Common statistics for graphs

William Cohen

Why I'm talking about graphs

- Lots of large data *is* graphs
  - Facebook, Twitter, citation data, and other *social* networks
  - The web, the blogosphere, the semantic web, Freebase, Wikipedia, Twitter, and other *information* networks
  - Text corpora (like RCV1), large datasets with discrete feature values, and other *bipartite* networks
    - nodes = documents or words
    - links connect document → word or word → document
  - Computer networks, biological networks (proteins, ecosystems, brains, …), …
  - Heterogeneous networks with multiple types of nodes
    - people, groups, documents

A question

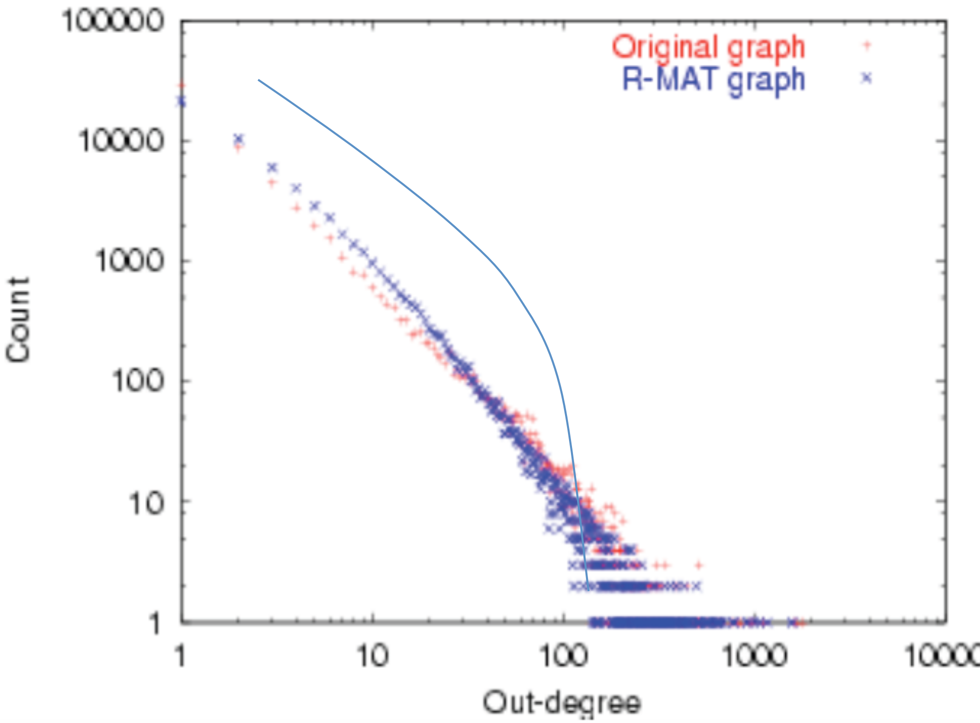
- How do you explore a dataset?
  - compute statistics (e.g., feature histograms, conditional feature histograms, correlation coefficients, …)
  - sample and inspect
    - run a bunch of small-scale experiments
- How do you explore a graph?
  - compute statistics (degree distribution, …)
  - sample and inspect
    - how do you sample? non-trivial!

	network	type	#vertices n	#edges m	mean degree z	mean distance between vertices ℓ	α	$C^{(1)}$	$C^{(2)}$
social	film actors	undirected	449 913	25 516 482	113.43	3.48	2.3	0.20	0.78
	company directors	undirected	7 673	55 392	14.44	4.60	–	0.59	0.88
	math coauthorship	undirected	253 339	496 489	3.92	7.57	–	0.15	0.34
	physics coauthorship	undirected	52 909	245 300	9.27	6.19	–	0.45	0.56
	biology coauthorship	undirected	1 520 251	11 803 064	15.53	4.92	–	0.088	0.60
	telephone call graph	undirected	47 000 000	80 000 000	3.16		2.1		
	email messages	directed	59 912	86 300	1.44	4.95	1.5/2.0		0.16
	email address books	directed	16 881	57 029	3.38	5.22	–	0.17	0.13
	student relationships	undirected	573	477	1.66	16.01	–	0.005	0.001
	sexual contacts	undirected	2 810				3.2		
information	WWW `nd.edu`	directed	269 504	1 497 135	5.55	11.27	2.1/2.4	0.11	0.29
	WWW Altavista	directed	203 549 046	2 130 000 000	10.46	16.18	2.1/2.7		
	citation network	directed	783 339	6 716 198	8.57		3.0/–		
	Roget's Thesaurus	directed	1 022	5 103	4.99	4.87	–	0.13	0.15
	word co-occurrence	undirected	460 902	17 000 000	70.13		2.7		0.44
technological	Internet	undirected	10 697	31 992	5.98	3.31	2.5	0.035	0.39
	power grid	undirected	4 941	6 594	2.67	18.99	–	0.10	0.080
	train routes	undirected	587	19 603	66.79	2.16	–		0.69
	software packages	directed	1 439	1 723	1.20	2.42	1.6/1.4	0.070	0.082
	software classes	directed	1 377	2 213	1.61	1.51	–	0.033	0.012
	electronic circuits	undirected	24 097	53 248	4.34	11.05	3.0	0.010	0.030
	peer-to-peer network	undirected	880	1 296	1.47	4.28	2.1	0.012	0.011
biological	metabolic network	undirected	765	3 686	9.64	2.56	2.2	0.090	0.67
	protein interactions	undirected	2 115	2 240	2.12	6.80	2.4	0.072	0.071
	marine food web	directed	135	598	4.43	2.05	–	0.16	0.23
	freshwater food web	directed	92	997	10.84	1.90	–	0.20	0.087
	neural network	directed	307	2 359	7.68	3.97	–	0.18	0.28

Degree distribution

- Plot cumulative degree
  – X axis is degree
  – Y axis is #nodes that have degree at least *k*
- Typically use a log-log scale
  – Straight lines are a power law; normal curve dives to zero at some point
  – Left: trust network in epinions web site from Richardson & Domingos

