Probability Distributions on Structured Objects

September 17, 2013

Reminder

- HW1 is due at 11:59pm tonight

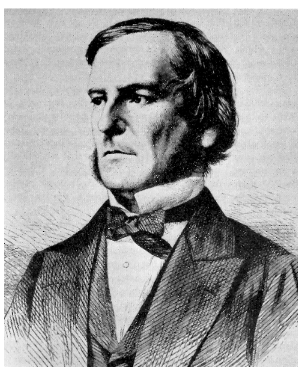
- There was some ambiguity in this assignment
- The TAs gave a lot of help, but in general, learning to work from incomplete specs is important

Probability Outline

- Why probability?
- Probability review
- Multinomials vs. exponential parameterization
- Locally vs. globally normalized models & partition functions
- Examples

Why Probability?

- Probability formalizes
  - The concept of **models**
  - The concept of **data**
  - The concept of **learning**
  - The concept of **prediction** (inference)



Probability is expectation founded upon partial knowledge.

Why Probability?

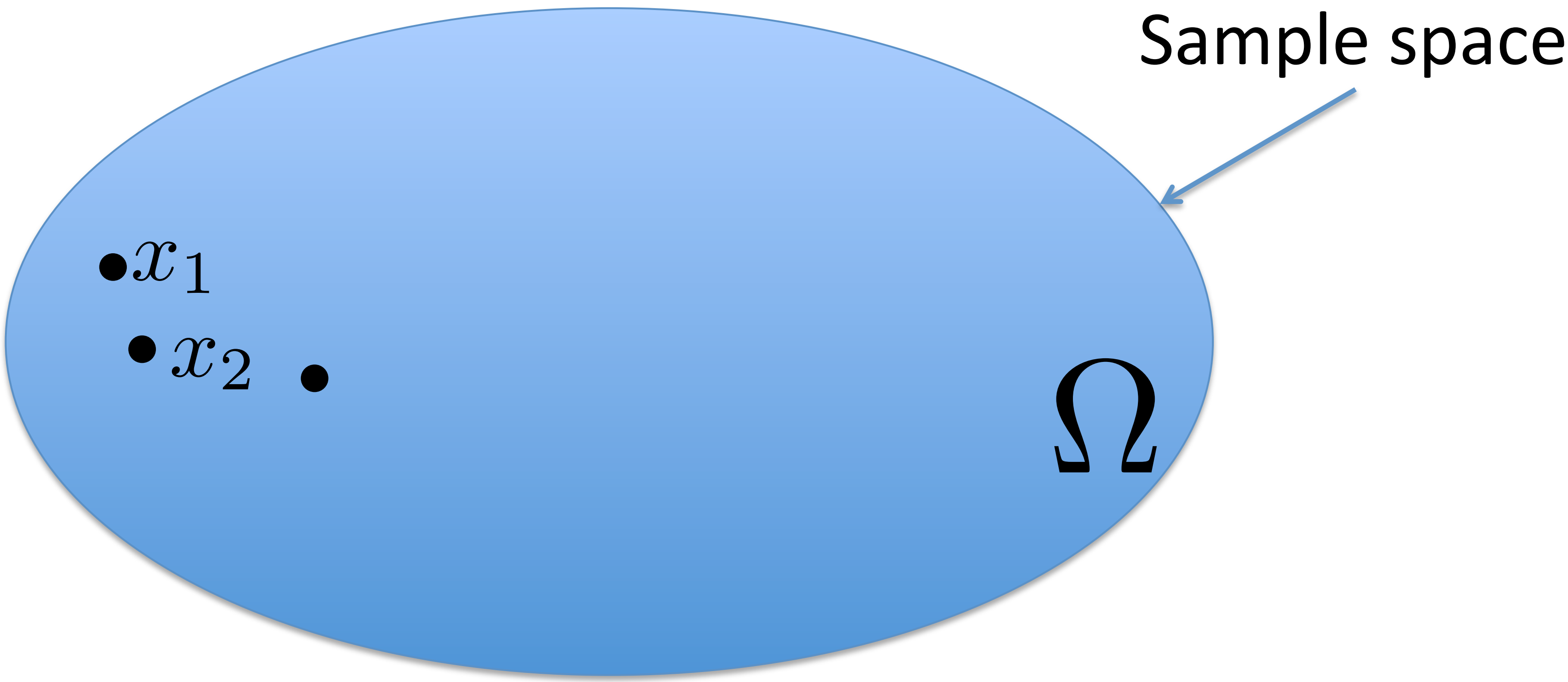
- What might we have partial knowledge about?
  - The state of the world (test data)
  - The reliability of our training data
  - The correctness of our model
  - The values of our parameters

$$p(x \mid \text{partial knowledge})$$

What is a Probability?

