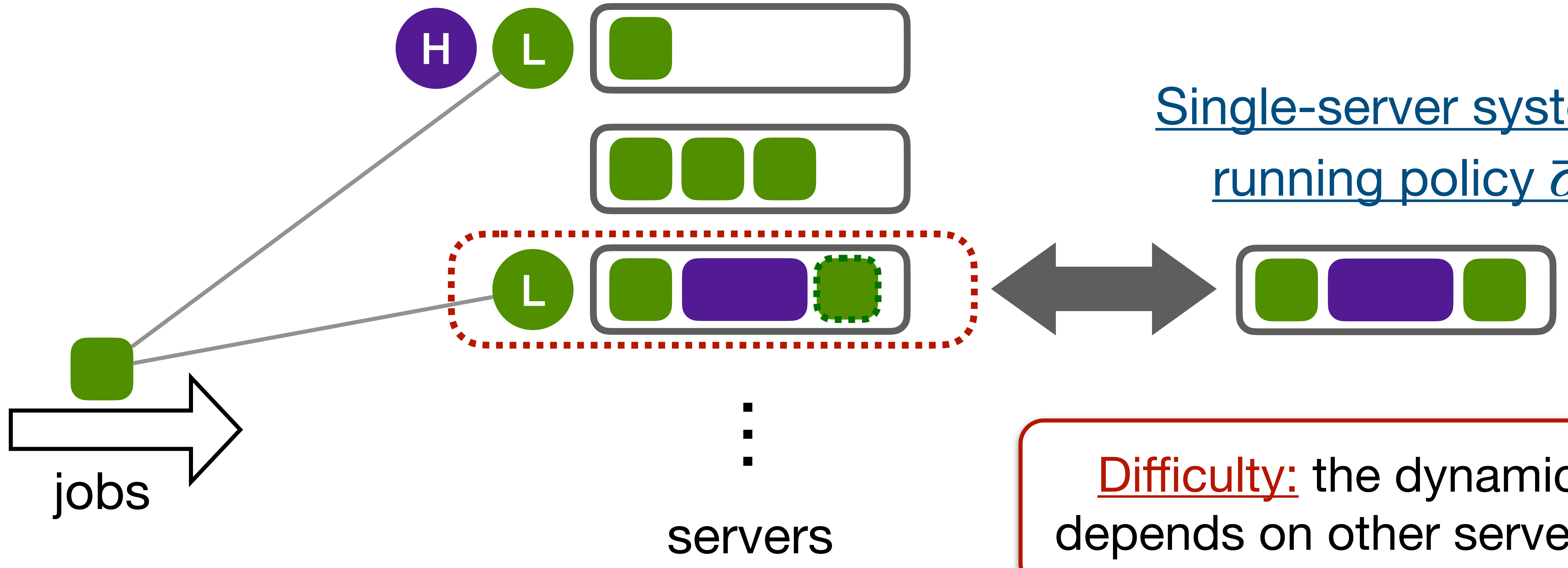
Do job arrivals affect $\widetilde{K}_i$?

Key proof idea 1

Will show that regular servers in the original system $\approx \overline{N}$ independent single-server systems



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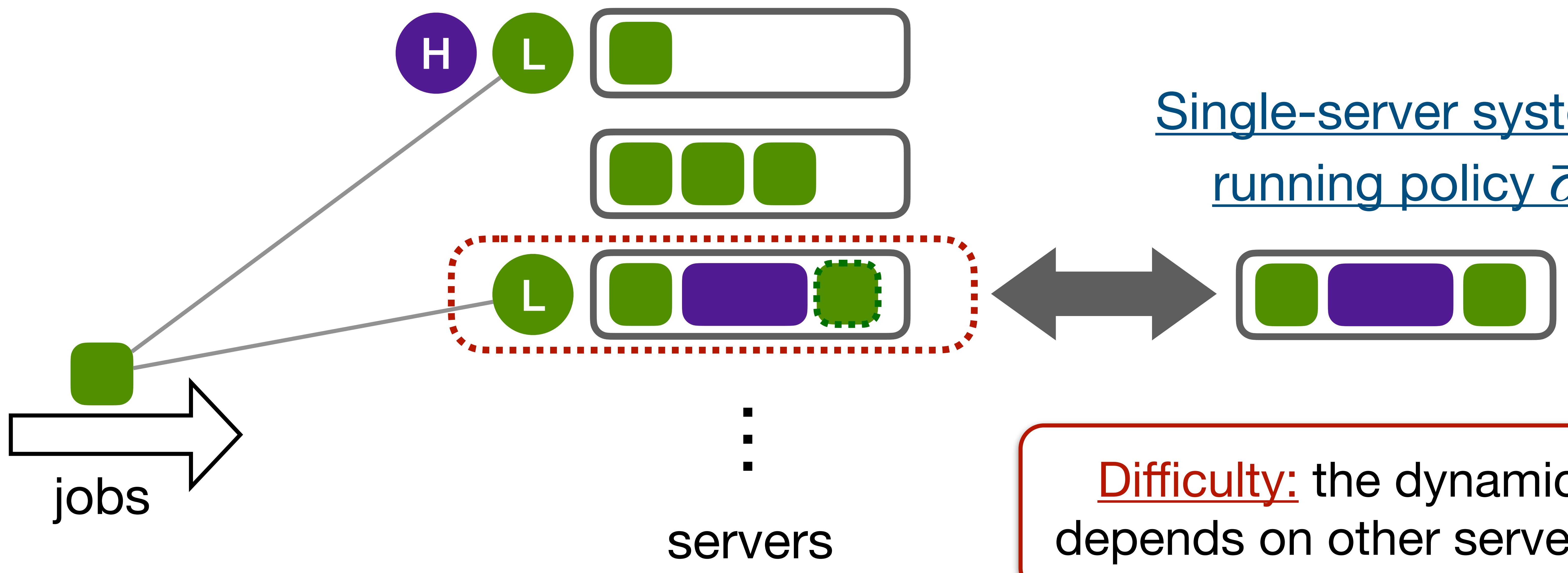

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Do token overflows affect $\widetilde{K}_i$?

Arrivals & token overflows do not affect $\widetilde{K}_i$

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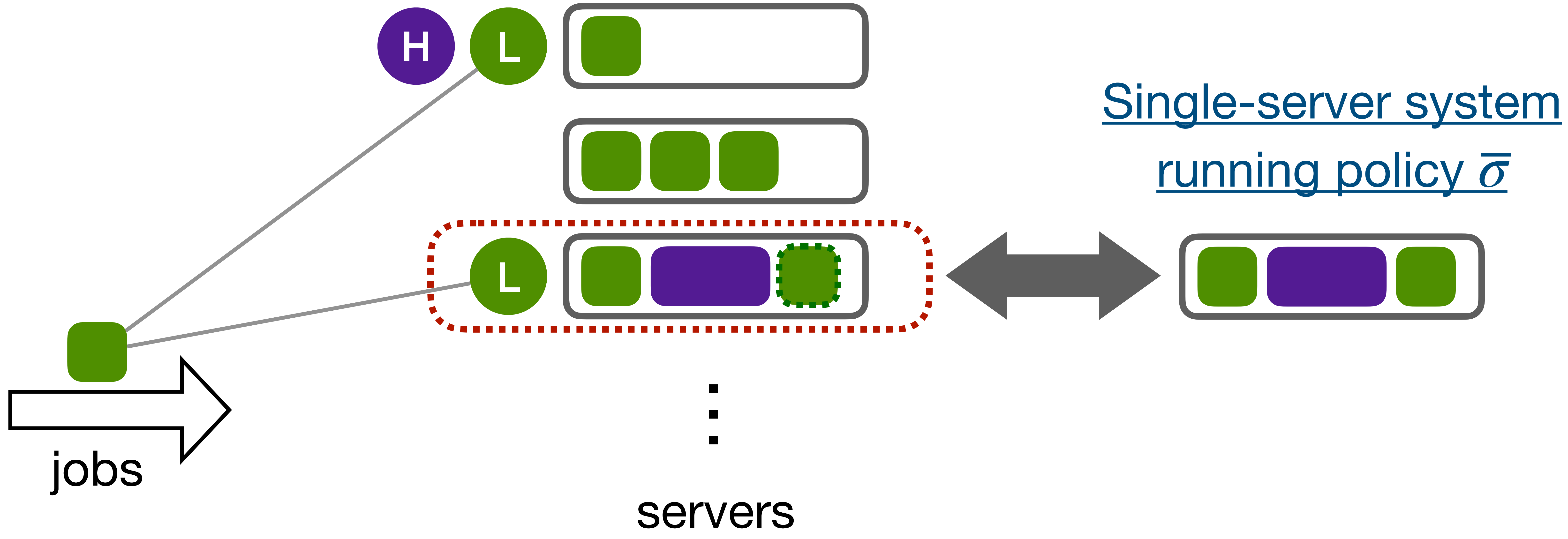

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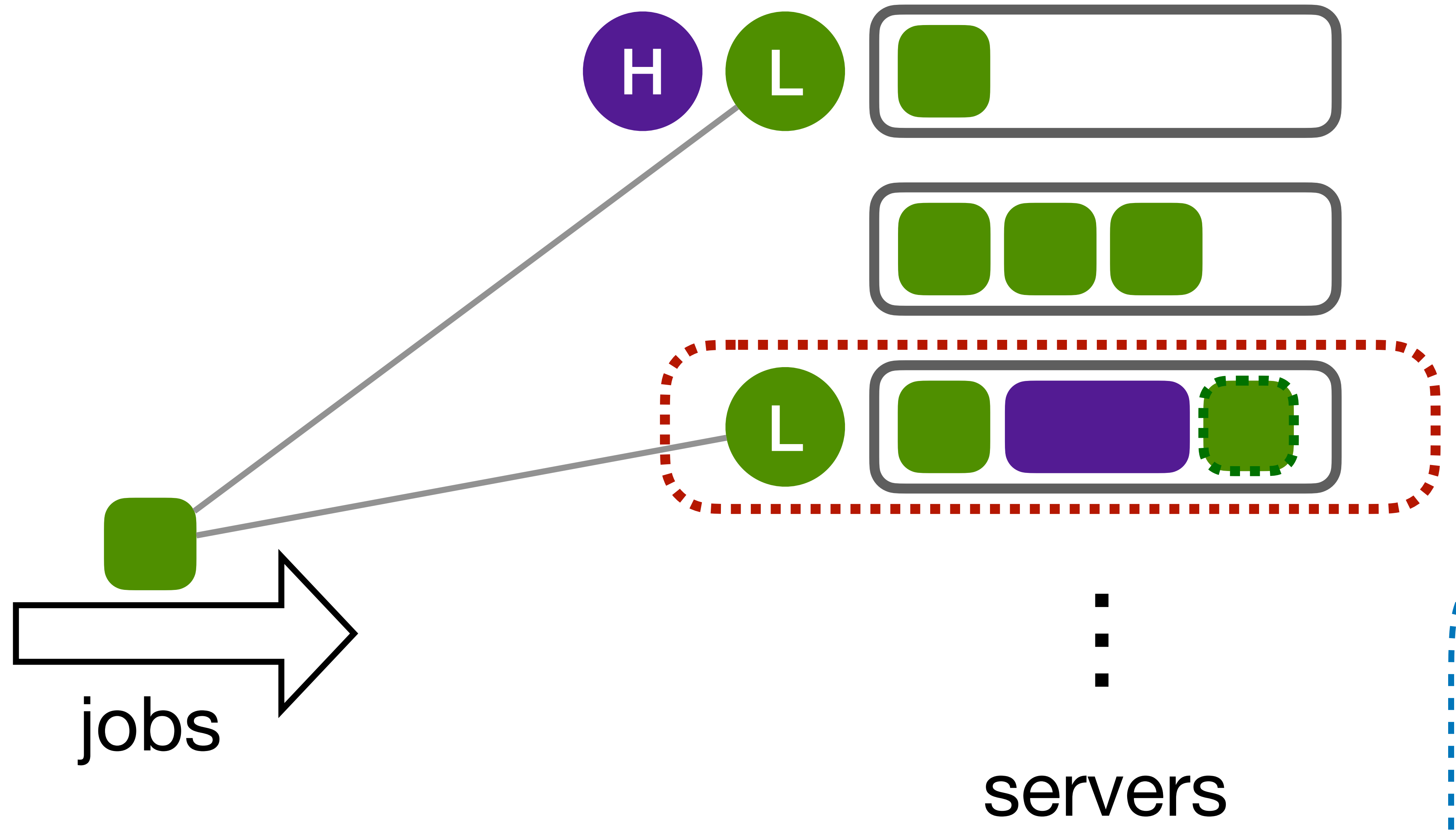
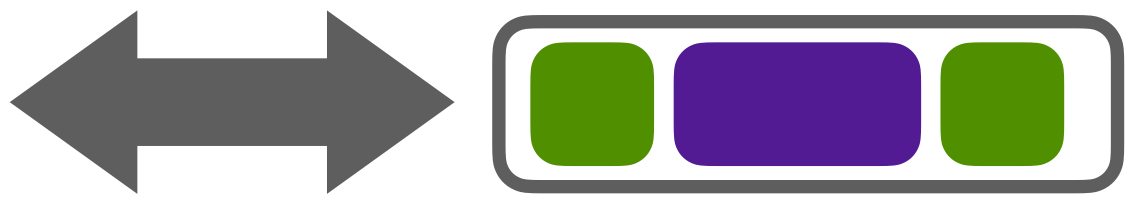


Single-server system
running policy $\overline{\sigma}$



Dynamics in the original system v.s.
Dynamics in $\overline{N}$ independent single-server systems

Idea: for each type i, consider

$\widetilde{K}_i$ = # jobs + # virtual jobs + # tokens

Arrivals & token overflows do not affect $\widetilde{K}_i$

Key proof idea 1

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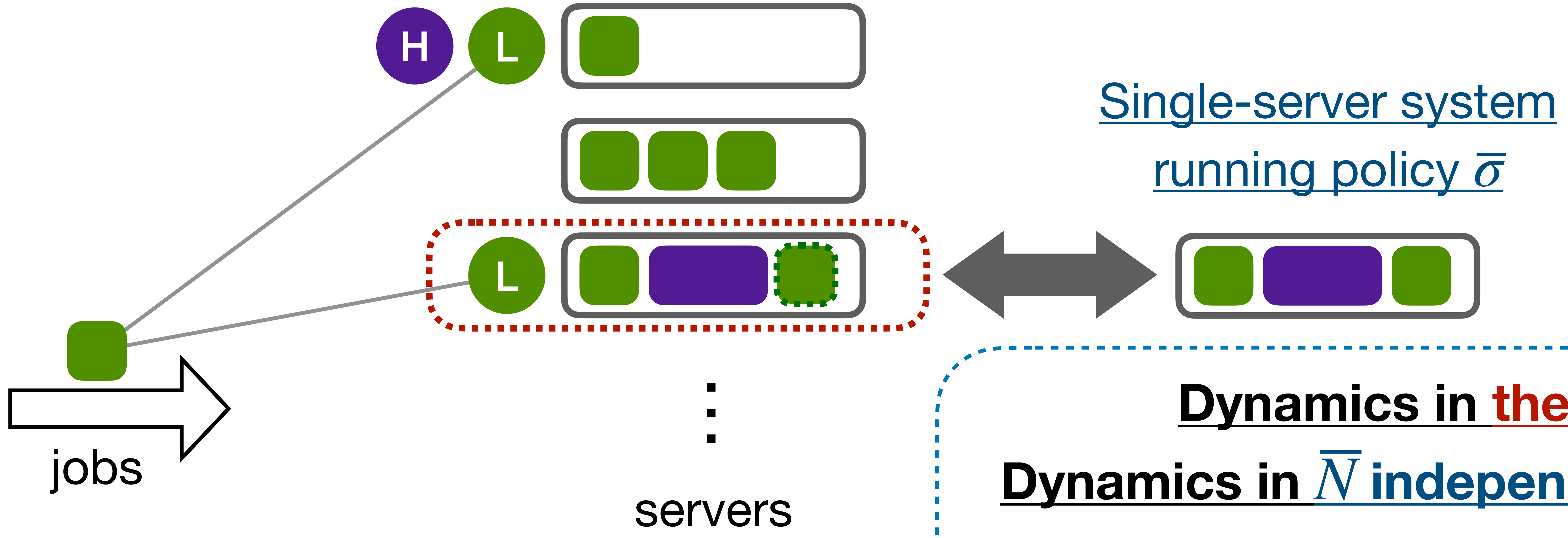

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Dynamics in the original system v.s.