count



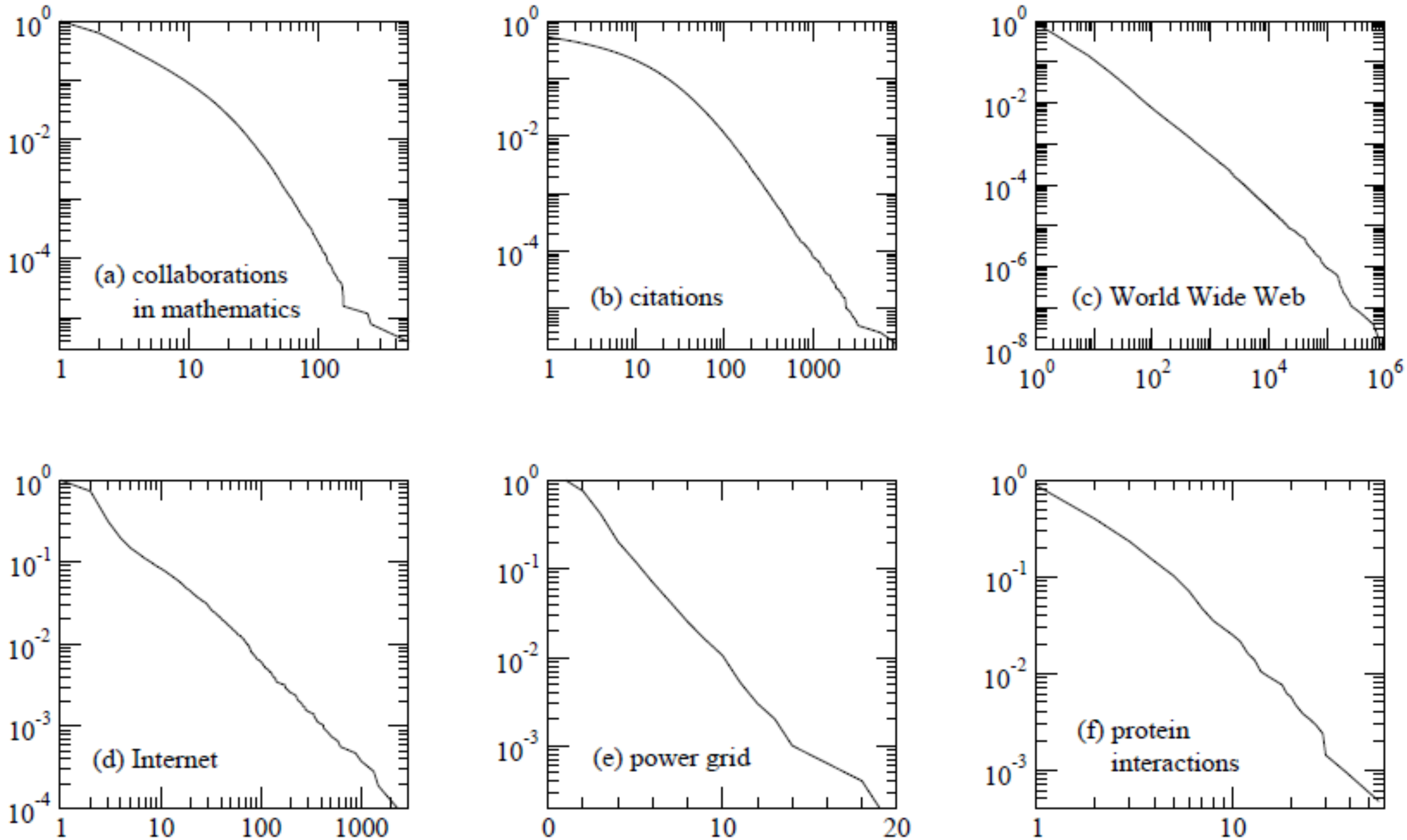


FIG. 6 Cumulative degree distributions for six different networks. The horizontal axis for each panel is vertex degree k (or in-degree for the citation and Web networks, which are directed) and the vertical axis is the cumulative probability distribution of degrees, i.e., the fraction of vertices that have degree greater than or equal to k. The networks shown are: (a) the collaboration network of mathematicians [182]; (b) citations between 1981 and 1997 to all papers cataloged by the Institute for Scientific Information [351]; (c) a 300 million vertex subset of the World Wide Web, *circa* 1999 [74]; (d) the Internet at the level of autonomous systems, April 1999 [86]; (e) the power grid of the western United States [416]; (f) the interaction network of proteins in the metabolism of the yeast *S. Cerevisiae* [212]. Of these networks, three of them, (c), (d) and (f), appear to have power-law degree distributions, as indicated by their approximately straight-line forms on the doubly logarithmic scales, and one (b) has a power-law tail but deviates markedly from power-law behavior for small degree. Network (e) has an exponential degree distribution (note the log-linear scales used in this panel) and network (a) appears to have a truncated power-law degree distribution of some type, or possibly two separate power-law regimes with different exponents.

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Homophily

- Another def'n: excess edges between common neighbors of *v*

$$CC(v) = \frac{\#\,\text{triangles connected to } v}{\#\,\text{pairs connected to } v}$$

$$CC(V,E) = \frac{1}{|V|}\sum_{v} CC(v)$$

$$\boxed{CC'(V,E) = \frac{\#\,\text{triangles in graph}}{\#\,\text{length 3 paths in graph}}}$$

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An important question

- How do you explore a dataset?
  - compute statistics (e.g., feature histograms, conditional feature histograms, correlation coefficients, …)
  - sample and inspect
    - run a bunch of small-scale experiments
- How do you explore a graph?
  - compute statistics (degree distribution, …)
  - sample and inspect
    - how do you sample?

Sampling from Large Graphs

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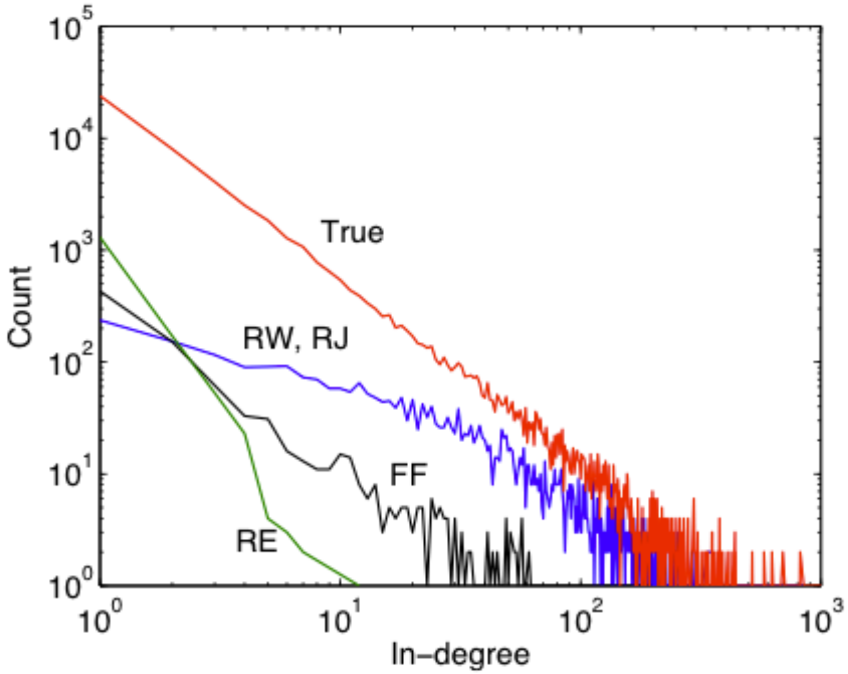
Brief summary

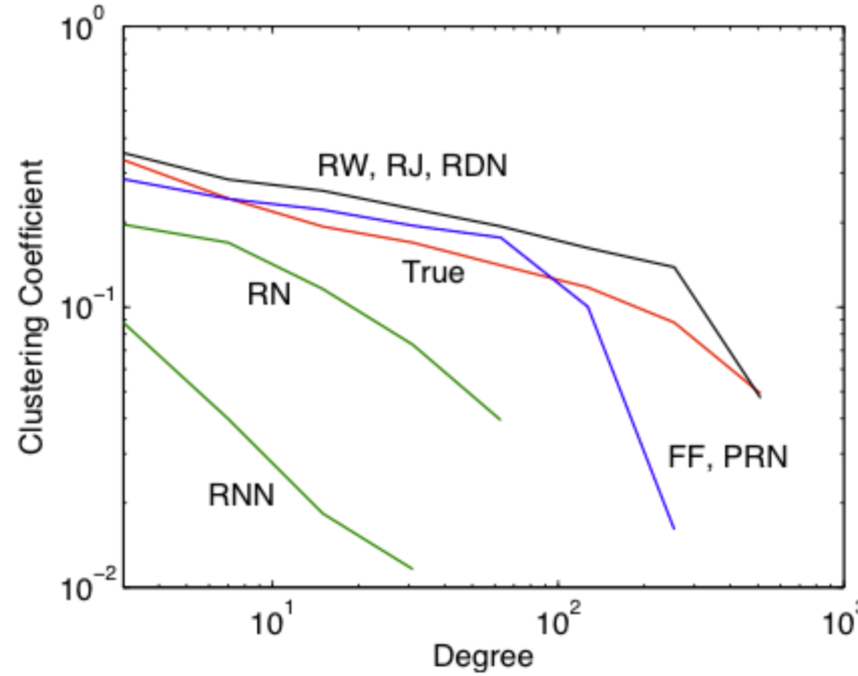
- Define *goals* of sampling:
  - "scale-down" – find G'<G with similar statistics
  - "back in time": for a growing G, find G'<G that is similar (statistically) to an earlier version of G
- Experiment on real graphs with plausible sampling methods, such as
  - RN – random nodes, sampled uniformly
  - ...
- See how well they perform

Brief summary

- Experiment on real graphs with plausible sampling methods, such as
  - RN – random nodes, sampled uniformly
    - RPN – random nodes, sampled by PageRank
    - RDP – random nodes sampled by in-degree
  - RE – random edges
  - RJ – run PageRank's "random surfer" for *n* steps
  - RW – run RWR's "random surfer" for *n* steps
  - FF – repeatedly pick *r(i)* neighbors of *i* to "burn", and then recursively sample from them

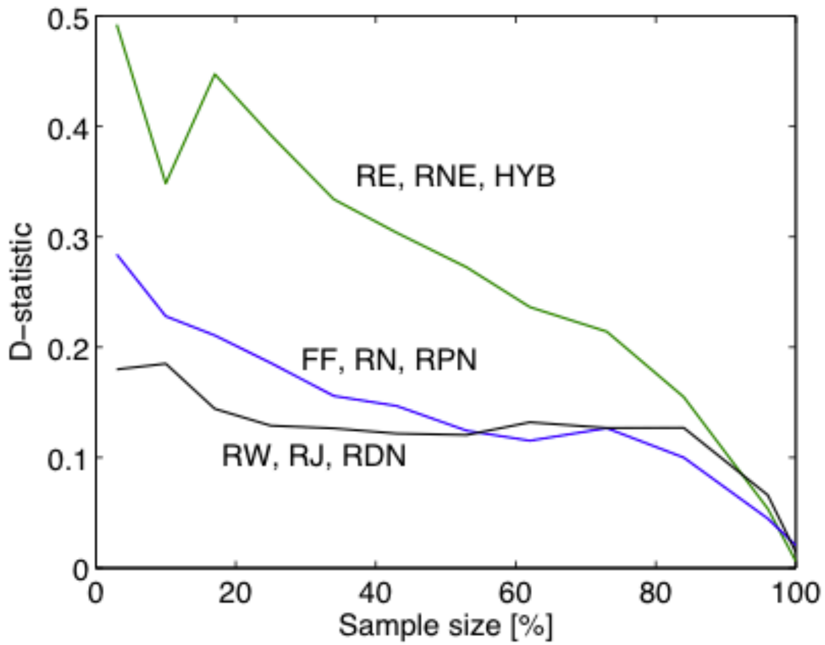


(a) S1: In-degree

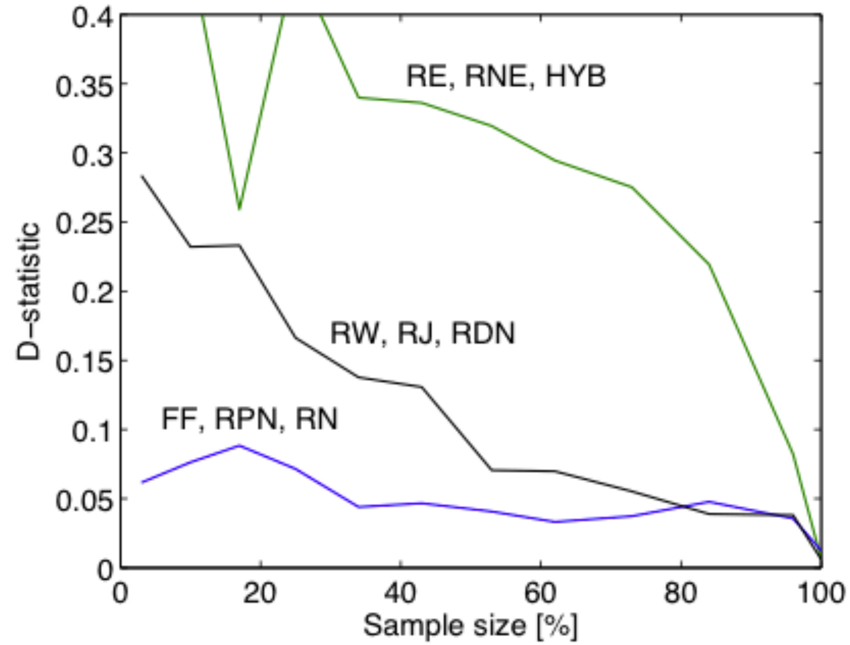


(b) S9: Clustering coef.

10% sample – pooled on five datasets



(a) Scale-down



(b) Back-in-time

d-statistic measures disagreement between distributions

- D=max{|F(x)-F'(x)|} where F, F' are cdf's
- max over nine different statistics

	Static graph patterns							
	in-deg	out-deg	wcc	scc	hops	sng-val	sng-vec	clust
RN	0.084	0.145	0.814	0.193	0.231	0.079	0.112	0.327
RPN	**0.062**	**0.097**	0.792	0.194	**0.200**	0.048	0.081	0.243
RDN	0.110	0.128	0.818	0.193	0.238	0.041	0.048	0.256
RE	0.216	0.305	**0.367**	0.206	0.509	0.169	0.192	0.525
RNE	0.277	0.404	0.390	0.224	0.702	0.255	0.273	0.709
HYB	0.273	0.394	0.386	0.224	0.683	0.240	0.251	0.670
RNN	0.179	0.014	0.581	0.206	0.252	0.060	0.255	0.398
RJ	0.132	0.151	0.771	0.215	0.264	0.076	0.143	**0.235**
RW	0.082	0.131	0.685	0.194	0.243	0.049	**0.033**	**0.243**
FF	0.082	0.105	0.664	0.194	**0.203**	**0.038**	0.092	**0.244**

Local Graph Partitioning using PageRank Vectors

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Fan Chung
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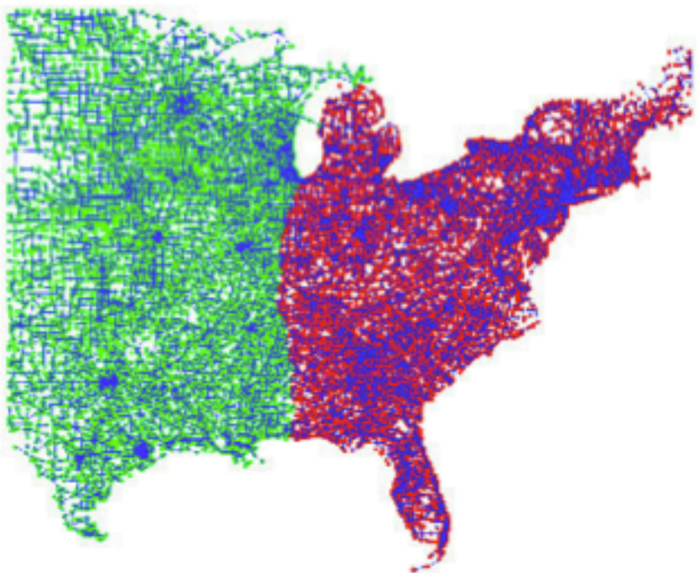
Kevin Lang
Yahoo! Research