Dynamics in $\overline{N}$ independent single-server systems

- A server without tokens in the original system generates tokens and has job transitions in the same way as a single-server system

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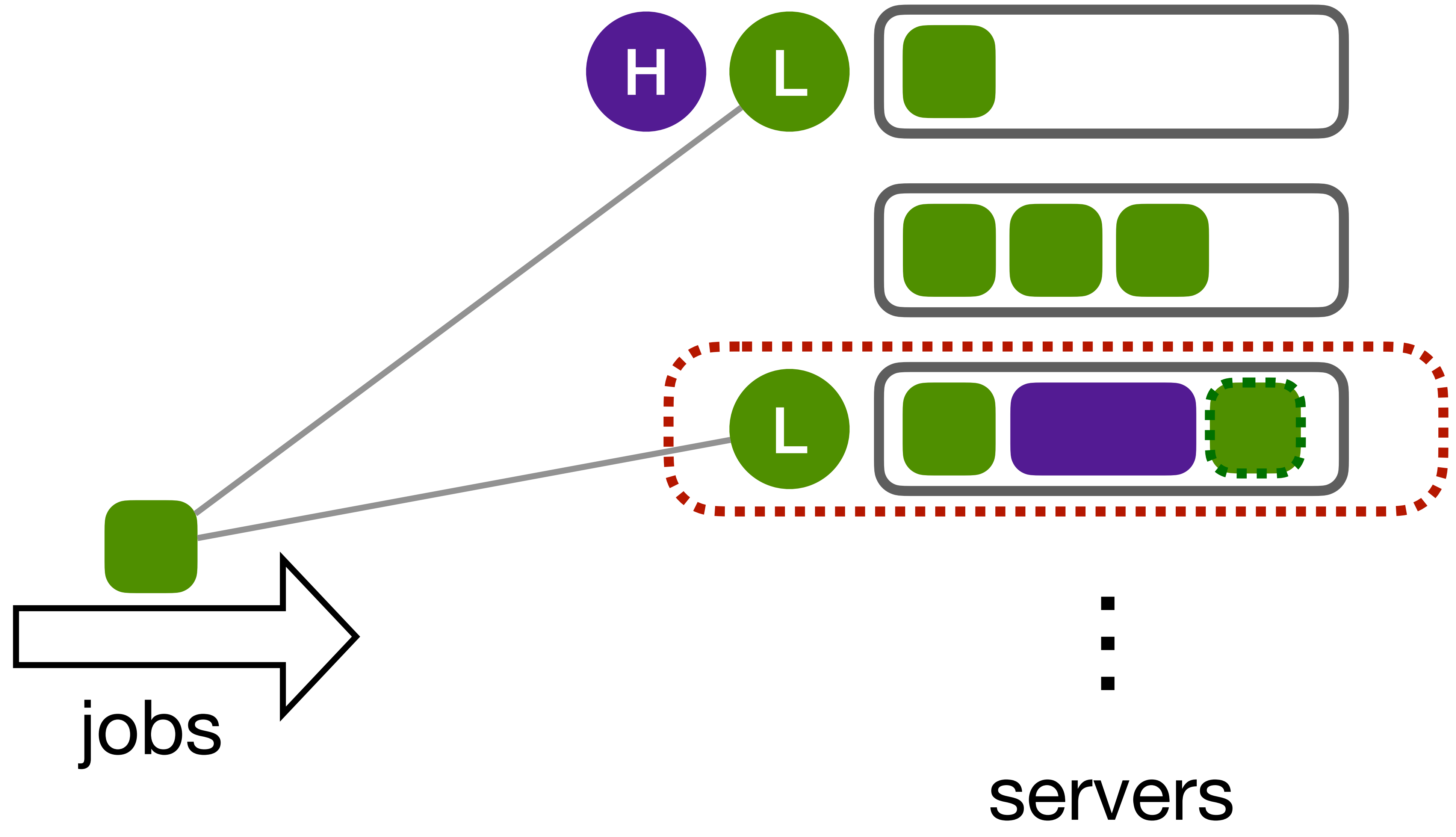

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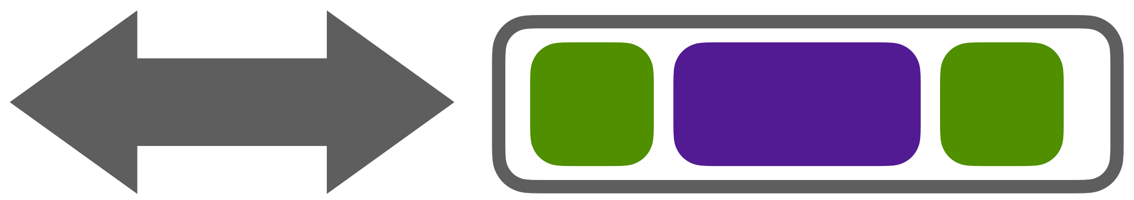

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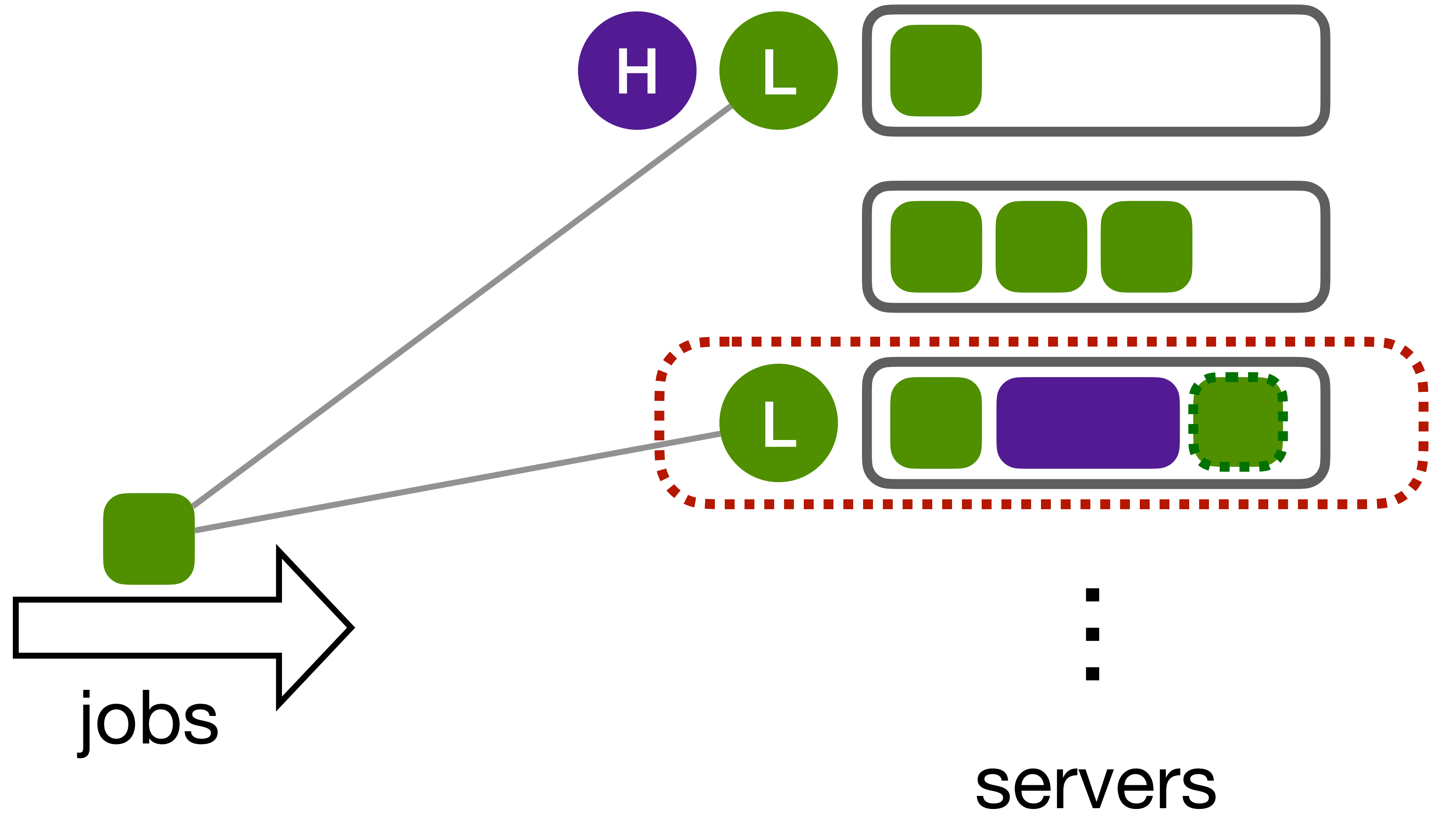
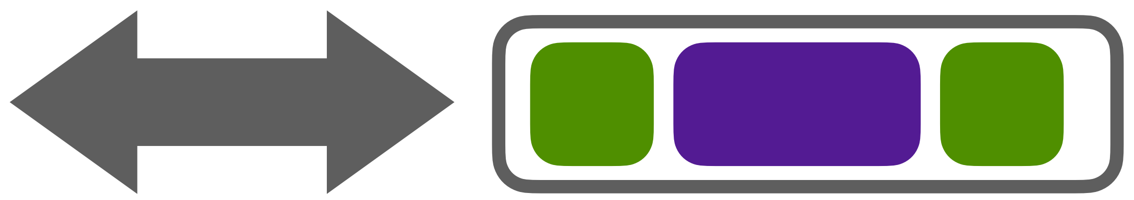


Single-server system running policy $\bar{\sigma}$



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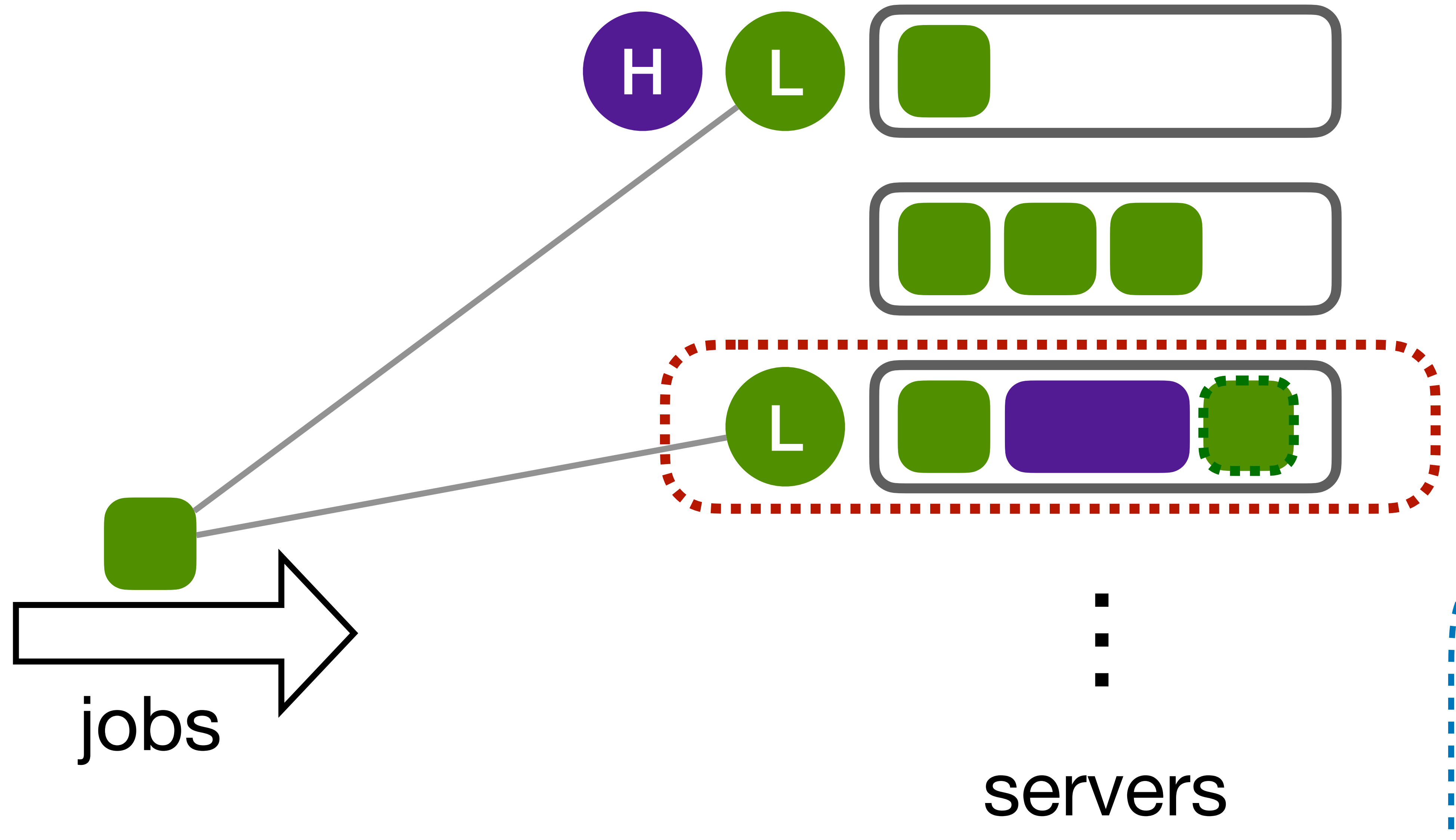

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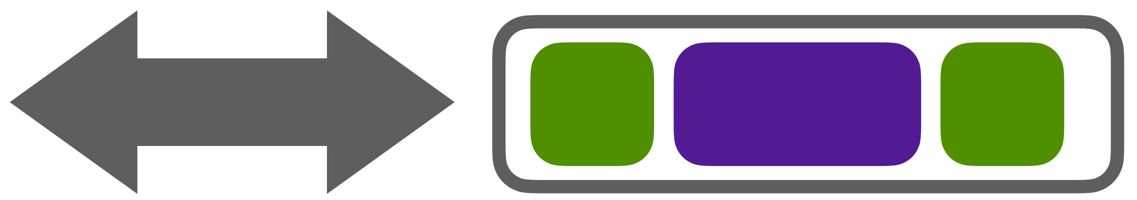

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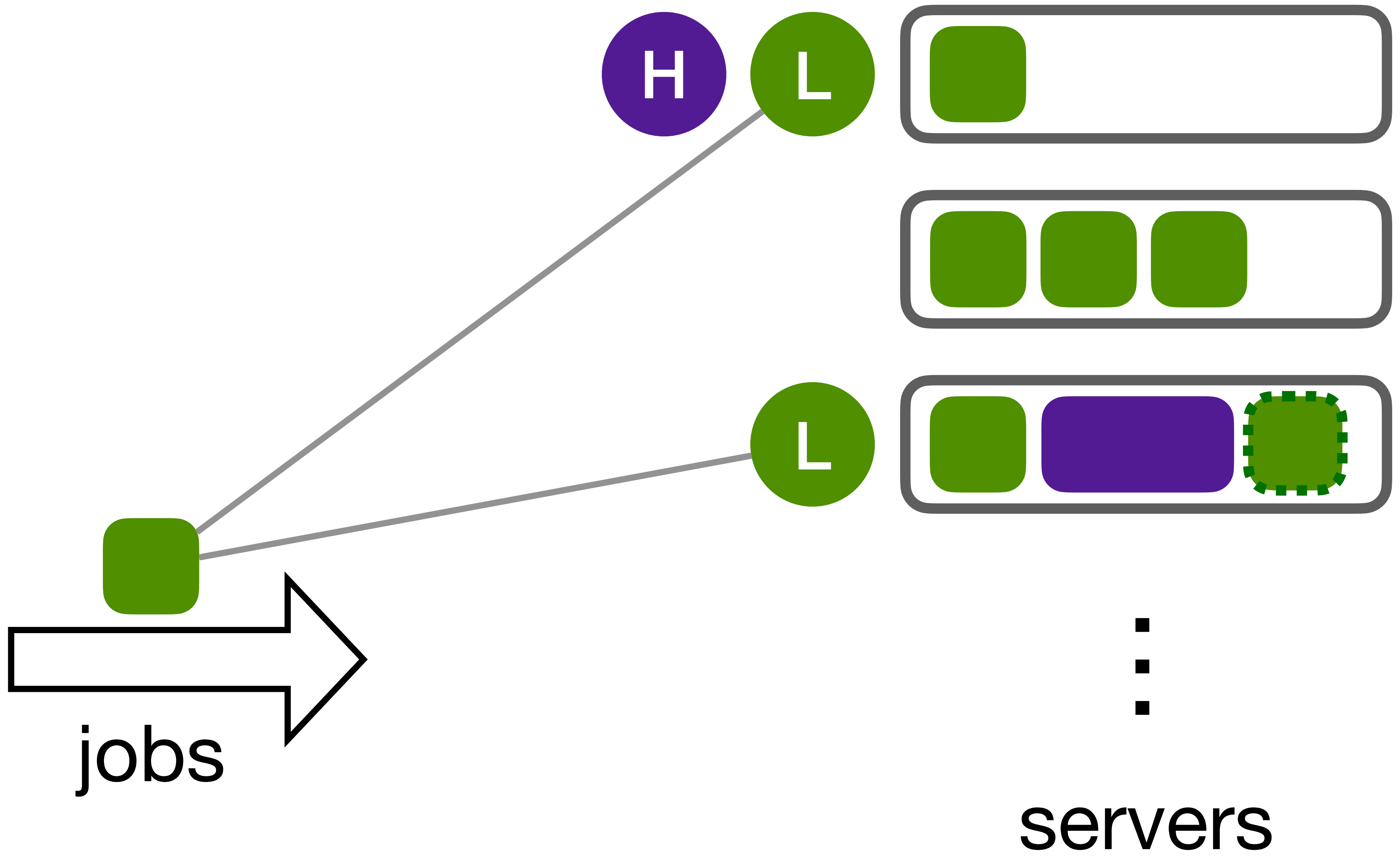
$\Rightarrow$ Using Stein's method, $d_W\left(\widetilde{K}^{1:\overline{N}}, \overline{K}^{1:\overline{N}}\right) = O\left(r^{0.5}\right)$

Key proof idea 2



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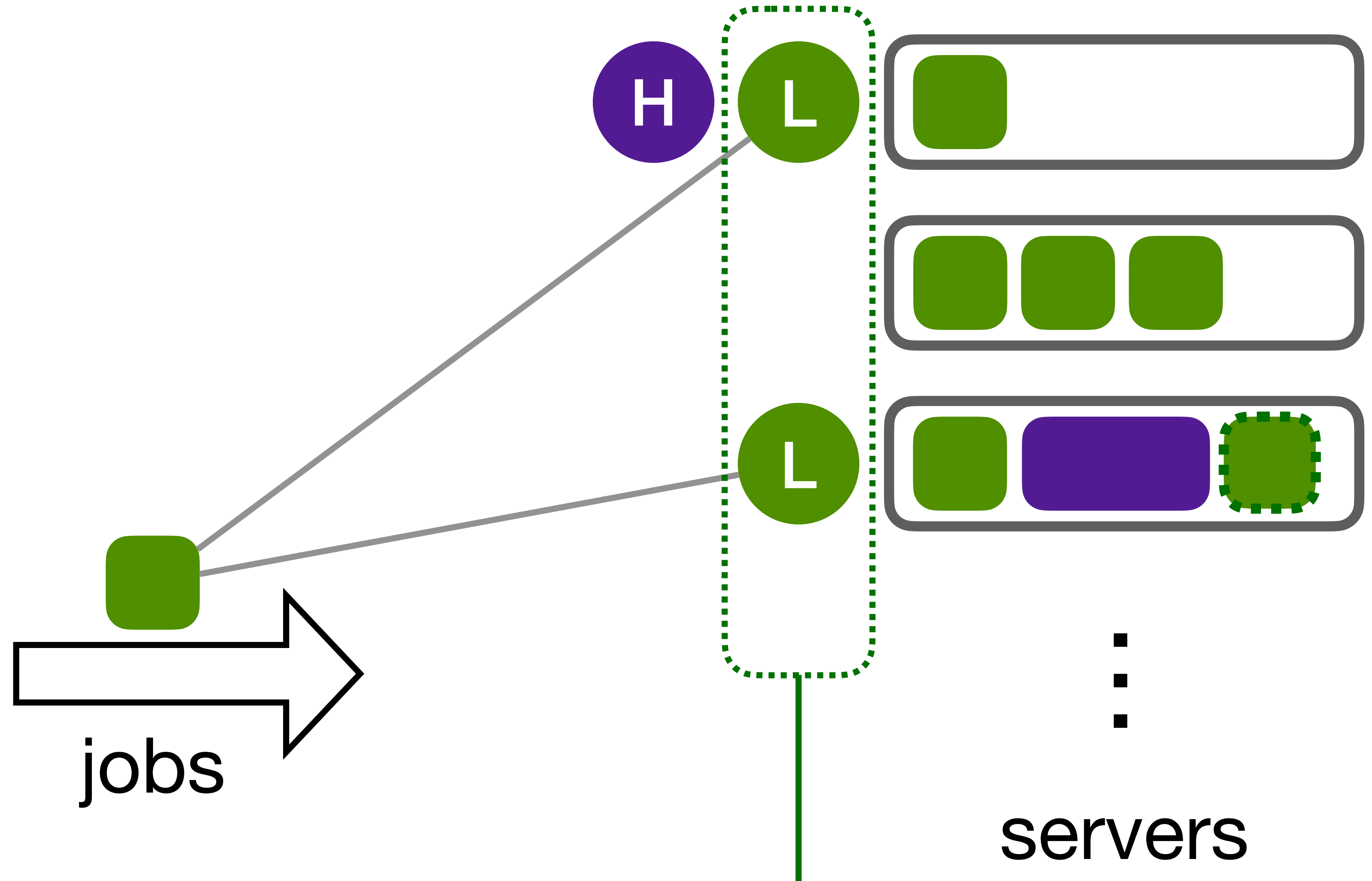

When the # tokens of a type $> \sqrt{r}$, remove the overflow tokens and generate virtual jobs

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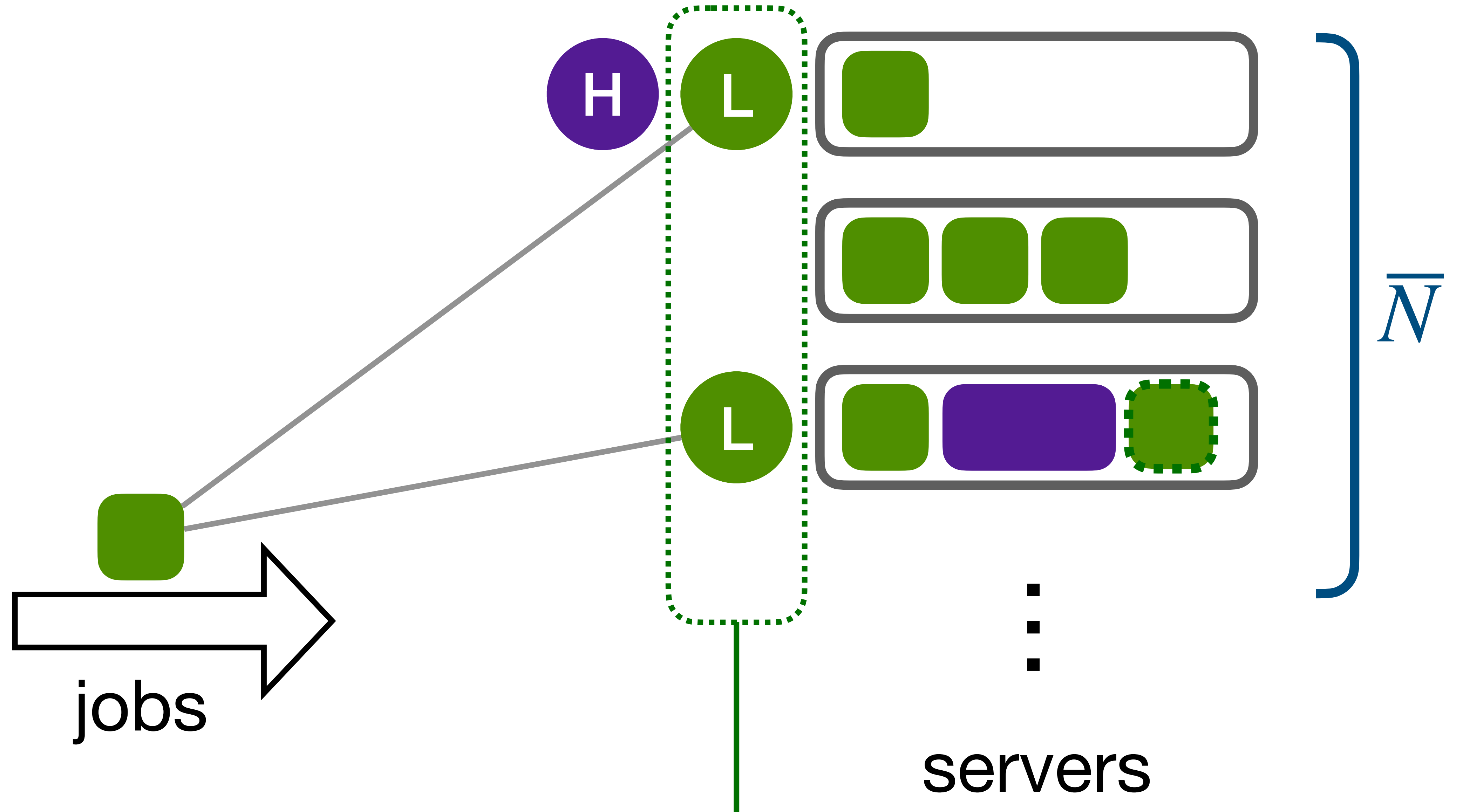

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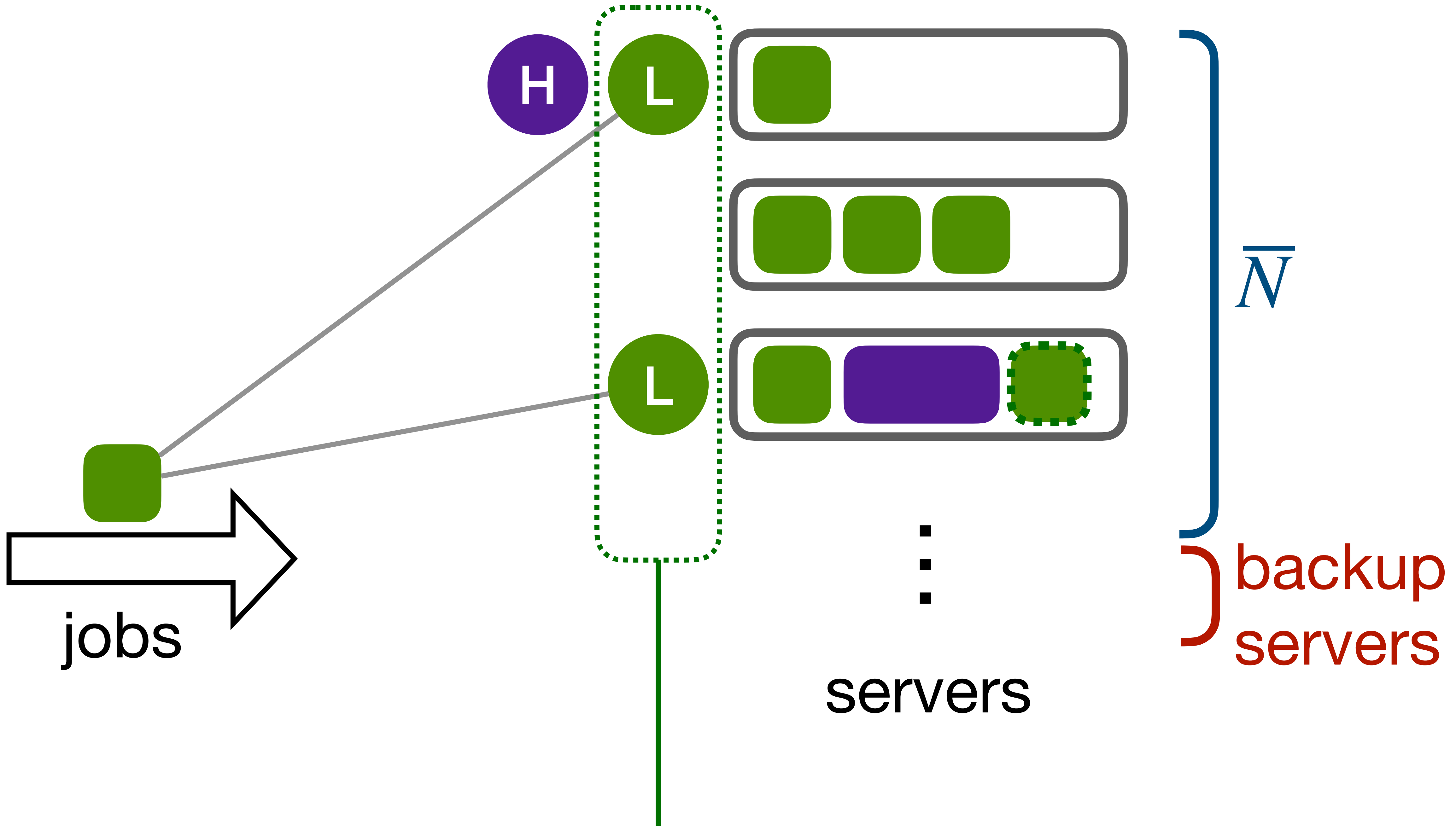

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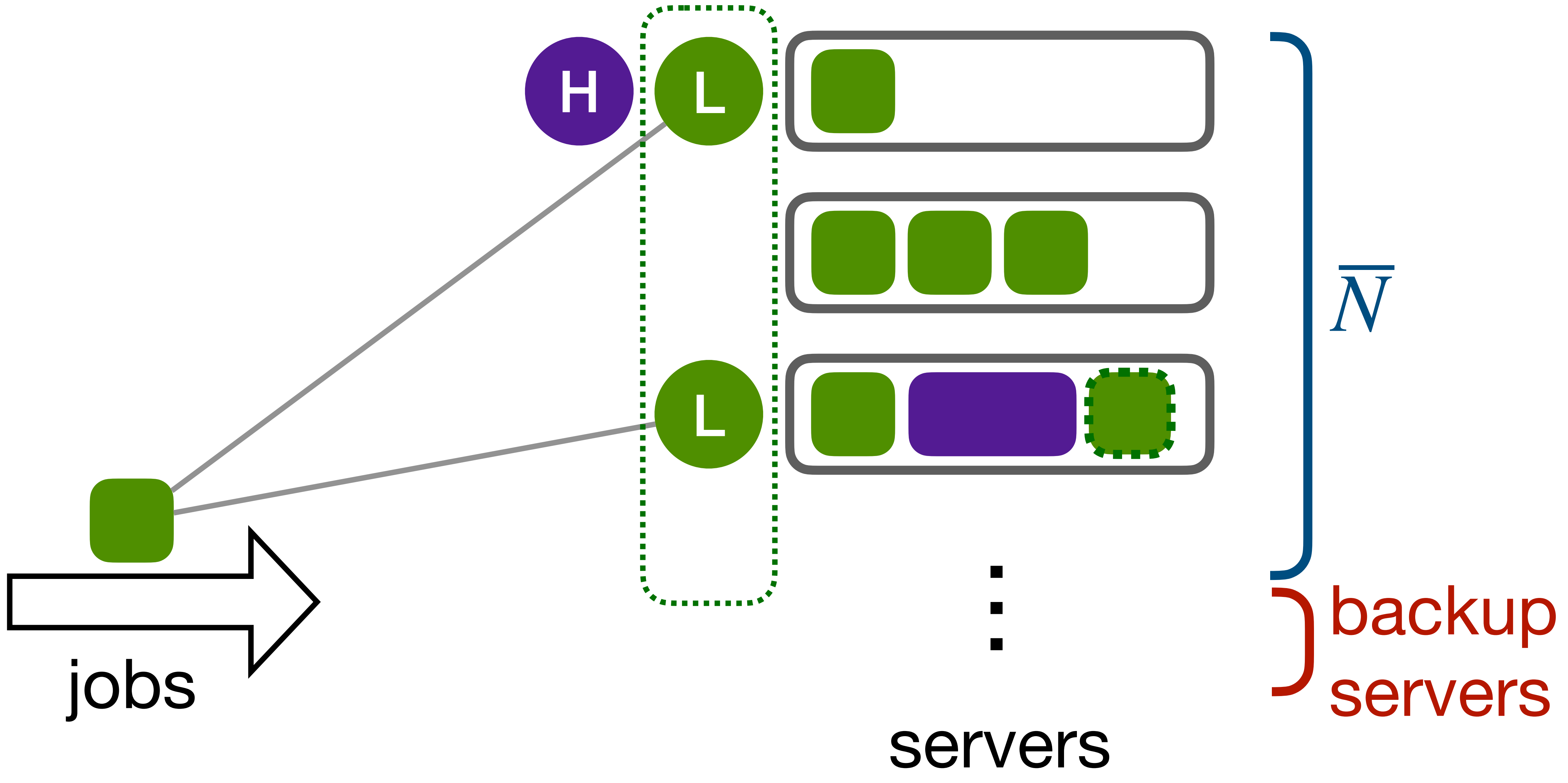

Will show that # virtual jobs $= O\left(\sqrt{r}\right)$,

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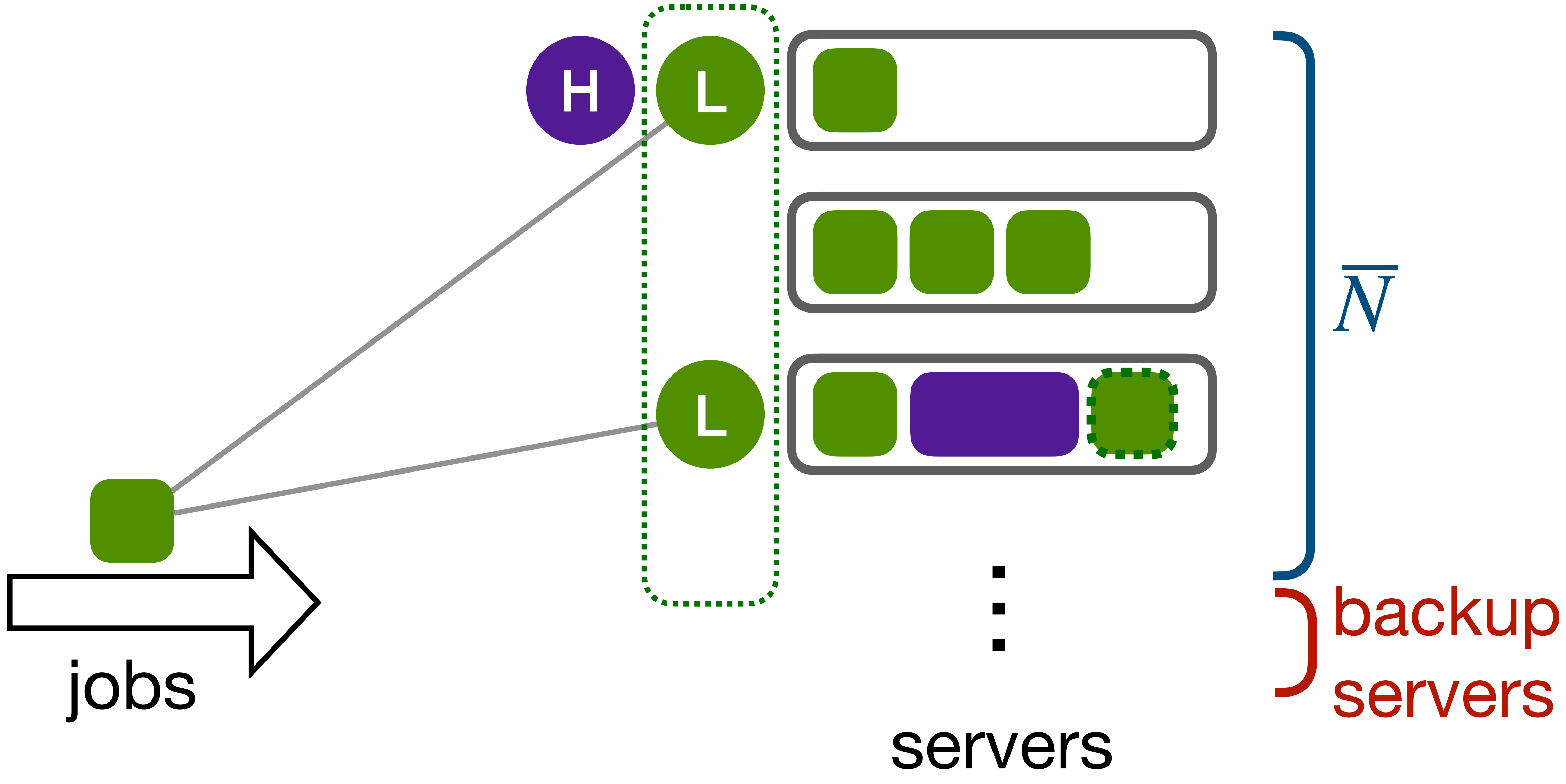

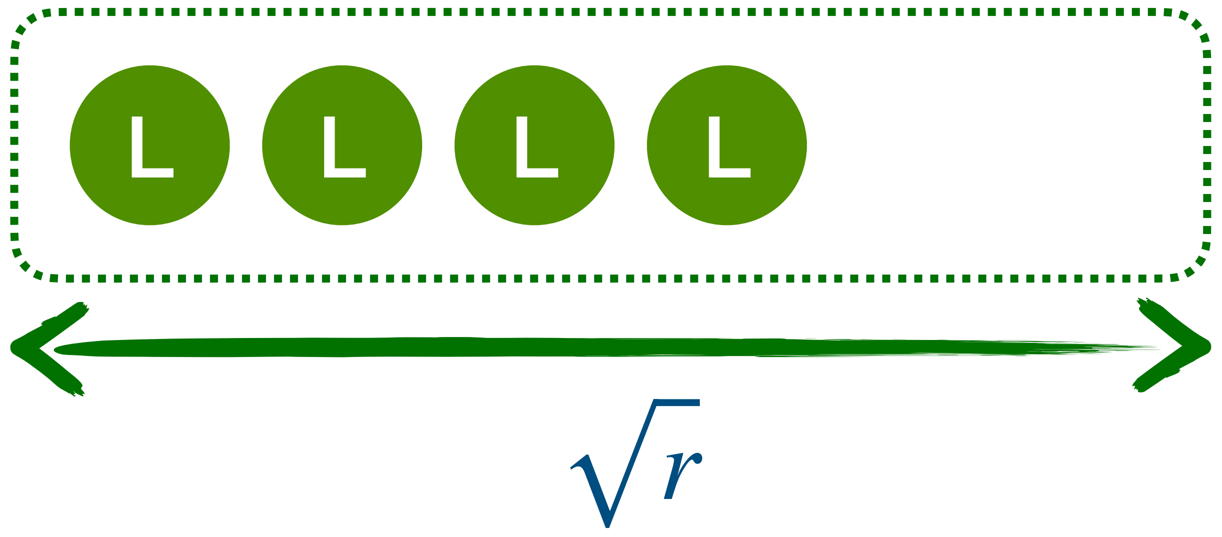

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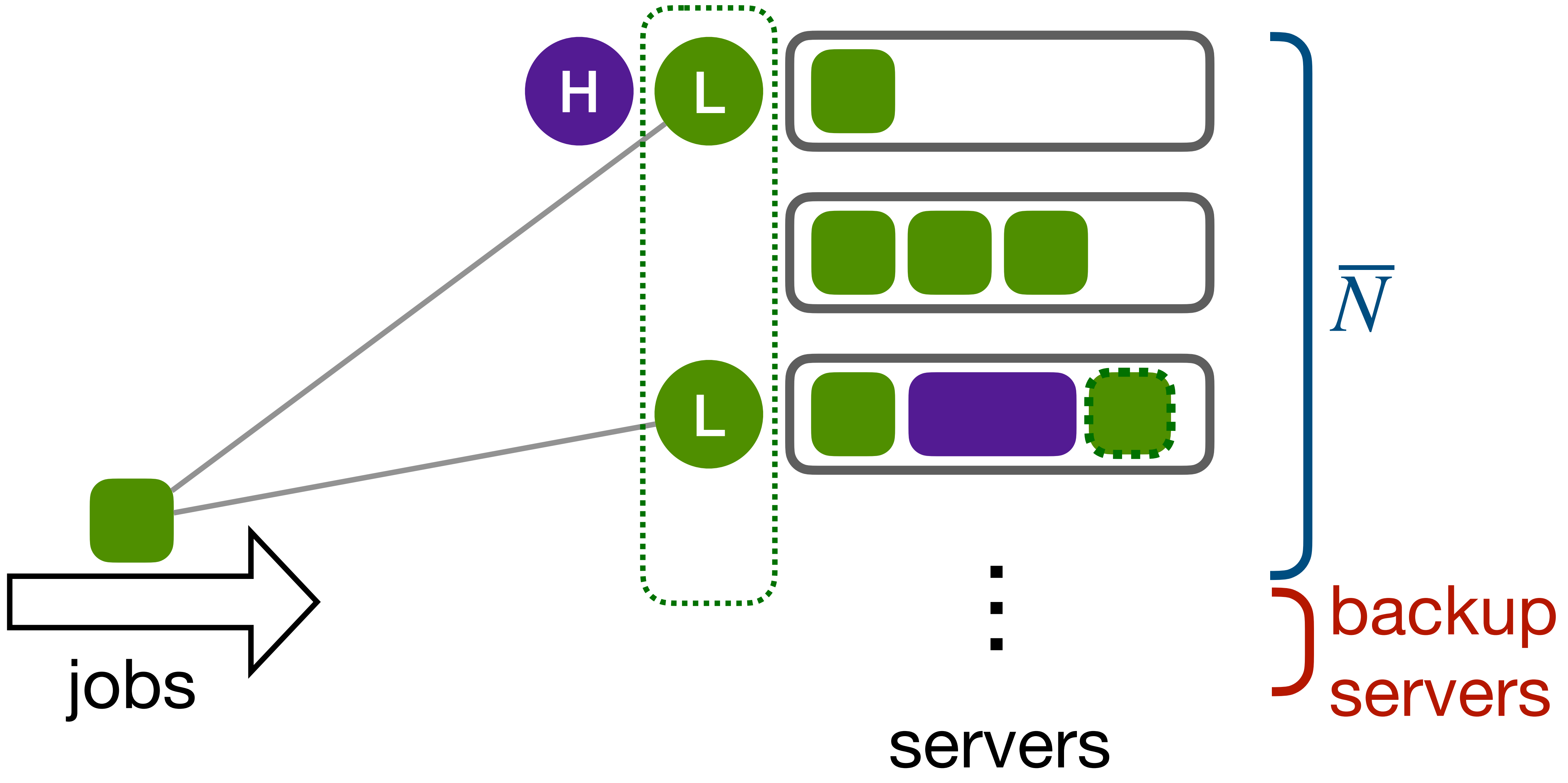