FOCS 2006

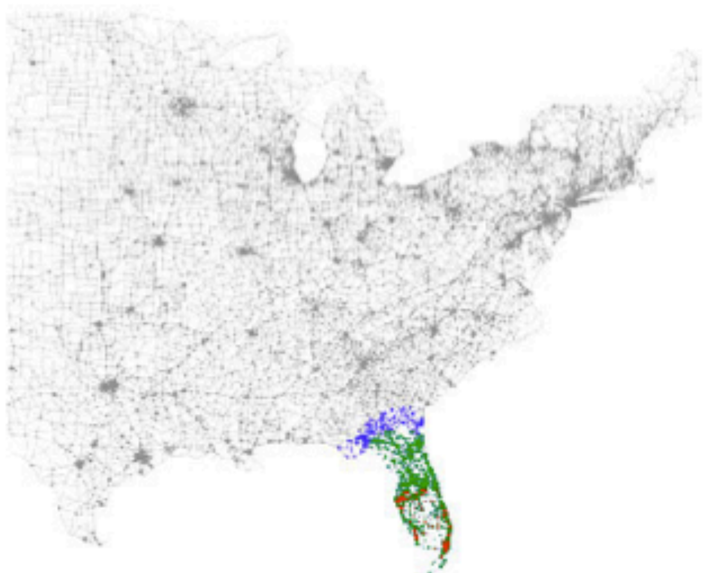
What is Local Graph Partitioning?

A local graph partitioning algorithm finds a small cut near the given seed(s) with running time depending only on the size of the output.

Global

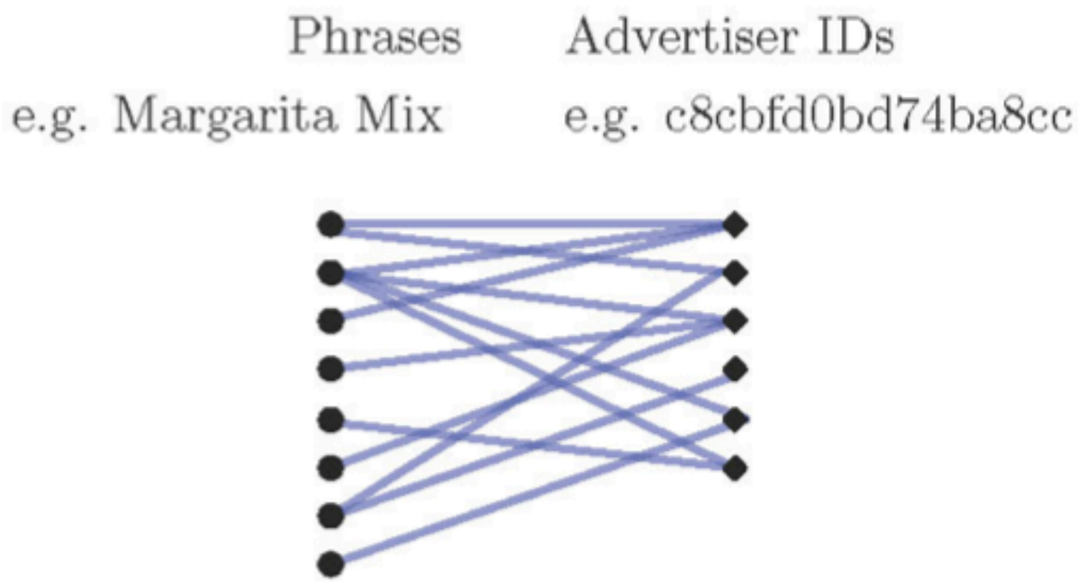


Local



What is Local Graph Partitioning?

A bidding graph from Yahoo sponsored search



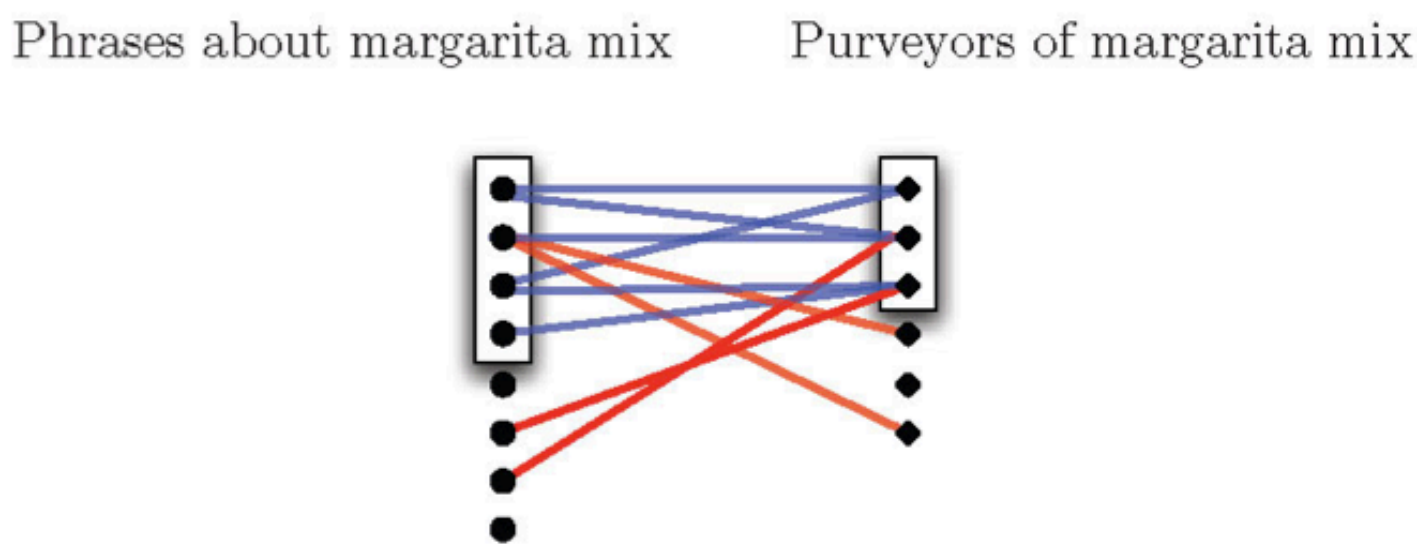
On the left are search phrases, on the right are advertisers. Each edge represents a bid by an advertiser on a phrase.

$400K$ phrases, $200K$ advertisers, and 2 million edges.

What is Local Graph Partitioning?

Submarkets in bidding graph

The bidding graph has submarkets, sets of bidders and phrases that interact mostly with each other.