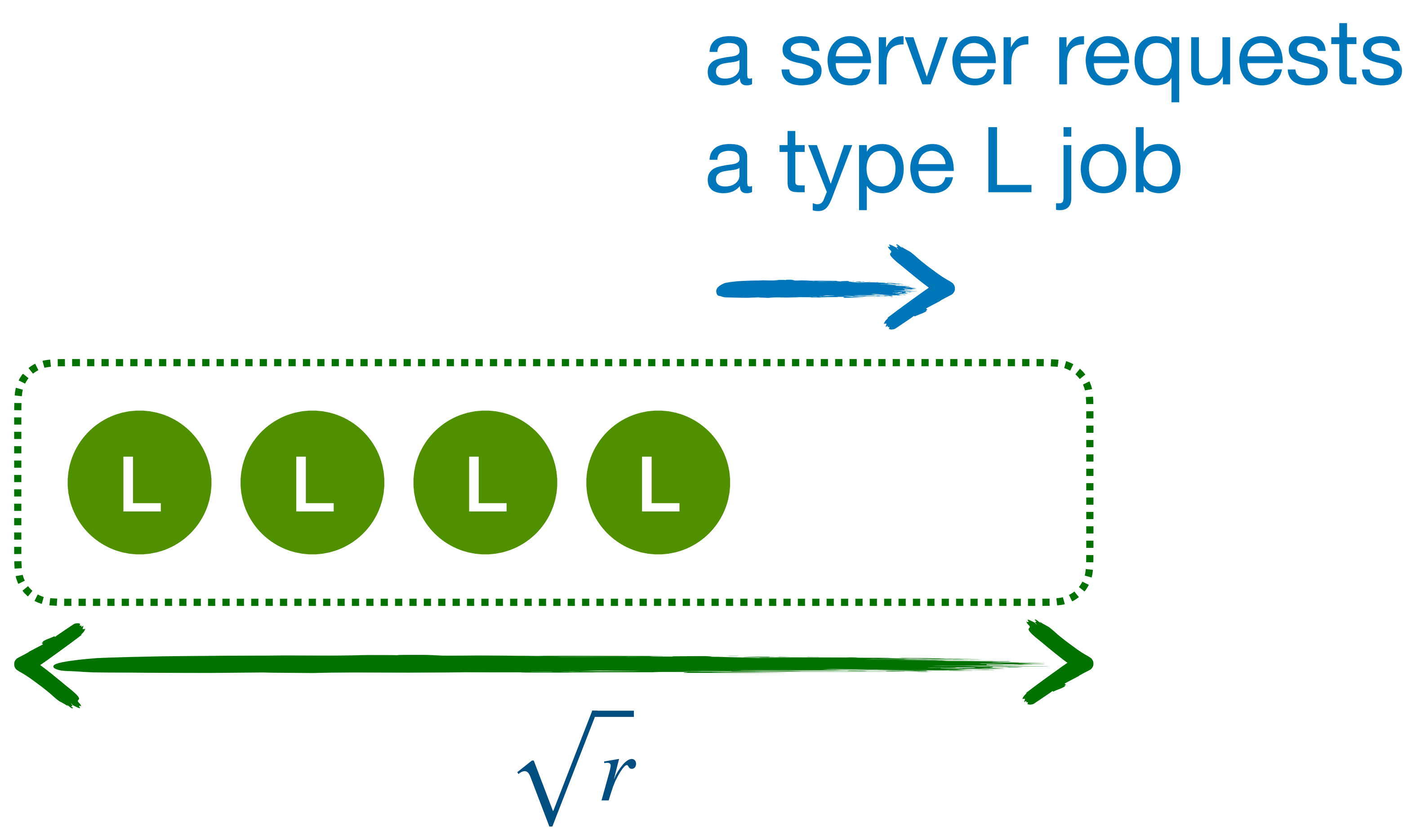

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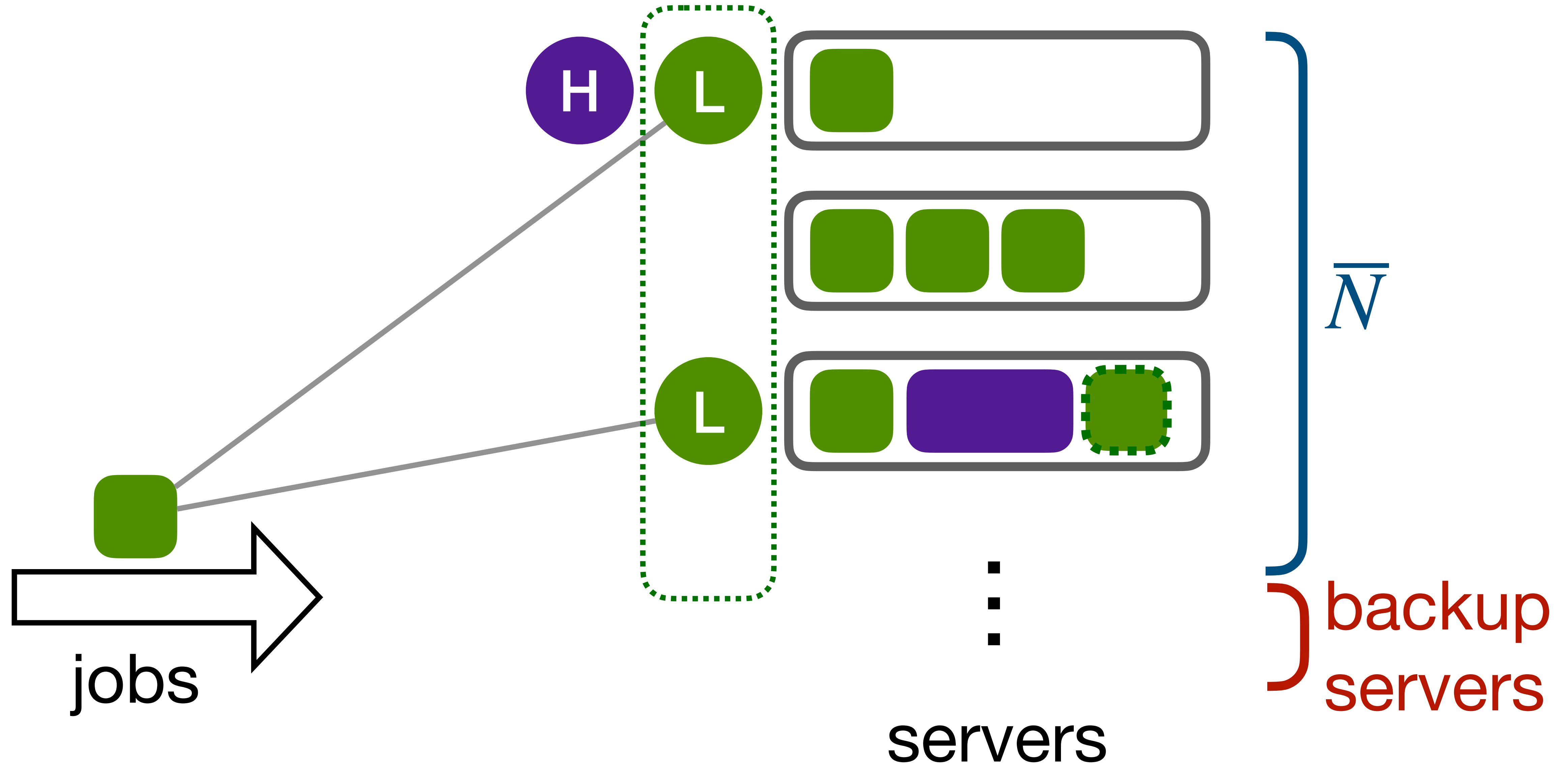

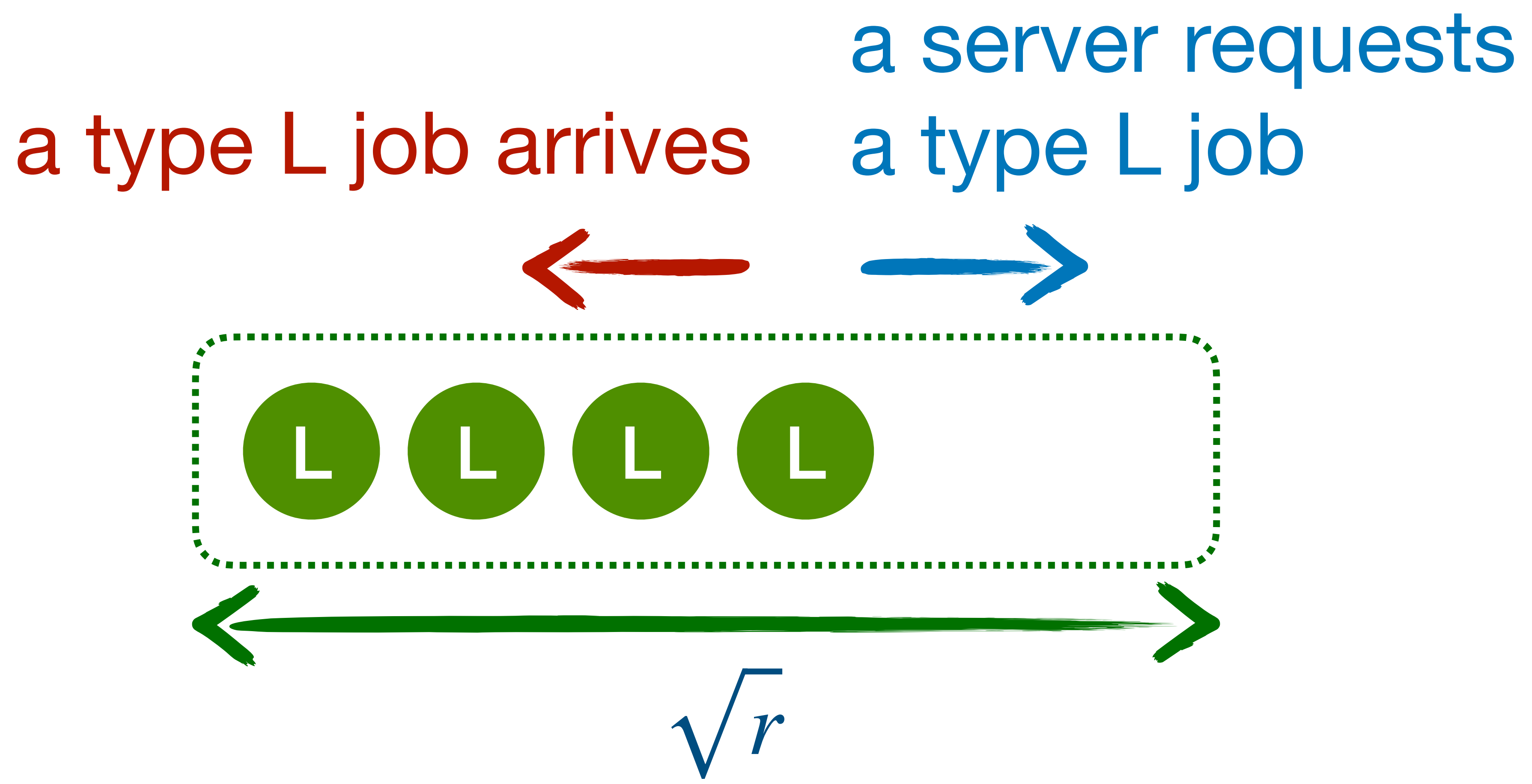

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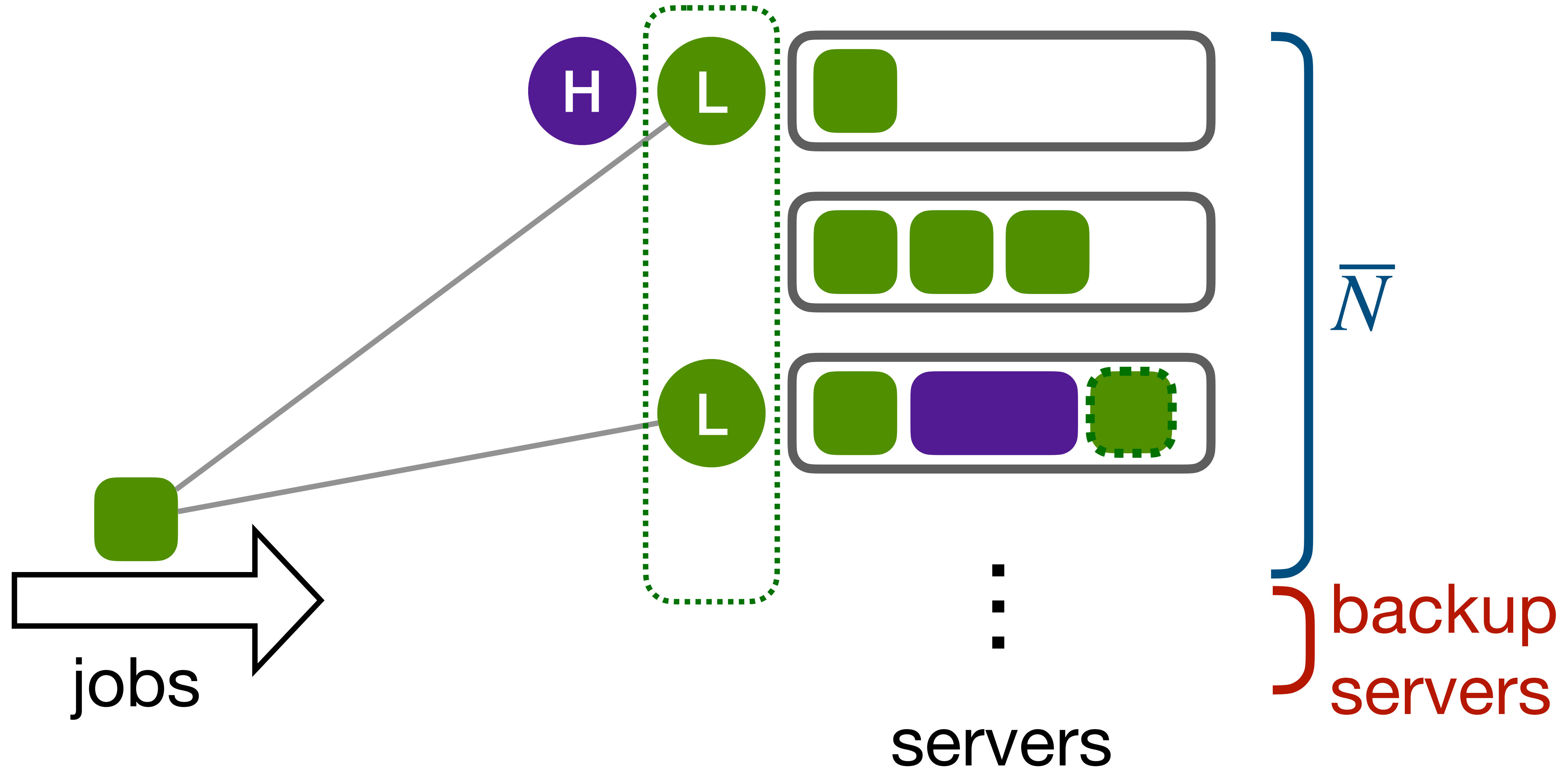
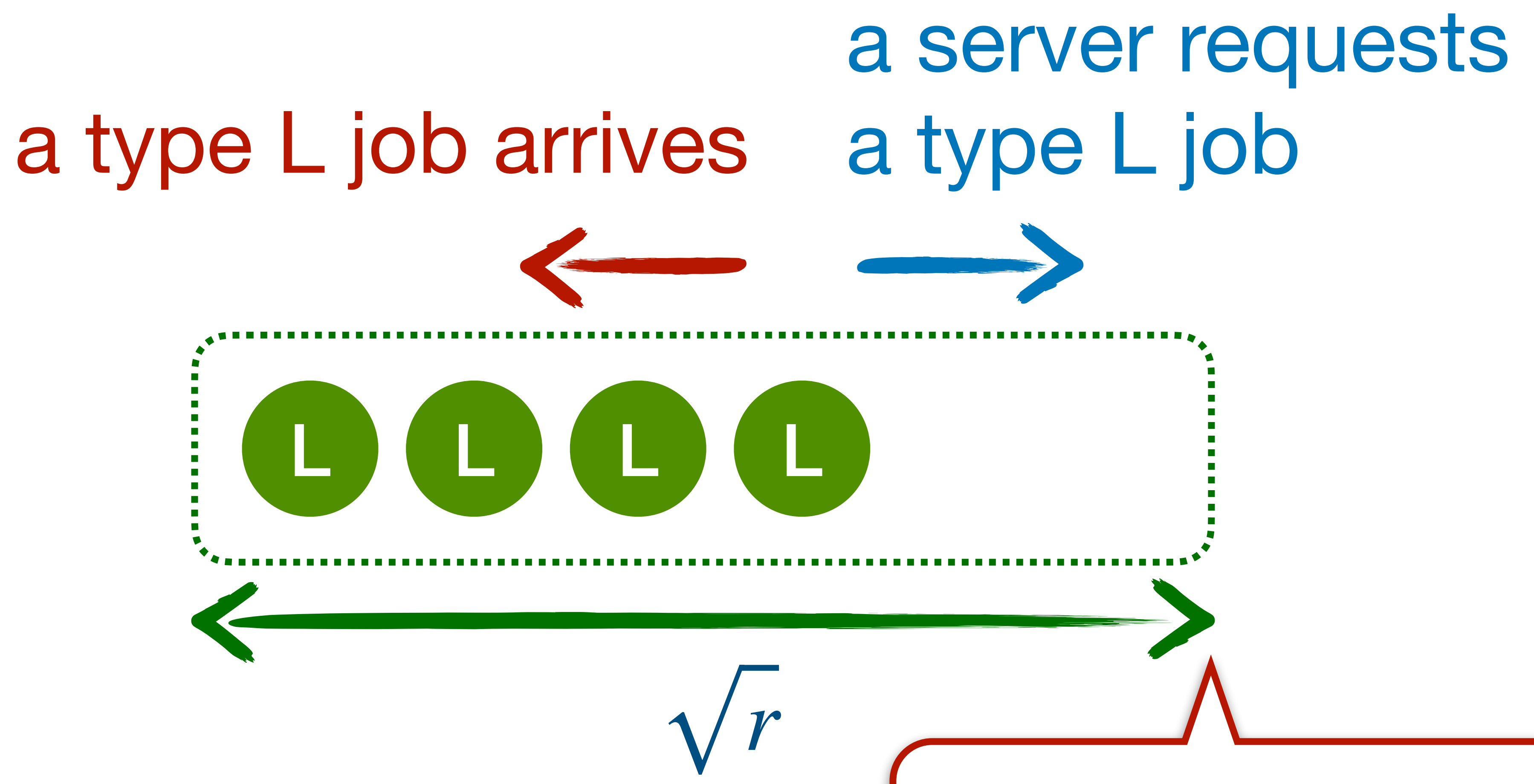
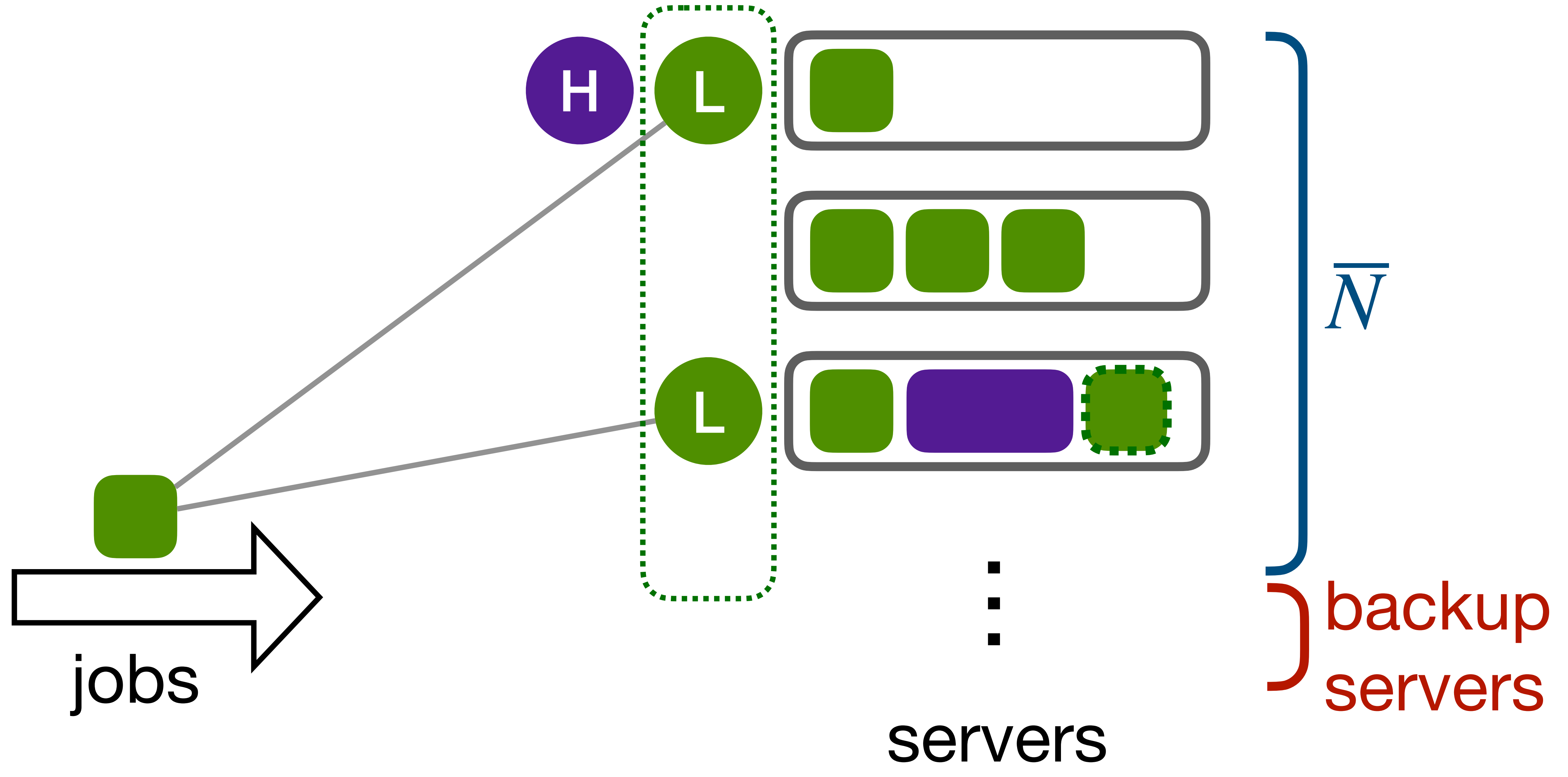




What happens when # tokens hits $\sqrt{r}$?

Key proof idea 2



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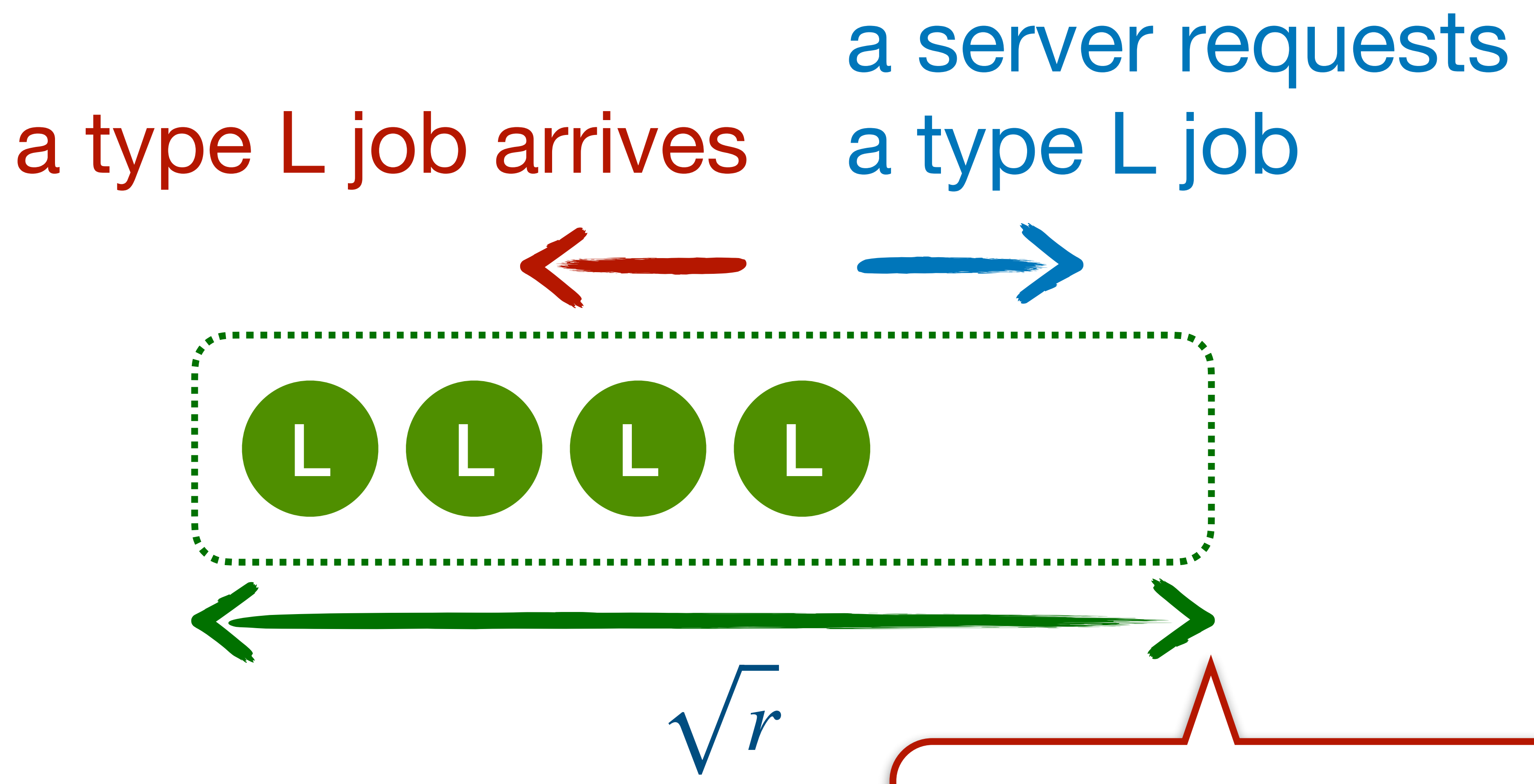

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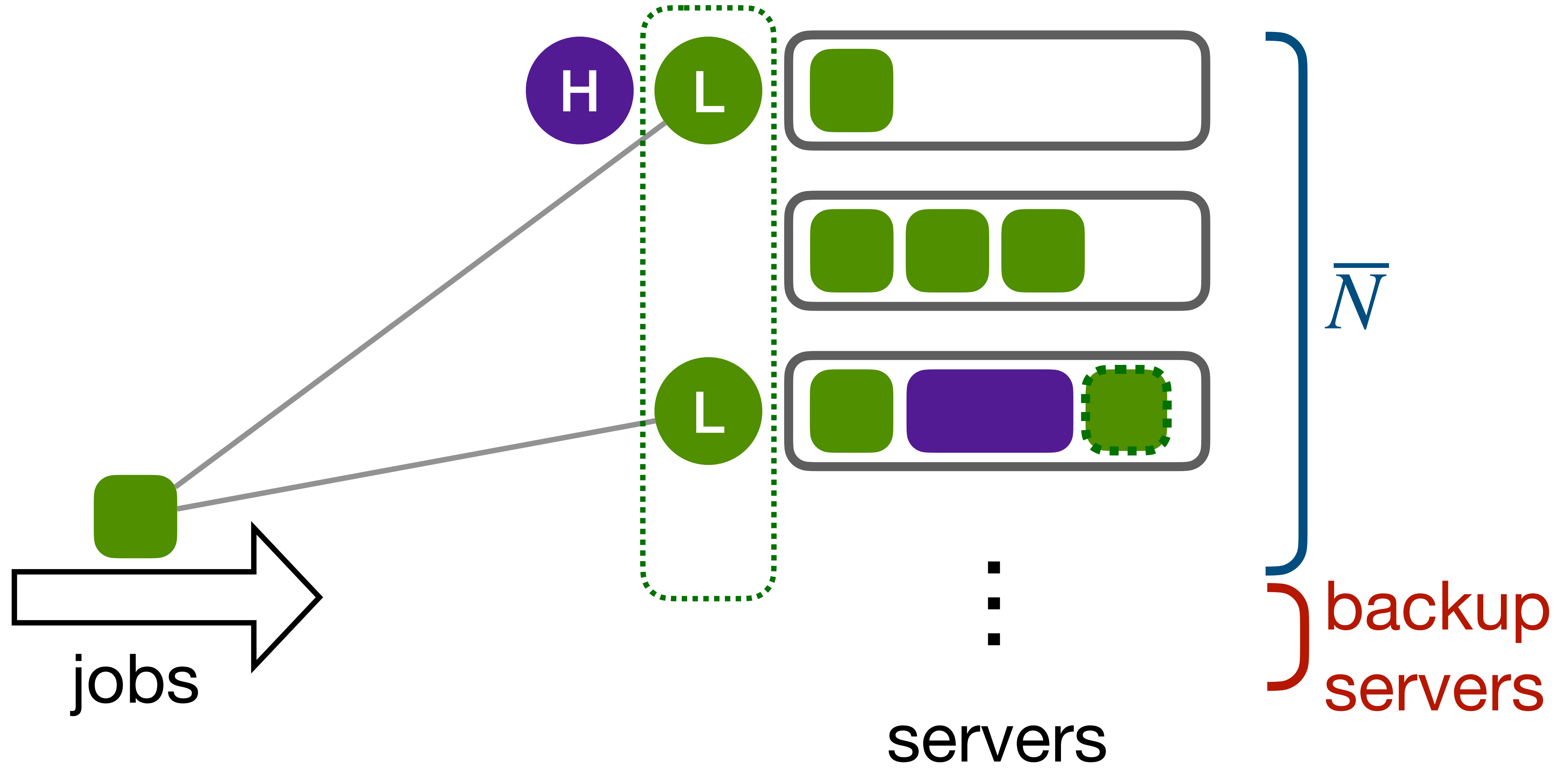
Generate a virtual job

Key proof idea 2



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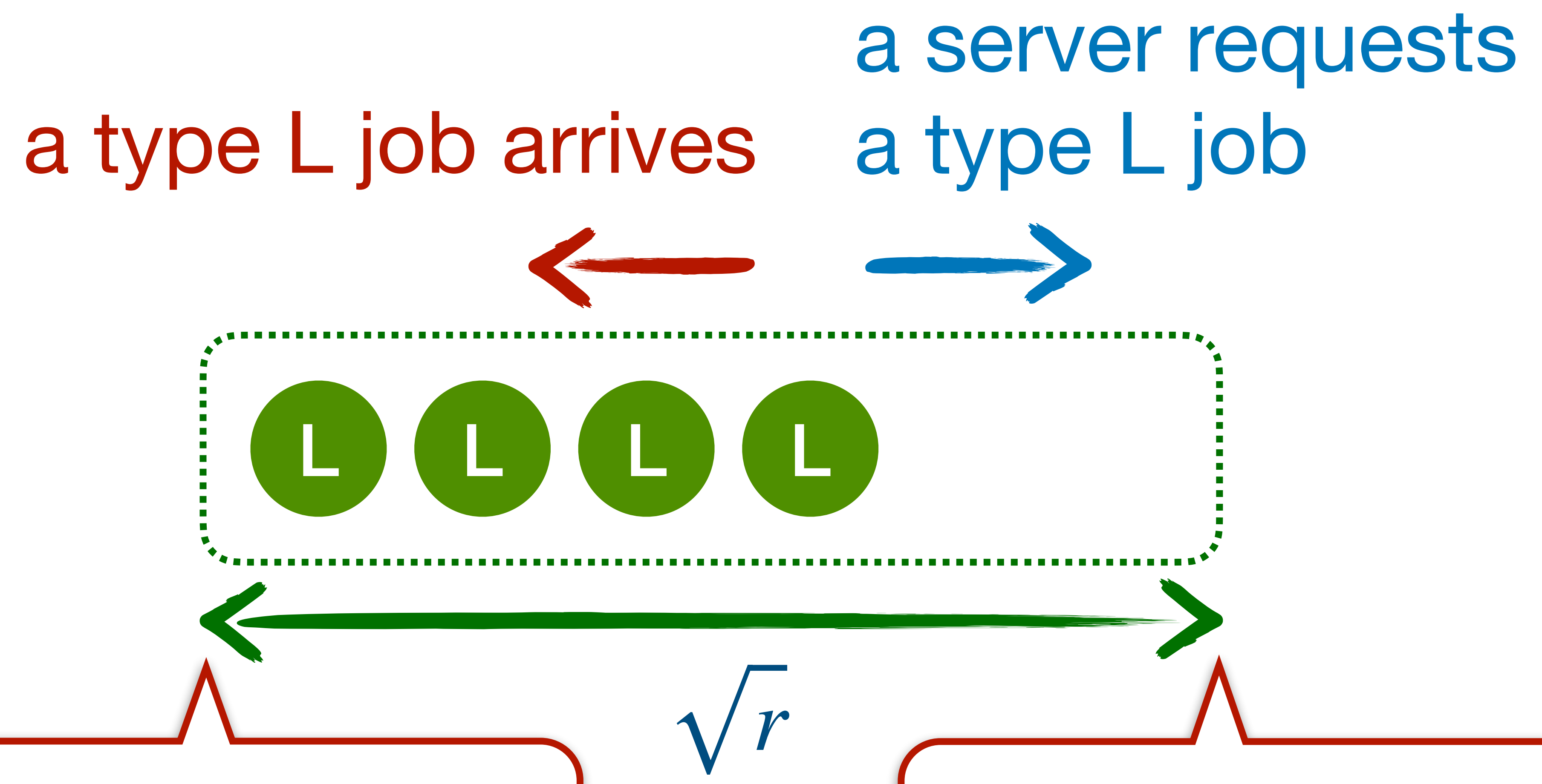

What happens when # tokens hits 0?

What happens when # tokens hits $\sqrt{r}$?