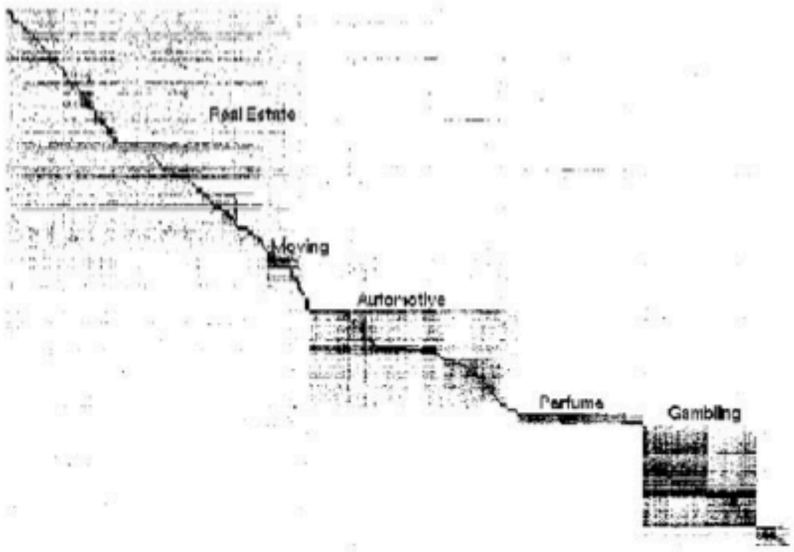


These sets of vertices (containing both advertisers and phrases) have small conductance.

What is Local Graph Partitioning?

Submarkets in the bigging graph

The bidding graph has numerous submarkets, related to real estate, flower delivery, hotels, gambling, ...



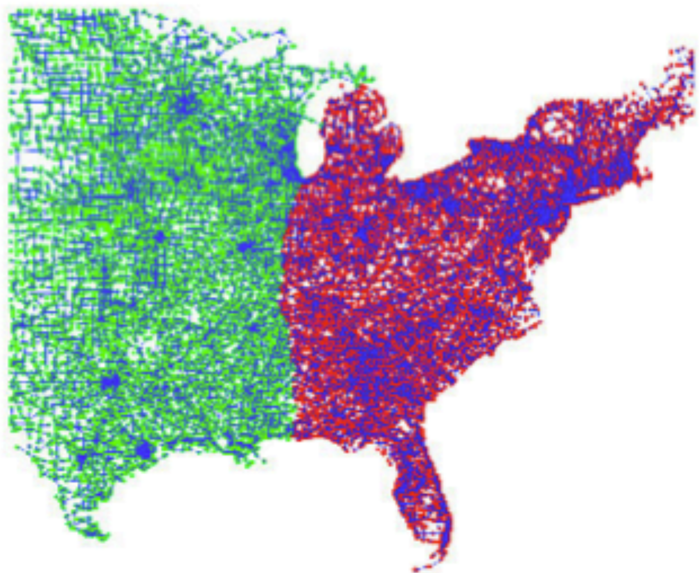
It is useful to identify these submarkets.

- Find groups of related phrases to suggest to advertisers.
- Find small submarkets for testing and experimentation.

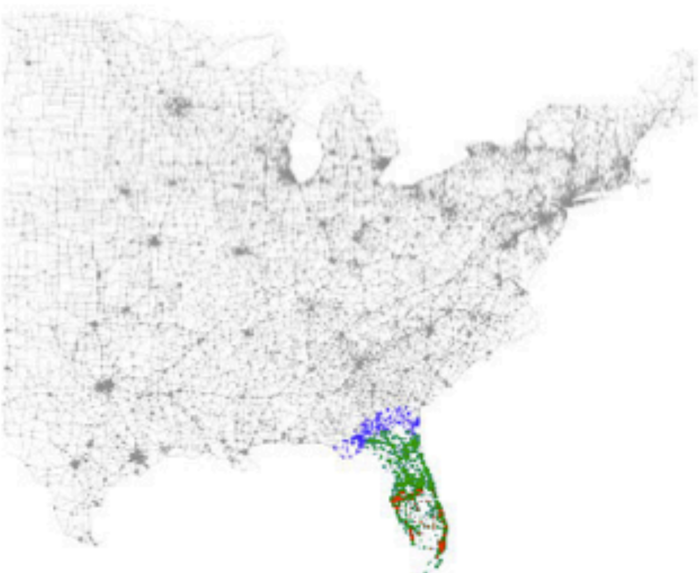
What is Local Graph Partitioning?

A local graph partitioning algorithm finds a small cut near the given seed(s) with running time depending only on the size of the output.

Global

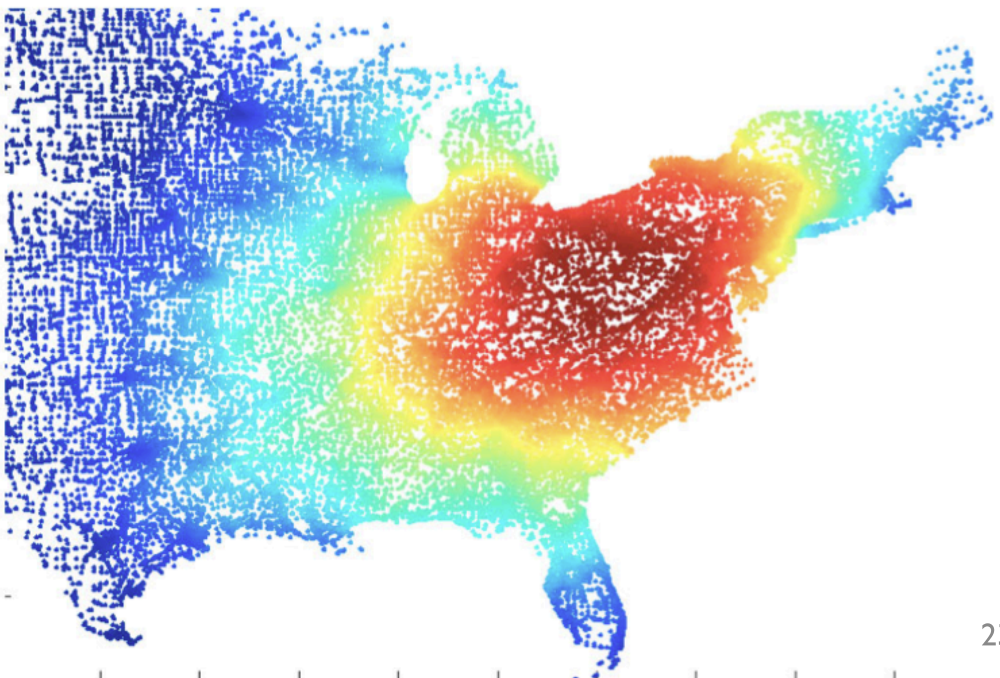


Local



Key idea: a "sweep"

- Order all vertices in some way $v_{i,1}$, $v_{i,2}$, ….
  - Say, by personalized PageRank from a seed
- Pick a prefix $v_{i,1}$, $v_{i,2}$, …. $v_{i,k}$ that is "best"
  - ….