Generate a virtual job

Key proof idea 2

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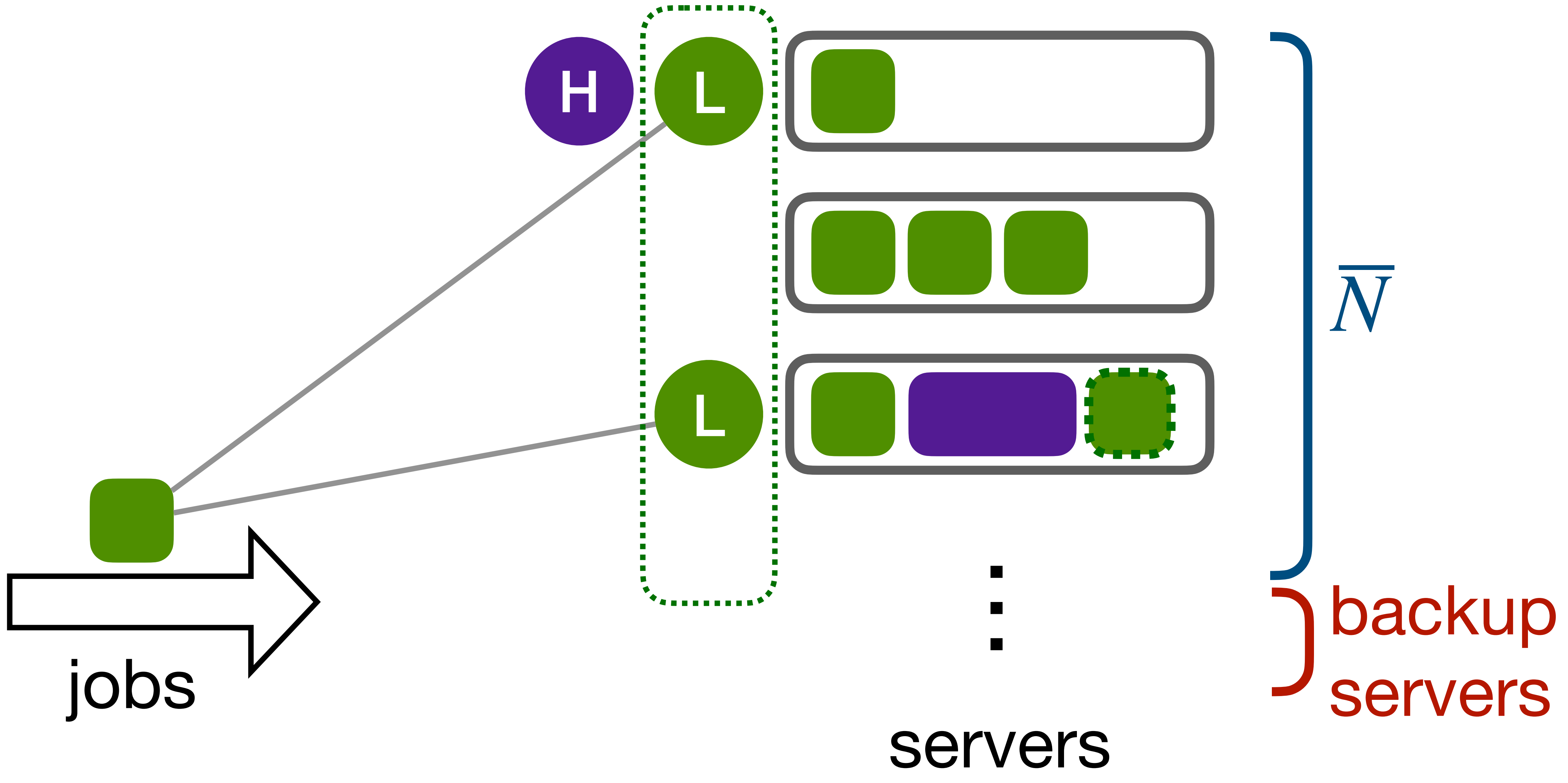

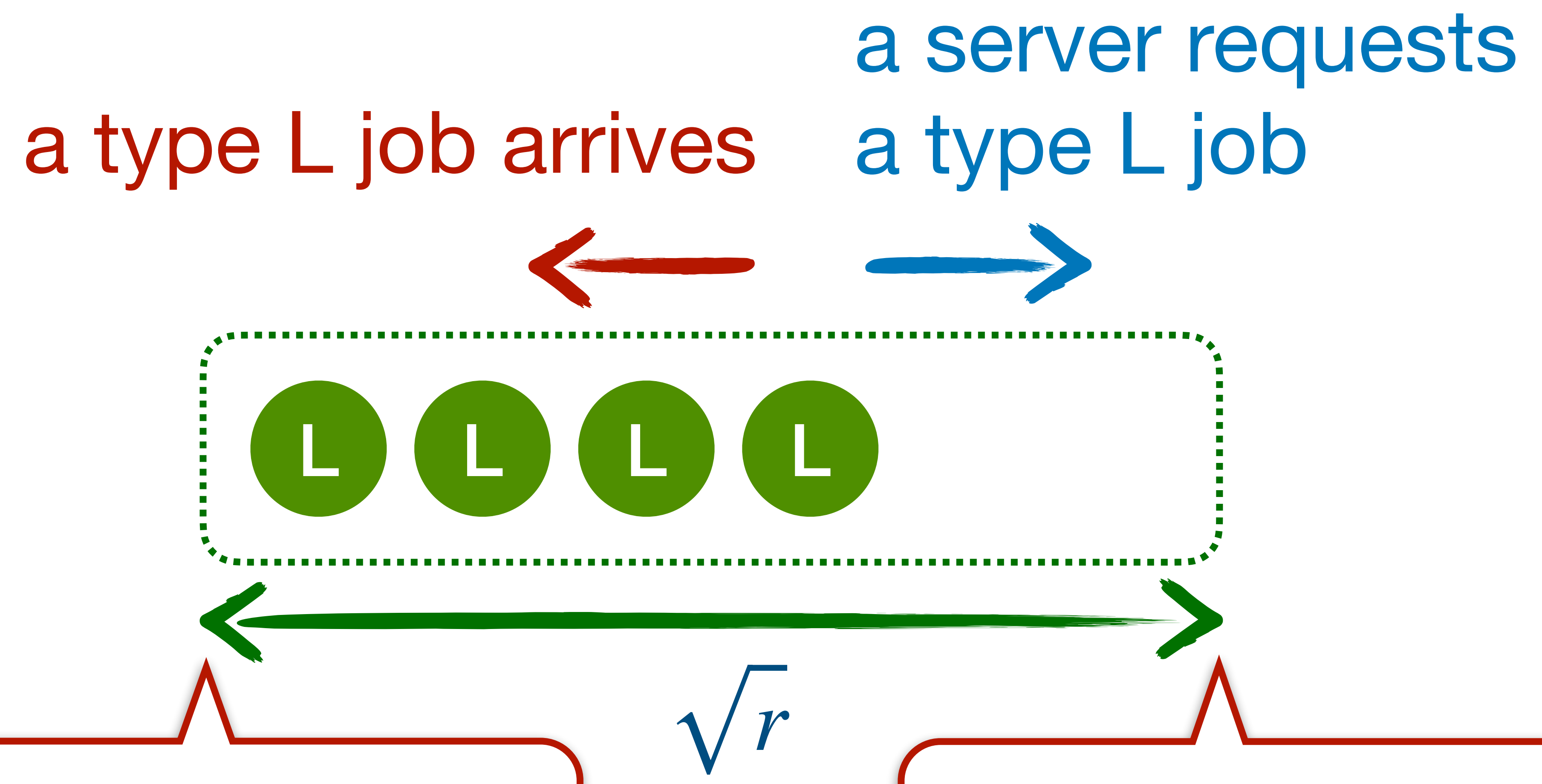

What happens when # tokens hits 0?

Generate a job to backup servers

What happens when # tokens hits $\sqrt{r}$?

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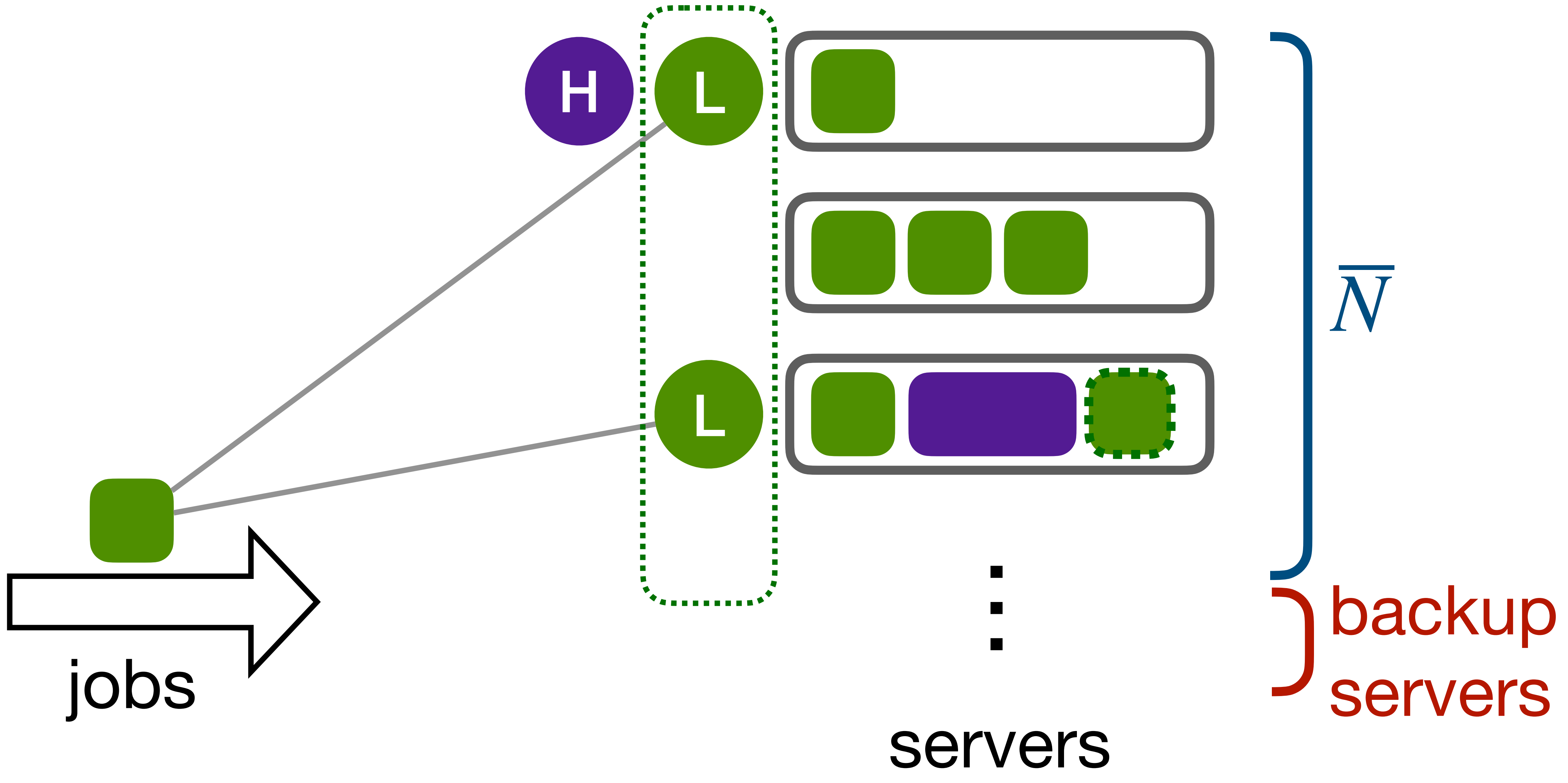
Key proof idea 2



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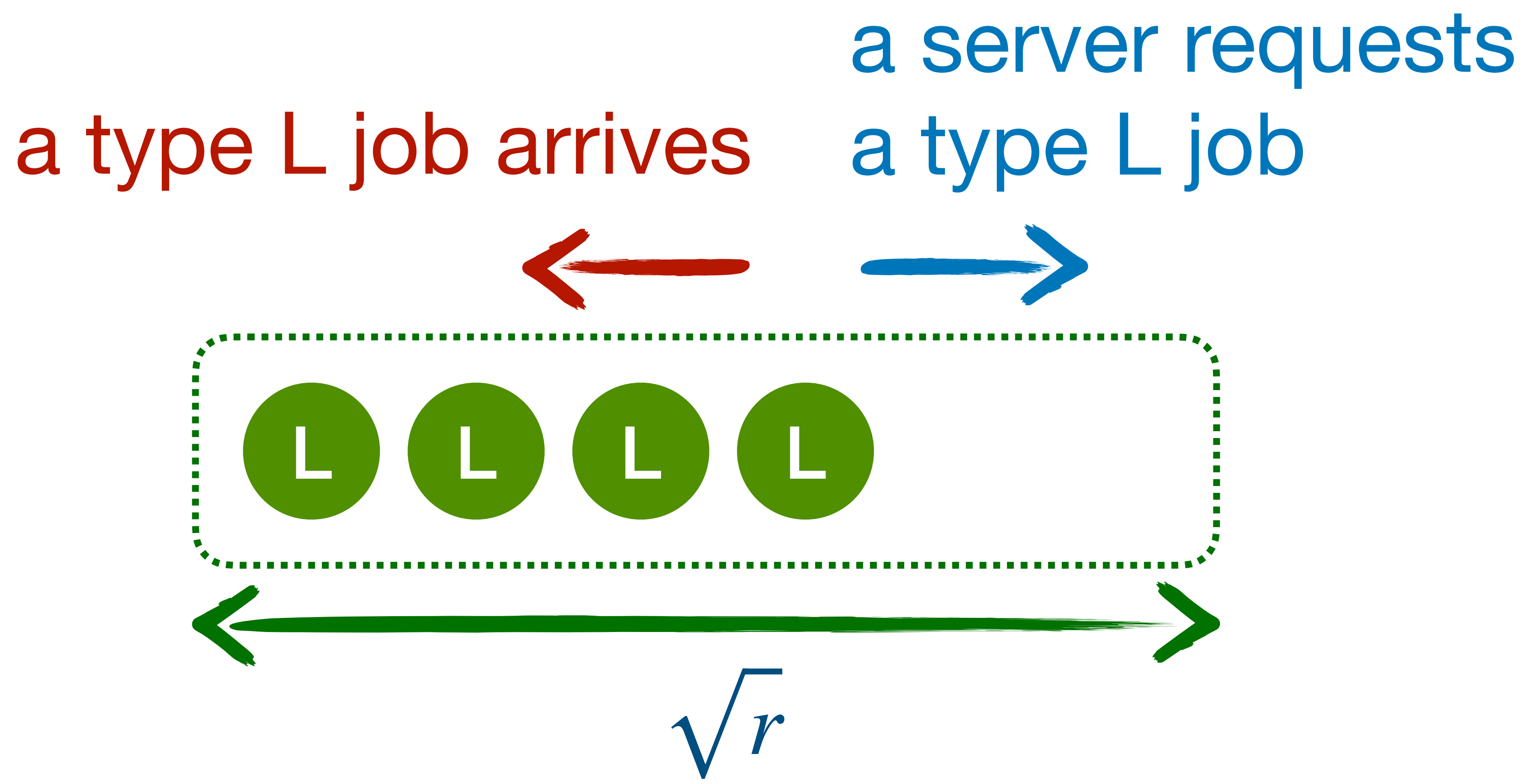

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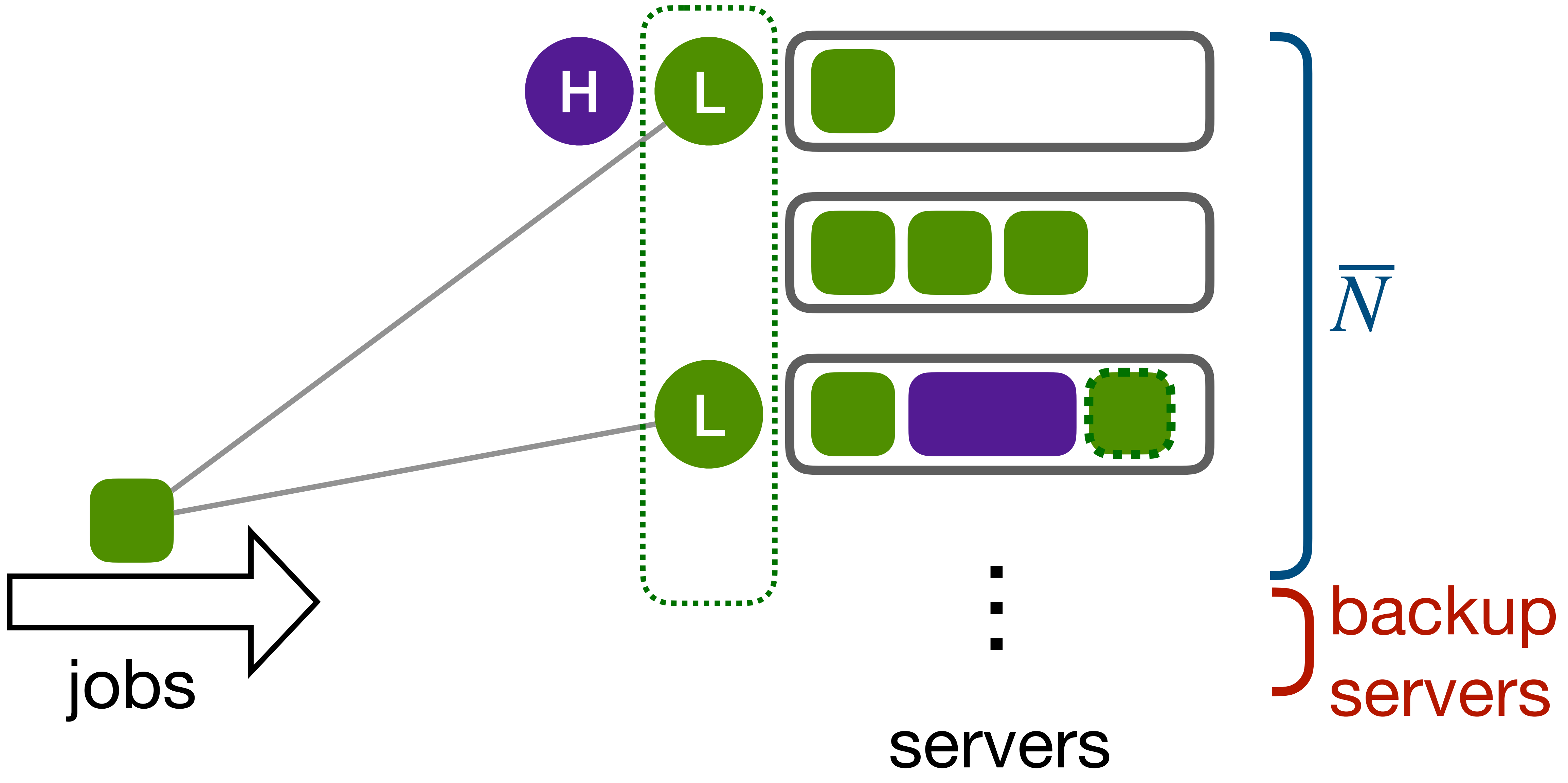

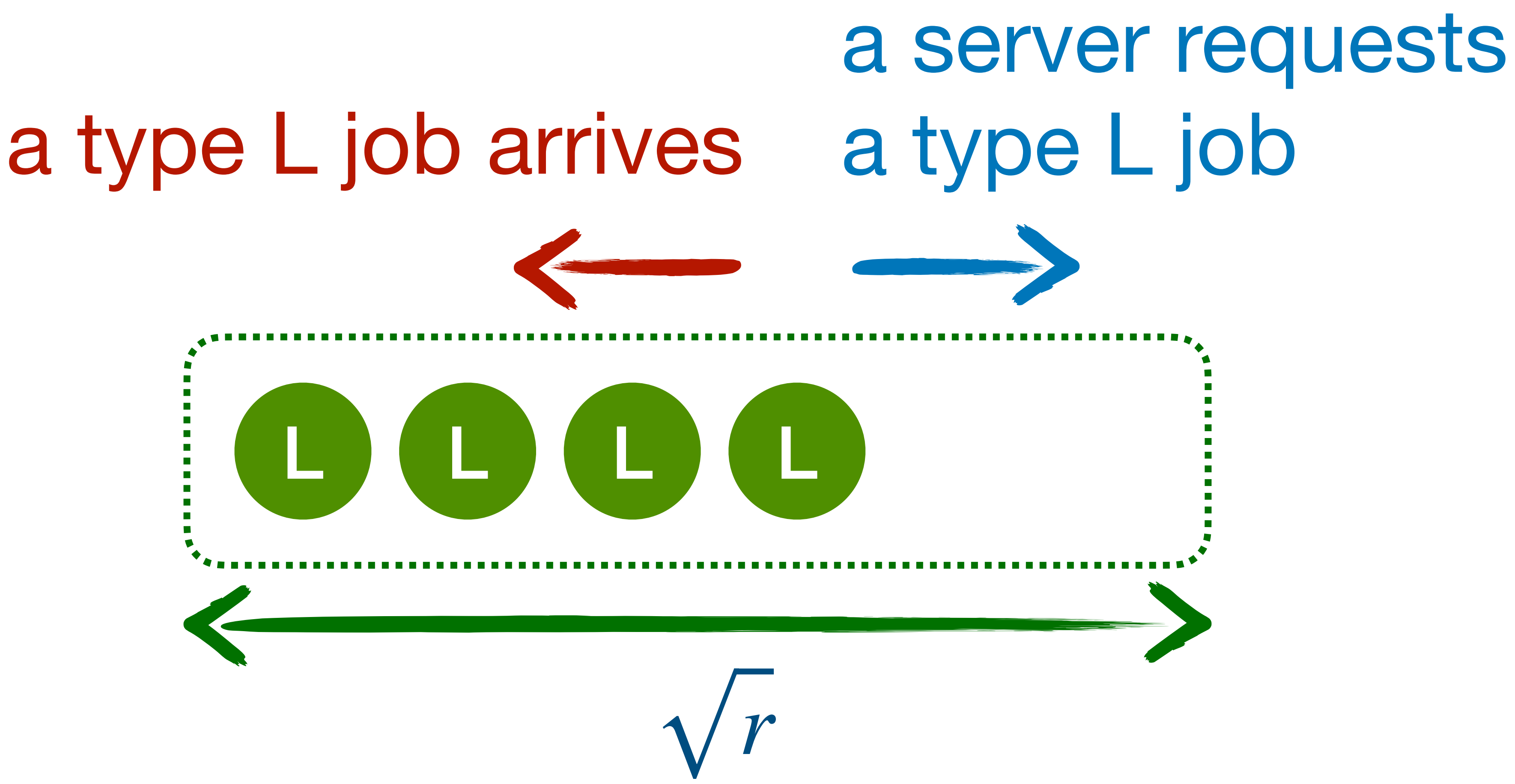

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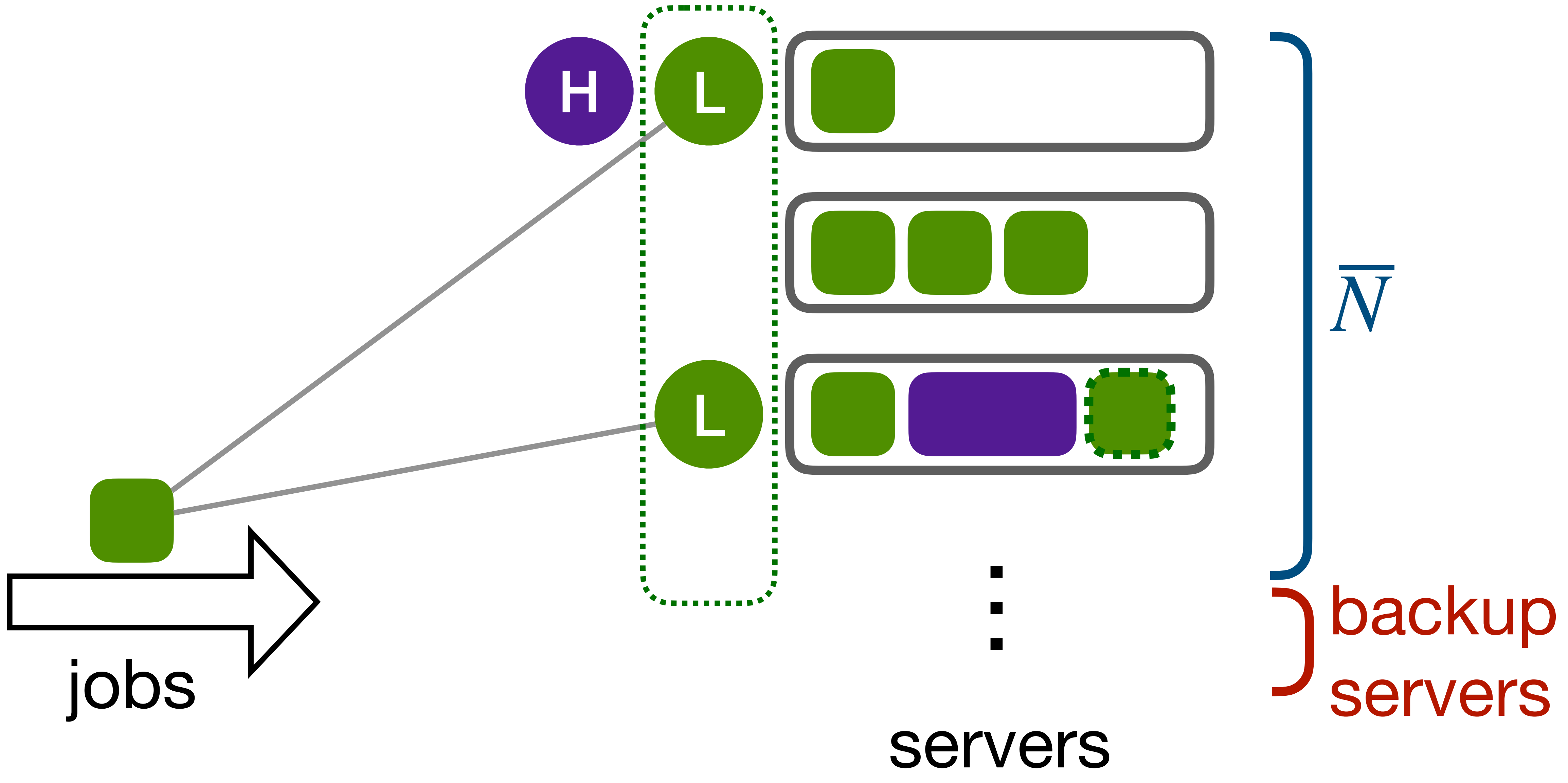

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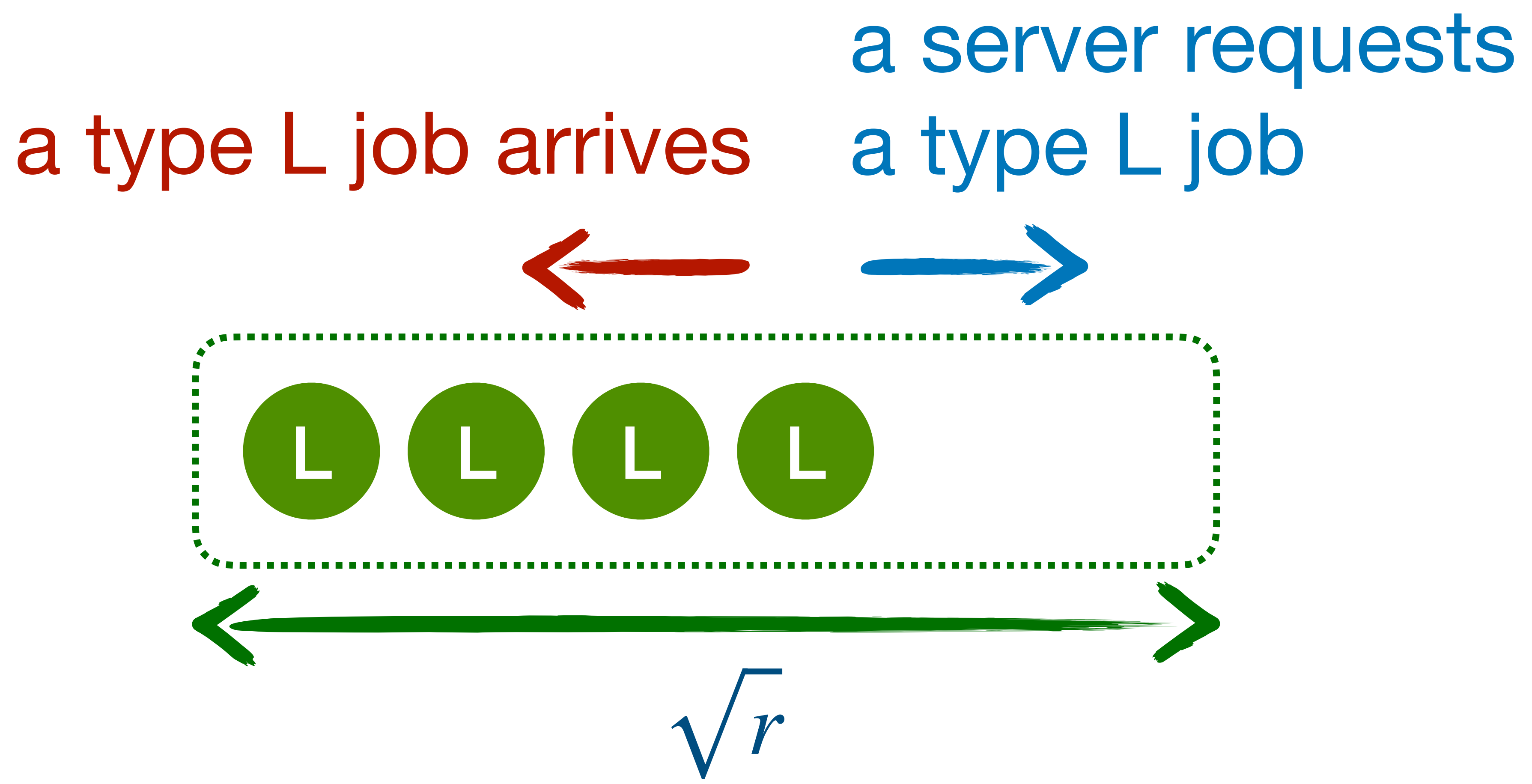
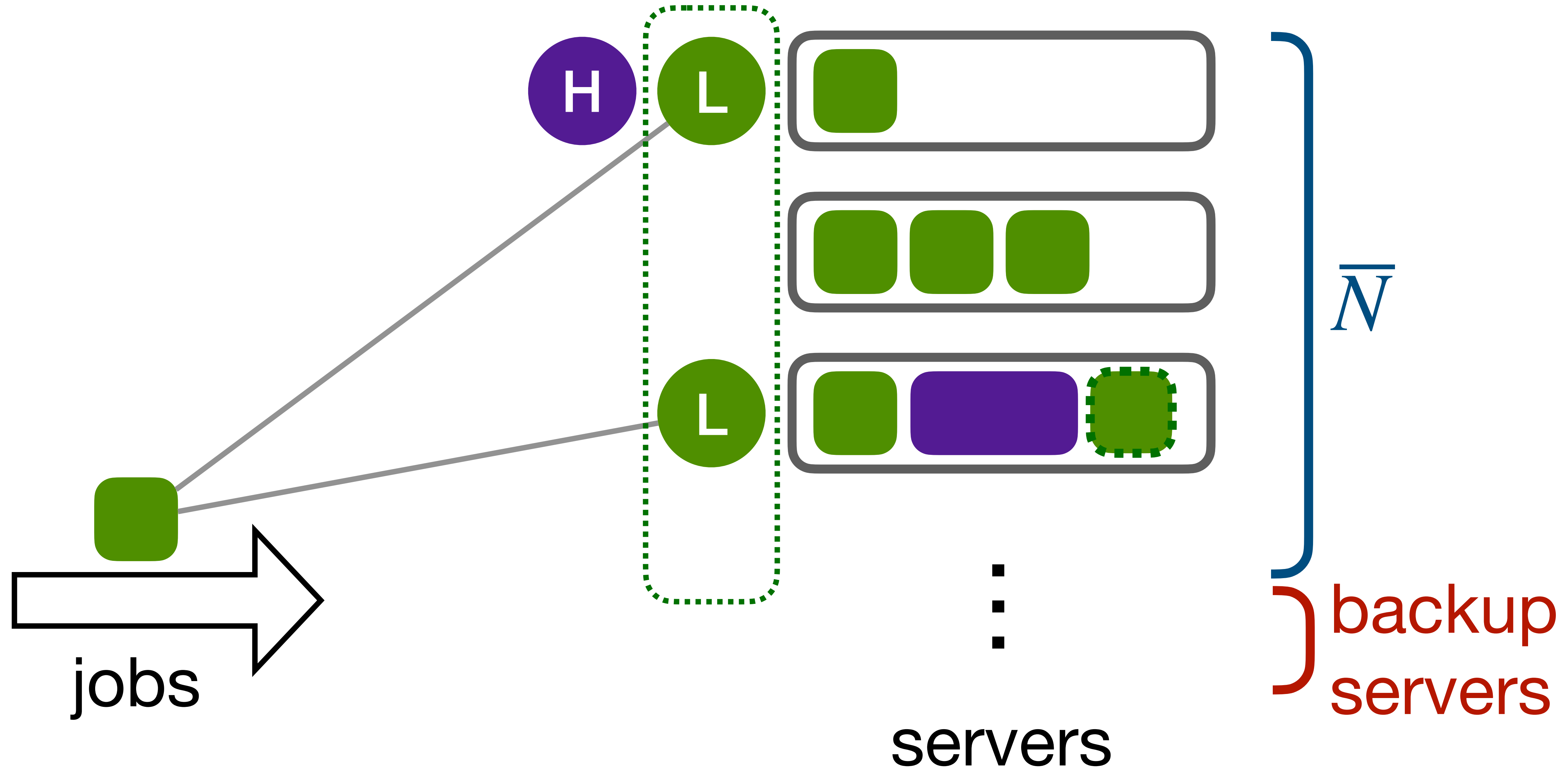


- An almost balanced random walk

Key proof idea 2



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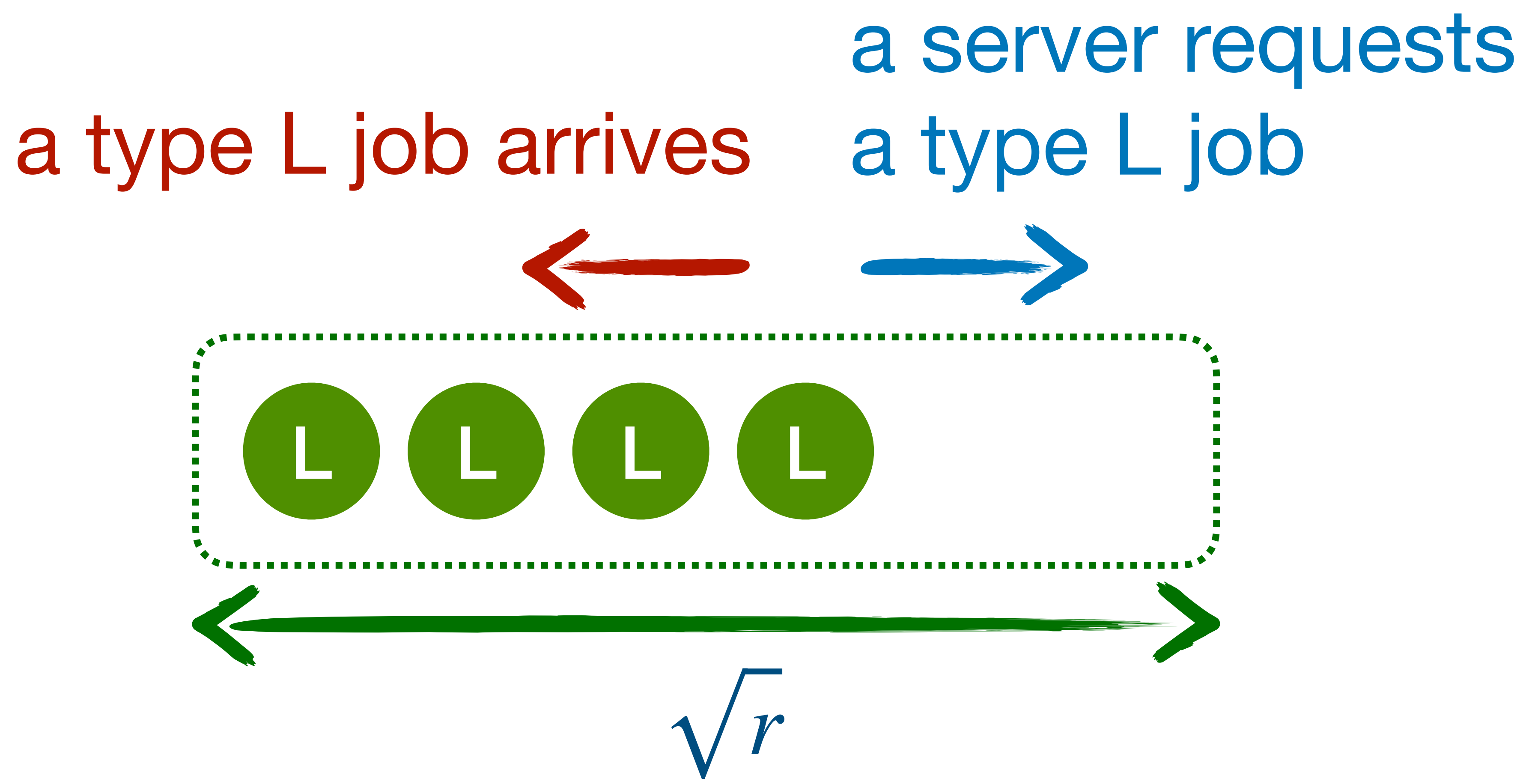

- An almost balanced random walk
- Stationary distribution ≈ uniform on $\{0, 1, \ldots, \sqrt{r}\}$