What is a "good" subgraph?

$$\partial(S) = \{\{x, y\} \in E \mid x \in S, y \notin S\}$$

the edges leaving S

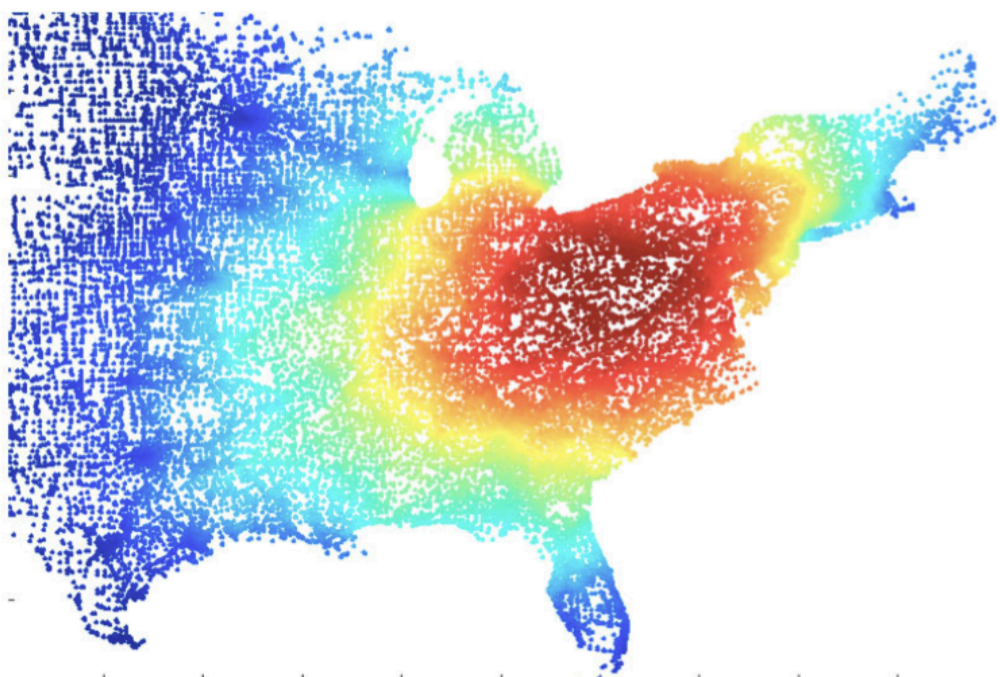
$$\Phi(S) = \frac{|\partial(S)|}{\min\left(\mathrm{vol}(S), 2m - \mathrm{vol}(S)\right)}.$$

- vol(S) is sum of deg(x) for x in S
- for small S: Prob(random *edge* leaves S)

Key idea: a "sweep"

- Order all vertices in some way $v_{i,1}$, $v_{i,2}$, ….
  - Say, by personalized PageRank from a seed
- Pick a prefix $S = \{ v_{i,1}, v_{i,2}, \ldots v_{i,k} \}$ that is "best"
  - Minimal "conductance" $\phi(S)$

You can re-compute conductance incrementally as you add a new vertex so the sweep is fast



Main results of the paper

1. An *approximate* personalized PageRank computation that only touches nodes “near” the seed
   – but has small error relative to the true PageRank vector
2. A proof that a sweep over the approximate PageRank vector finds a cut with conductance sqrt(α ln m)
   – unless no good cut exists
      - no subset S contains significantly more pass in the approximate PageRank than in a uniform distribution

	Static graph patterns							
	in-deg	out-deg	wcc	scc	hops	sng-val	sng-vec	clust
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Result 2 explains Jure & Christos's experimental results with RW sampling:

- RW approximately picks up a *random subcommunity* (maybe with some extra nodes)
- Features like clustering coefficient, degree should be representative of the graph as a whole…
  - which is roughly a mixture of subcommunities

Main results of the paper

1. An *approximate* personalized PageRank computation that only touches nodes "near" the seed
   - but has small error relative to the true PageRank vector

This is a very useful technique to know about…

Random Walks

G : a graph

P : transition probability matrix

$$P(u,v) = \begin{cases} \dfrac{1}{d_u} & if\ u : \ v, \\ 0 & otherwise. \end{cases}$$

d_u := the degree of u.

A lazy walk:

$$W = \frac{I+P}{2}$$

avoids messy "dead ends"….

Random Walks: PageRank

A (bored) surfer

- either surf a random webpage

  with probability α

- or surf a linked webpage

  with probability $1-\alpha$

α : the jumping constant

$$p = \alpha\left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right) + (1-\alpha)\, pW$$

Random Walks: PageRank

Two equivalent ways to define PageRank $p = pr(\alpha, s)$

(1) $$p = \alpha s + (1-\alpha) pW$$

(2) $$p = \alpha \sum_{t=0}^{\infty} (1-\alpha)^t (sW^t)$$

$s = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$ ⟹ the (original) PageRank

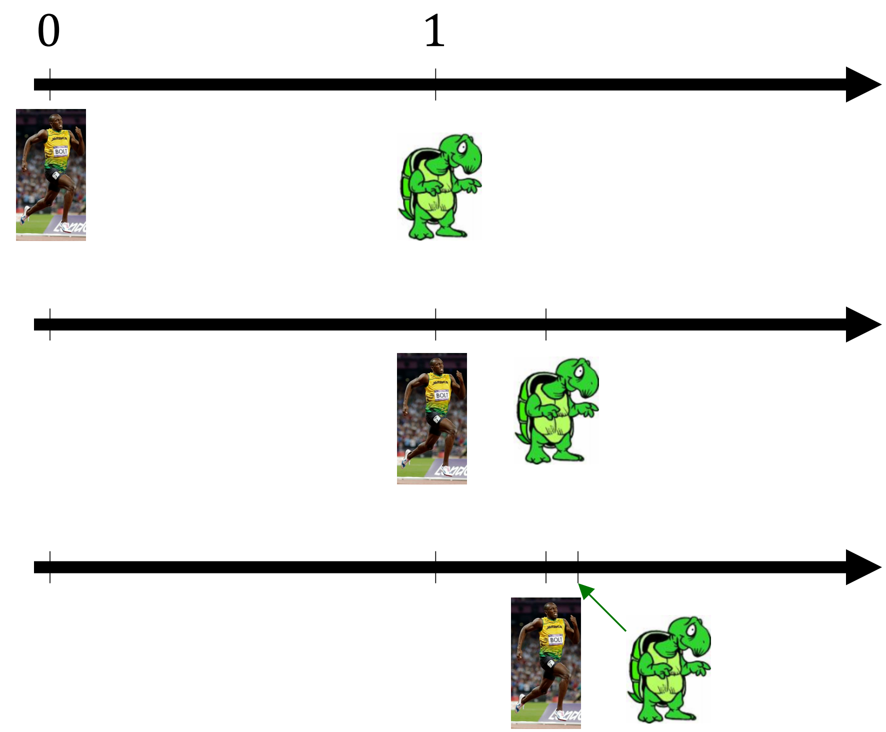
$s =$ some "seed", e.g., $(1, 0, \ldots, 0)$

⟹ personalized PageRank

Flashback: Zeno's paradox

$$1+0.1+0.01+0.001+0.0001+\ldots = ?$$

- Usain Bolt and the tortoise have a race
- Bolt is 10x faster
- Tortoise has a 1m head start at time 0



- So, when Bolt gets to 1m the tortoise is at 1.1m
- So, when Bolt gets to 1.1m the tortoise is at 1.11m …
- So, when Bolt gets to 1.11m the tortoise is at 1.111m … and Lance will *never* catch up -?

unresolved until calculus was invented

Zeno: powned by telescoping sums

Let *x* be less than 1. Then

$$y = 1 + x + x^2 + x^3 + \ldots + x^n$$

$$y(1-x) = (1 + x + x^2 + x^3 + \ldots + x^n)(1-x)$$

$$y(1-x) = (1 - \cancel{x}) + (\cancel{x} - \cancel{x^2}) + (\cancel{x^2} - x^3) + \ldots + (x^n - x^{n+1})$$

$$y(1-x) = 1 - x^{n+1}$$

$$y = \frac{1 - x^{n+1}}{(1-x)}$$

$$y \approx (1-x)^{-1}$$

Example: x=0.1, and 1+0.1+0.01+0.001+…. = 1.11111 = 10/9.