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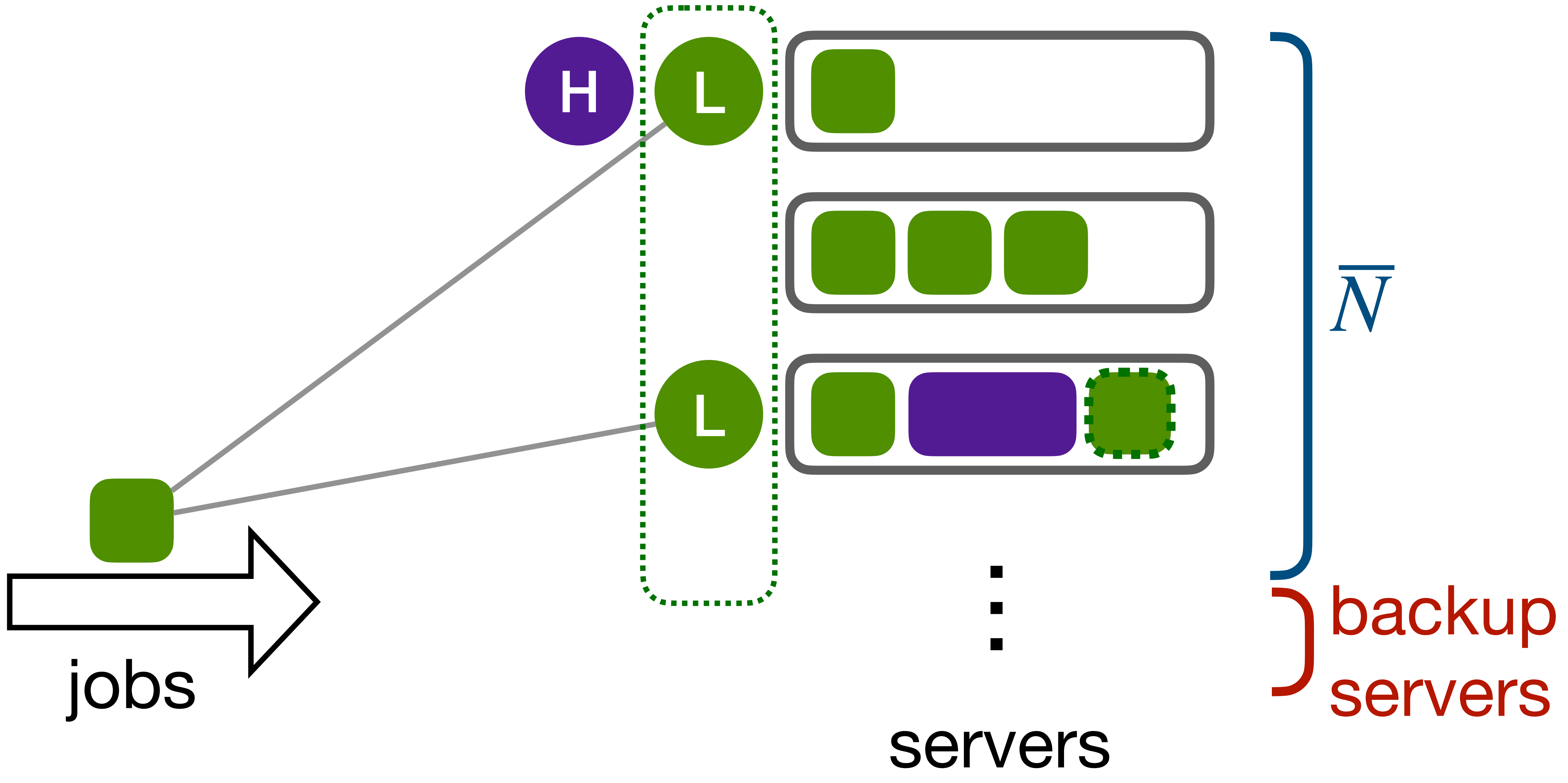

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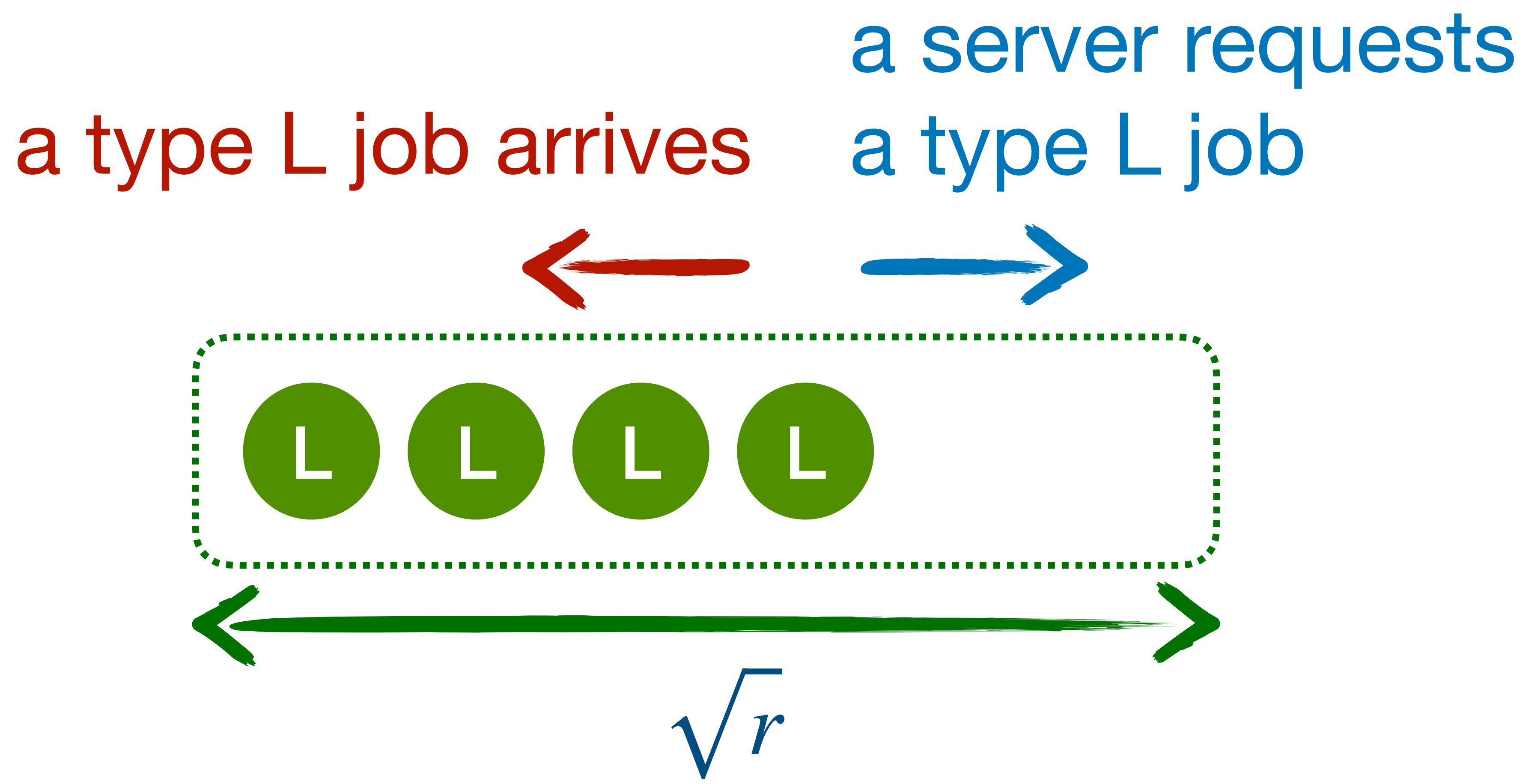
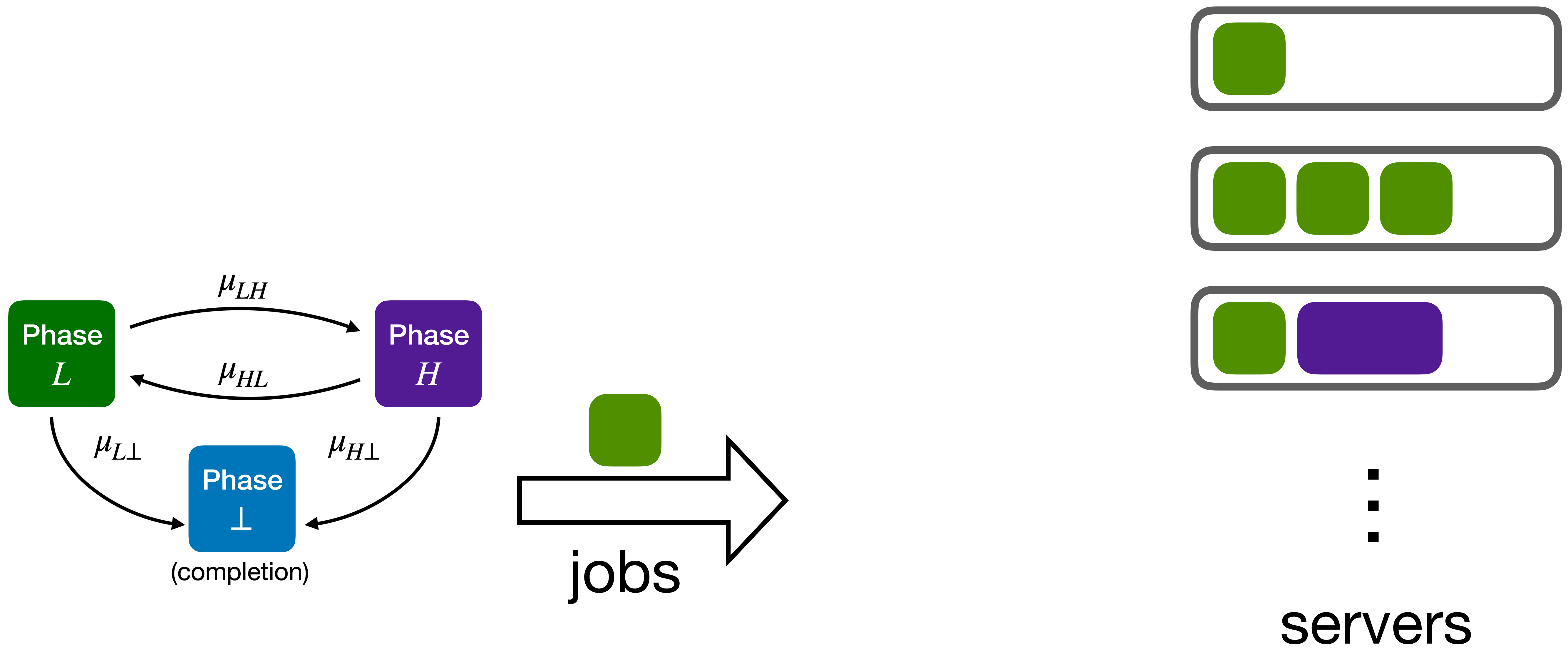


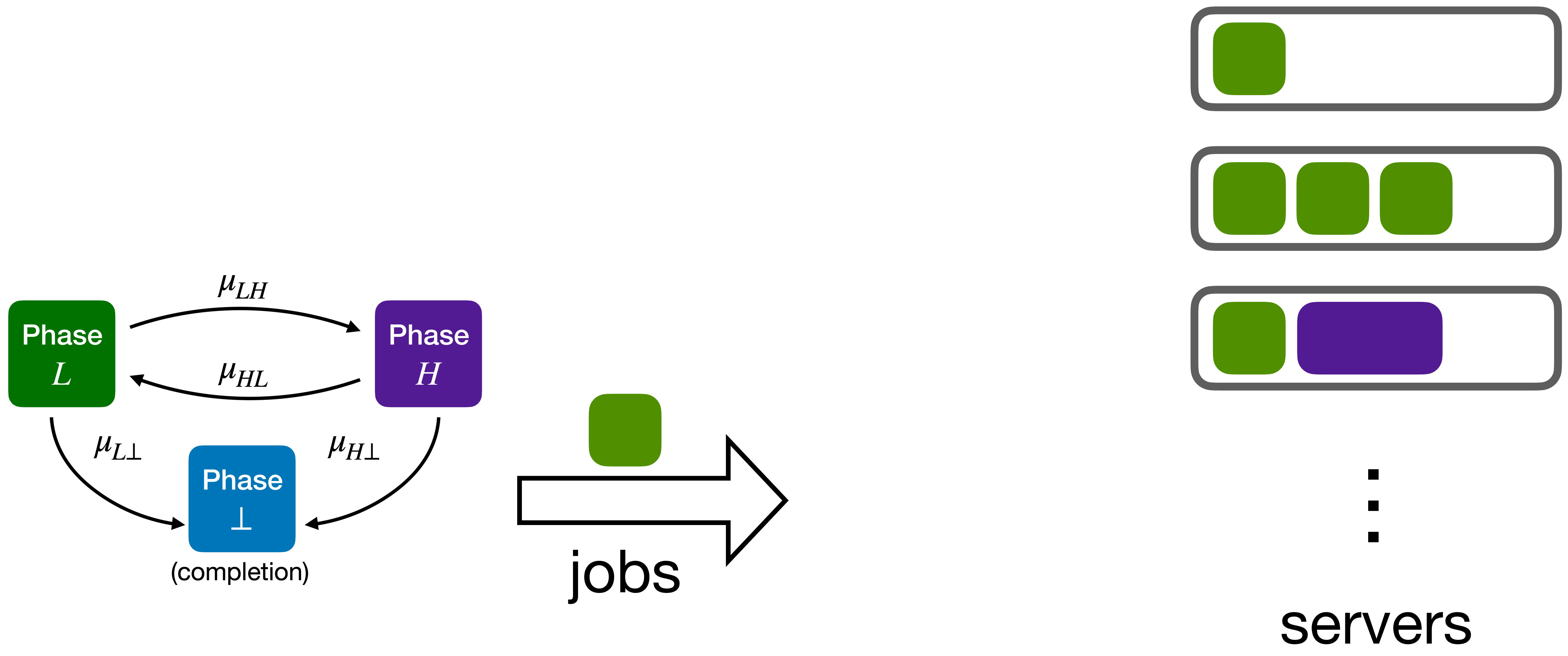
- An almost balanced random walk
- Stationary distribution ≈ uniform on {0, 1, …, $\sqrt{r}$}
- Rate of generating virtual jobs
  ≈ rate of sending jobs to backup servers
  ≈ arrival rate$/\sqrt{r} = O\left(\sqrt{r}\right)$

Summary


μ_{LH}
Phase L
μ_{HL}
Phase H
$\mu_{L\perp}$
$\mu_{H\perp}$
Phase $\perp$
(completion)
jobs
servers

Summary

- We considered the problem of assigning jobs to servers when jobs have time-varying resource requirements



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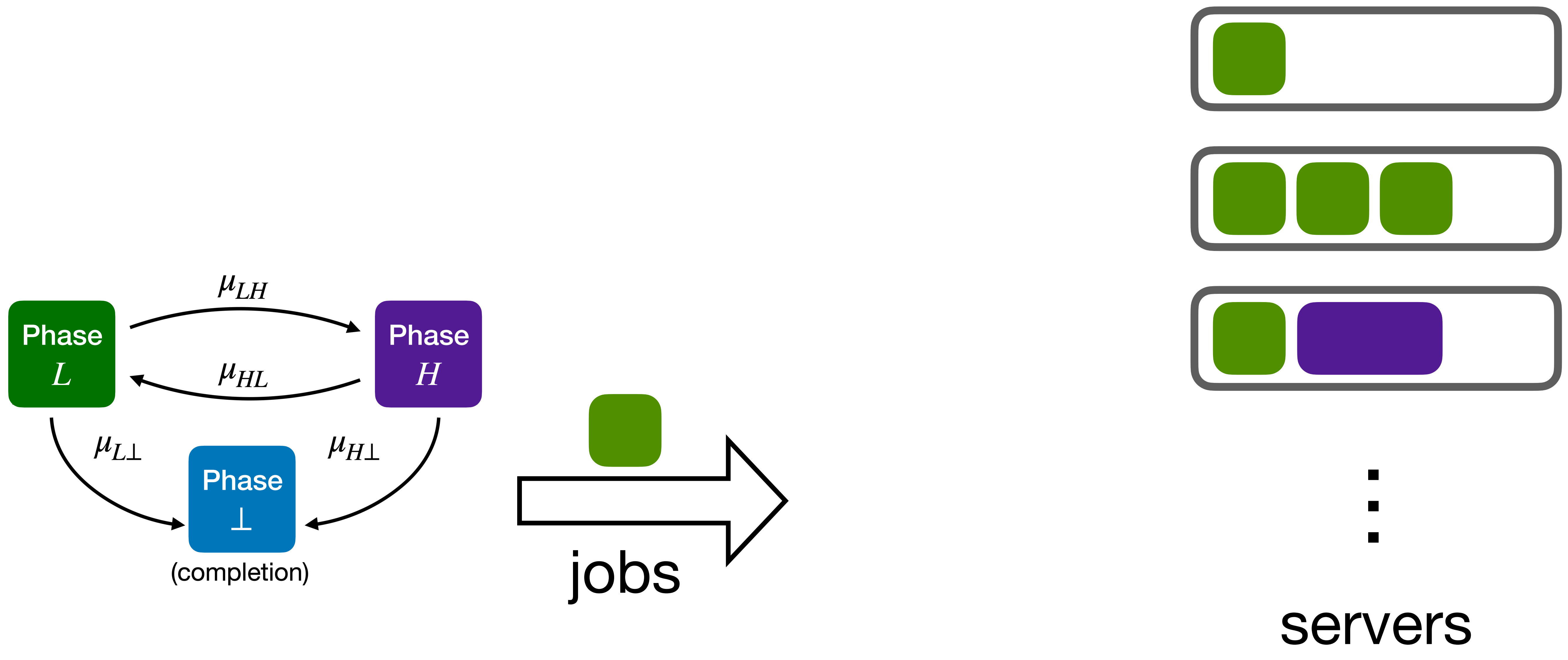

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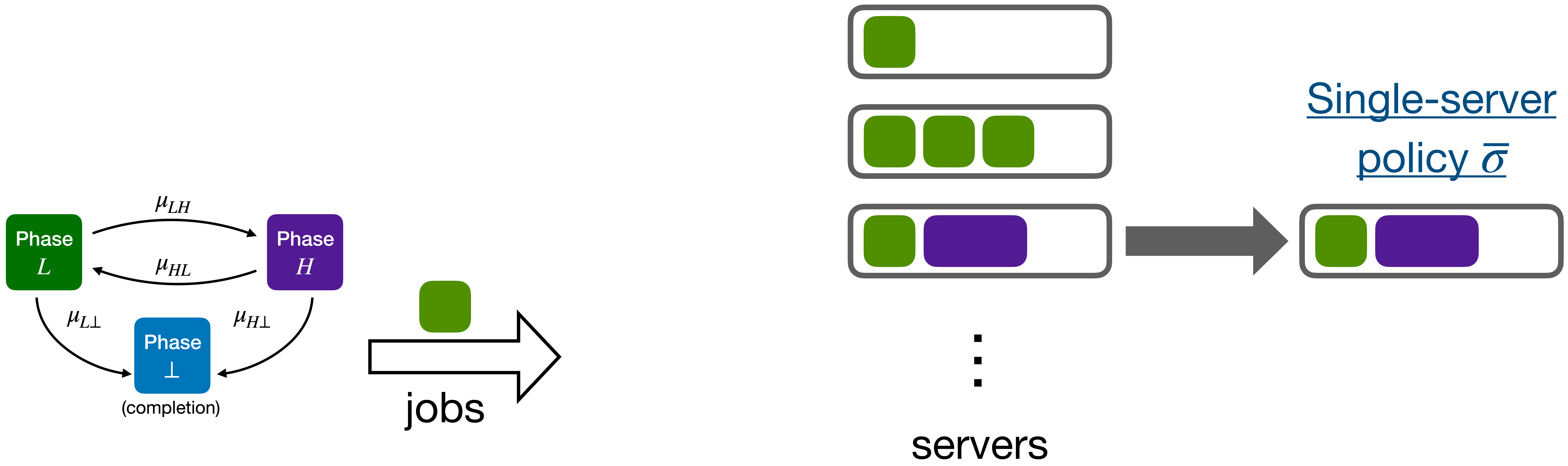

Summary

- We considered the problem of assigning jobs to servers when jobs have time-varying resource requirements
- We designed an asymptotically optimal policy
- We proposed a policy-conversion framework that allows us to reduce the policy-design problem to that in a single-server system
- A highlight of the framework is the meta-algorithm, JOIN-REQUESTING-SERVER (JRS), that converts a single-server policy to a policy in the original system