Graph = Matrix

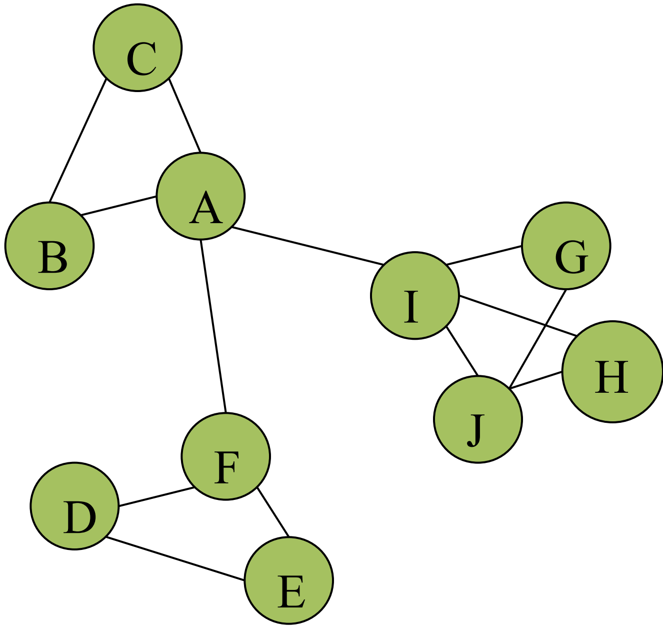
Vector = Node → Weight

M

	A	B	C	D	E	F	G	H	I	J
A	_	1	1			1				
B	1	_	1							
C	1	1	_							
D				_	1	1				
E				1	_	1				
F	1			1	1	_				
G							_		1	1
H								_	1	1
I							1	1	_	1
J							1	1	1	_

V

	A
A	3
B	2
C	3
D	
E	
F	
G	
H	
I	
J	



M

Racing through a graph?

Let W[*i,j*] **be Pr(walk to** *j* **from** *i***)and let** *α* **be less than 1. Then:**

$$\mathbf{Y} = \mathbf{I} + \alpha\mathbf{W} + (\alpha\mathbf{W})^2 + (\alpha\mathbf{W})^3 + \ldots(\alpha\mathbf{W})^n$$

$$\mathbf{Y}(\mathbf{I} - \alpha\mathbf{W}) = (\mathbf{I} + \alpha\mathbf{W} + (\alpha\mathbf{W})^2 + (\alpha\mathbf{W})^3 + \ldots)(\mathbf{I} - \alpha\mathbf{W})$$

$$\mathbf{Y}(\mathbf{I} - \alpha\mathbf{W}) = (\mathbf{I} - \cancel{\alpha\mathbf{W}}) + (\cancel{\alpha\mathbf{W}} - (\cancel{\alpha\mathbf{W}})^2 + \ldots)(\mathbf{I} - \alpha\mathbf{W})$$

$$\mathbf{Y}(\mathbf{I} - \alpha\mathbf{W}) = \mathbf{I} - (\alpha\mathbf{W})^{n+1}$$

$$\mathbf{Y} \approx (\mathbf{I} - \alpha\mathbf{W})^{-1} \qquad \mathbf{Y}[i,j] = \frac{1}{Z}\Pr(j \mid i)$$

The matrix **(I-** *α***W)** is the *Laplacian* of *α***W.**

Generally the Laplacian is **(D- A)** where **D**[*i,i*] is the degree of *i* in the adjacency matrix **A.**

Random Walks: PageRank

Two equivalent ways to define PageRank $p=pr(\alpha,s)$

(1) $$p = \alpha s + (1-\alpha) p W$$

(2) $$p = \alpha \sum_{t=0}^{\infty} (1-\alpha)^t (sW^t)$$

$s = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$ $\Longrightarrow$ the (original) PageRank

$s =$ some "seed", e.g., $(1, 0, \ldots, 0)$

$\Longrightarrow$ personalized PageRank

Approximate PageRank: Key Idea

By definition PageRank is fixed point of:

$$\mathrm{pr}(\alpha, s) = \alpha s + (1-\alpha)\mathrm{pr}(\alpha, s)W,$$

Claim:

$$\mathrm{pr}(\alpha, s) = \alpha s + (1-\alpha)\mathrm{pr}(\alpha, sW).$$

Proof:

define a matrix for the pr operator:
$R_\alpha s = \mathrm{pr}(\alpha, s)$

$$\begin{aligned} R_\alpha &= \alpha \sum_{t=0}^{\infty} (1-\alpha)^t W^t \\ &= \alpha \left(I + \sum_{u=1}^{\infty} (1-\alpha)^u W^u \right) \\ &= \alpha I + (1-\alpha) W \sum_{t=0}^{\infty} (1-\alpha)^t W^t \\ &= \alpha I + (1-\alpha) W R_\alpha \end{aligned}$$

Approximate PageRank: Key Idea

By definition PageRank is fixed point of:

$$\mathrm{pr}(\alpha, s) = \alpha s + (1 - \alpha)\mathrm{pr}(\alpha, s)W,$$

Claim:

$$\mathrm{pr}(\alpha, s) = \alpha s + (1 - \alpha)\mathrm{pr}(\alpha, sW).$$

Proof:

$$\begin{aligned} R_\alpha &= \alpha \sum_{t=0}^{\infty} (1 - \alpha)^t W^t \\ &= \alpha I + (1 - \alpha) W R_\alpha. \end{aligned}$$

$$\begin{aligned} \mathrm{pr}(\alpha, s) &= s R_\alpha \\ &= \alpha s + (1 - \alpha) \underline{sW} R_\alpha \\ &= \alpha s + (1 - \alpha)\mathrm{pr}(\alpha, sW). \end{aligned}$$

Approximate PageRank: Key Idea

By definition PageRank is fixed point of:

$$\mathrm{pr}(\alpha, s) = \alpha s + (1-\alpha)\mathrm{pr}(\alpha, s)W,$$

Claim:

$$\mathrm{pr}(\alpha, s) = \alpha s + (1-\alpha)\mathrm{pr}(\alpha, sW).$$

Recursively compute PageRank of "neighbors of *s*" *(=sW)*, then adjust

Key idea in apr:

- do this "recursive step" repeatedly
- focus on nodes where finding PageRank from neighbors will be useful

Approximate PageRank: Key Idea

$$\mathrm{pr}(\alpha, s) = \alpha s + (1-\alpha)\mathrm{pr}(\alpha, sW). \quad W = \frac{I+P}{2}$$

$\mathbf{push}_u(p, r)$:

1. Let $p' = p$ and $r' = r$, except for the following changes:
   (a) $p'(u) = p(u) + \alpha r(u)$.
   (b) $r'(u) = (1-\alpha)r(u)/2$.
   (c) For each v such that $(u, v) \in E$: $r'(v) = r(v) + (1-\alpha)r(u)/(2d(u))$.
2. Return (p', r').

- *p* is current approximation (start at **0**)
- *r* is set of "recursive calls to make"
  - residual error
  - start with all mass on *s*
- *u* is the node picked for the next call

Analysis

Lemma 1. *Let p' and r' be the result of the operation* $\texttt{push}_u$ *on p and r. Then*

$$p' + \mathrm{pr}(\alpha, r') = p + \mathrm{pr}(\alpha, r).$$

Proof of Lemma 1. After the push operation, we have

$$\begin{aligned} p' &= p + \alpha r(u)\chi_u. \\ r' &= r - r(u)\chi_u + (1-\alpha) r(u)\chi_u W. \end{aligned}$$

Using equation (5),

linearity

$$\begin{aligned} p + \mathrm{pr}(\alpha, r) &= p + \mathrm{pr}(\alpha, r - r(u)\chi_u) + \mathrm{pr}(\alpha, r(u)\chi_u) \\ &= p + \mathrm{pr}(\alpha, r - r(u)\chi_u) + [\alpha r(u)\chi_u + (1-\alpha)\mathrm{pr}(\alpha, r(u)\chi_u W)] \\ &= [p + \alpha r(u)\chi_u] + \mathrm{pr}(\alpha, [r - r(u)\chi_u + (1-\alpha) r(u)\chi_u W]) \\ &= p' + \mathrm{pr}(\alpha, r'). \end{aligned}$$

re-group & linearity

$pr(\alpha, r - r(u)\chi_u) + (1-\alpha)\, pr(\alpha, r(u)\chi_u W) = pr(\alpha, r - r(u)\chi_u + (1-\alpha)\ r(u)\chi_u W)$

$$\mathrm{pr}(\alpha, s) = \alpha s + (1-\alpha)\mathrm{pr}(\alpha, sW). \qquad (5)$$

Approximate PageRank: Algorithm

ApproximatePageRank (v, α, ϵ):

1. Let $p = \vec{0}$, and $r = \chi_v$.
2. While $\max_{u \in V} \frac{r(u)}{d(u)} \geq \epsilon$:
   (a) Choose any vertex u where $\frac{r(u)}{d(u)} \geq \epsilon$.
   (b) Apply $\mathbf{push}_u$ at vertex u, updating p and r.
3. Return p, which satisfies $p = \mathrm{apr}(\alpha, \chi_v, r)$ with $\max_{u \in V} \frac{r(u)}{d(u)} < \epsilon$.

$\mathbf{push}_u(p, r)$:

1. Let $p' = p$ and $r' = r$, except for the following changes:
   (a) $p'(u) = p(u) + \alpha r(u)$.
   (b) $r'(u) = (1 - \alpha) r(u)/2$.
   (c) For each v such that $(u, v) \in E$: $\quad r'(v) = r(v) + (1 - \alpha) r(u)/(2d(u))$.
2. Return (p', r').

Analysis

Lemma 1. *Let p' and r' be the result of the operation* $\mathbf{push}_u$ *on p and r. Then*

$$p' + \mathrm{pr}(\alpha, r') = p + \mathrm{pr}(\alpha, r).$$

So, at every point in the *apr* algorithm:

$$p + \mathrm{pr}(\alpha, r) = \mathrm{pr}(\alpha, \chi_v),$$

Also, at each point, $|r|_1$ **decreases** by $\alpha * \varepsilon *$degree(u), so:
after T push operations where degree(i-th u)=d_i, we know

$$\sum_i d_i \cdot \alpha\varepsilon \le 1 \quad \Rightarrow \quad \sum_{i=1}^{T} d_i \le \frac{1}{\epsilon\alpha}.$$

which bounds the size of *r* and *p*

Analysis

Theorem 1. $\texttt{ApproximatePageRank}(v, \alpha, \epsilon)$ *runs in time* $O(\frac{1}{\epsilon\alpha})$, *and computes an approximate PageRank vector* $p = \mathrm{apr}(\alpha, \chi_v, r)$ *such that the residual vector* r *satisfies* $\max_{u \in V} \frac{r(u)}{d(u)} < \epsilon$, *and such that* $\mathrm{vol}(\mathrm{Supp}(p)) \le \frac{1}{\epsilon\alpha}$.

With the invariant: $$p + \mathrm{pr}(\alpha, r) = \mathrm{pr}(\alpha, \chi_v),$$

This bounds the error of *p* relative to the PageRank vector.

Comments – API

`ApproximatePageRank` (v, α, ϵ):

1. Let $p = \vec{0}$, and $r = \chi_v$.
2. While $\max_{u \in V} \frac{r(u)}{d(u)} \geq \epsilon$:
   (a) Choose any vertex u where $\frac{r(u)}{d(u)} \geq \epsilon$.
   (b) Apply $\mathbf{push}_u$ at vertex u, updating p and r.
3. Return p, which satisfies $p = \text{apr}(\alpha, \chi_v, r)$ with $\max_{u \in V} \frac{r(u)}{d(u)} < \epsilon$.

p,r are hash tables – they are small (1/εα)

Could implement with API:
- List<Node> neighbor(Node *u*)
- int degree(Node *u*)

$\mathbf{push}_u(p, r)$:

1. Let $p' = p$ and $r' = r$, except for the following changes:
   (a) $p'(u) = p(u) + \alpha r(u)$.
   (b) $r'(u) = (1 - \alpha) r(u)/2$.
   (c) For each v such that $(u, v) \in E$: $r'(v) = r(v) + (1 - \alpha) r(u)/(2d(u))$.
2. Return (p', r').

push just needs *p, r,* and neighbors of *u*

d(v) = api.degree(*v*)

Comments - Ordering

`ApproximatePageRank` (v, α, ϵ):

1. Let $p = \vec{0}$, and $r = \chi_v$.
2. While $\max_{u \in V} \frac{r(u)}{d(u)} \geq \epsilon$:
   (a) Choose any vertex u where $\frac{r(u)}{d(u)} \geq \epsilon$.
   (b) Apply $\mathbf{push}_u$ at vertex u, updating p and r.
3. Return p, which satisfies $p = \mathrm{apr}(\alpha, \chi_v, r)$ with $\max_{u \in V} \frac{r(u)}{d(u)} < \epsilon$.

might pick the **largest** *r(u)/d(u)* … ***or…***

$\mathbf{push}_u(p, r)$:

1. Let $p' = p$ and $r' = r$, except for the following changes:
   (a) $p'(u) = p(u) + \alpha r(u)$.
   (b) $r'(u) = (1 - \alpha) r(u)/2$.
   (c) For each v such that $(u, v) \in E$: $r'(v) = r(v) + (1 - \alpha) r(u)/(2d(u))$.
2. Return (p', r').

Comments – Ordering for Scanning

`ApproximatePageRank` (v, α, ϵ):

1. Let $p = \vec{0}$, and $r = \chi_v$.
2. While $\max_{u \in V} \frac{r(u)}{d(u)} \geq \epsilon$:

Scan <u>repeatedly</u> through an adjacency-list encoding of the graph

For every line you read *u*, $v_1, \ldots, v_{d(u)}$ such that $r(u)/d(u) > \varepsilon$:

   (b) Apply $\mathbf{push}_u$ at vertex u, updating p and r.

3. Return p, which satisfies $p = \mathrm{apr}(\alpha, \chi_v, r)$ with $\max_{u \in V} \frac{r(u)}{d(u)} < \epsilon$.

benefit: storage is $O(1/\varepsilon\alpha)$ for the hash tables, avoids any *seeking*

Possible optimizations?

- Much faster than doing random access the first few scans, but then slower the last few
  - …there will be only a few 'pushes' per scan
- Optimizations you might imagine:
  - Parallelize?
  - Hybrid seek/scan:
    - Index the nodes in the graph on the first scan
    - Start seeking when you expect too few pushes to justify a scan
      - Say, less than one push/megabyte of scanning
  - Hotspots:
    - Save adjacency-list representation for nodes with a large $r(u)/d(u)$ in a separate file of "hot spots" as you scan
    - Then rescan that smaller list of "hot spots" until their score drops below threshold.

Putting this together

- Given a graph
  - that's too big for memory, and/or
  - that's only accessible via API
- …we can extract a *sample* in an interesting area
  - Run the apr/rwr from a seed node
  - Sweep to find a low-conductance subset
- Then
  - compute statistics
  - test out some ideas
  - visualize it…