- **Limiting (relative) frequency of events**
  - in repeated (identical) experiments
- **Degree of belief**
  - Subjective conception
  - 40% chance of rain tomorrow in Pittsburgh
- Viewpoint affects
  - interpretation
  - **not** rules of probability calculus themselves

Discrete Distributions



Discrete distribution: Ω is *finite* or *countable*, but no bigger

Discrete Distributions

$$\forall\ x \in \Omega, \quad f(x) \in [0,1]$$

$$\sum_{x \in \Omega} f(x) = 1$$

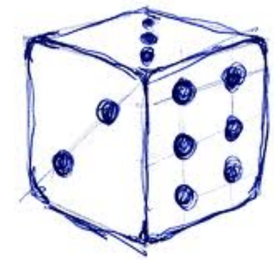
Probability mass function

An **event** is a subset (maybe one element) of the sample space, $E \subseteq \Omega$

$$P(E) = \sum_{x \in E} f(x)$$

Random Variables

A **random variable** is a function from a random event from a set of possible outcomes (Ω) and a probability distribution (ρ), a function from outcomes to probabilities.



$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

$$X(\omega) = \omega$$

$$\rho_X(x) = \begin{cases} \frac{1}{6} & \text{if } x = 1, 2, 3, 4, 5, 6 \\ 0 & \text{otherwise} \end{cases}$$

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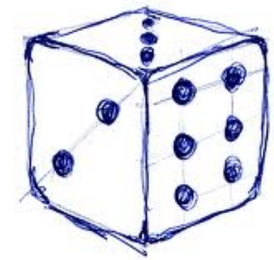

$$\Omega = \{1,2,3,4,5,6\}$$

$$Y(\omega) = \begin{cases} 0 & \text{if } \omega \in \{2,4,6\} \\ 1 & \text{otherwise} \end{cases}$$

$$\rho_Y(y) = \begin{cases} \frac{1}{2} & \text{if } y = 0,1 \\ 0 & \text{otherwise} \end{cases}$$

Sampling Notation

$$x = 4 \times z + 1.7$$

Expression

Variable

Sampling Notation

$$x = 4 \times z + 1.7$$

$$y \sim \mathrm{Distribution}(\boldsymbol{\theta})$$

Distribution

Random variable

Parameter

Sampling Notation

$$x = 4 \times z + 1.7$$

$$y \sim \text{Distribution}(\boldsymbol{\theta})$$

$$y' = y \times x$$

Random variable

Joint Probability

- Probability over multiple event types
- Tool for reasoning about dependent (correlated) events

A **joint probability distribution** is a probability distribution over r.v.'s with the following form:

$$Z = \begin{bmatrix} X(\omega) \\ Y(\omega) \end{bmatrix}$$

$$\sum_{x \in \mathcal{X}, y \in \mathcal{Y}} \rho_Z\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = 1 \qquad \rho_Z\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) \geq 0 \quad \forall x \in \mathcal{X}, y \in \mathcal{Y}$$

Joint Probability

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Words (→ $X(\omega)$)

Tags (→ $Y(\omega)$)

$$\sum_{x \in \mathcal{X}, y \in \mathcal{Y}} \rho_Z\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = 1 \qquad \rho_Z\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) \geq 0 \quad \forall x \in \mathcal{X}, y \in \mathcal{Y}$$

Joint Probability

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Words (→ $X(\omega)$)

Trees (→ $Y(\omega)$)

$$\sum_{x \in \mathcal{X}, y \in \mathcal{Y}} \rho_Z\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = 1 \qquad \rho_Z\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) \geq 0 \quad \forall x \in \mathcal{X}, y \in \mathcal{Y}$$

Joint Probability

- Probability over multiple event types
- Tool for reasoning about dependent (correlated) events

A **joint probability distribution** is a probability distribution over r.v.'s with the following form:

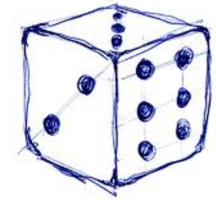
$$Z = \begin{bmatrix} X(\omega) \\ Y(\omega) \end{bmatrix}$$

$X(\omega)$ ← DNA sequence

$Y(\omega)$ ← Proteins

$$\sum_{x \in \mathcal{X}, y \in \mathcal{Y}} \rho_Z\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = 1 \qquad \rho_Z\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) \geq 0 \quad \forall x \in \mathcal{X}, y \in \mathcal{Y}$$

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

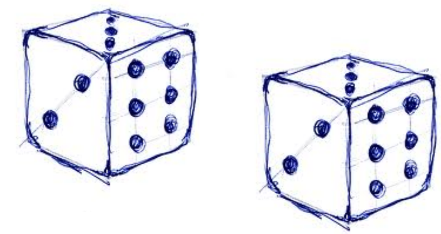


$$X(\omega) = \omega$$

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

$$X(\omega) = \omega$$

$$\begin{aligned}\Omega = \{&(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),\\&(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),\\&(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),\\&(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),\\&(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),\\&(6,1),(6,2),(6,3),(6,4),(6,5),(6,6),\}\end{aligned}$$



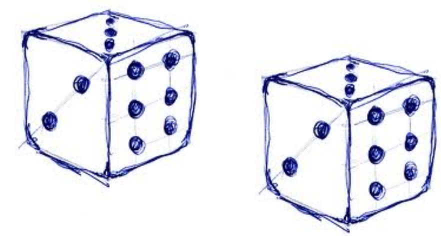
$$X(\omega) = \omega_1 \quad Y(\omega) = \omega_2$$

$$\rho_{X,Y}(x,y) = \begin{cases} \frac{1}{36} & \text{if } (x,y) \in \Omega \\ 0 & \text{otherwise} \end{cases}$$

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

$$X(\omega) = \omega$$

$$\Omega = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),\\ (2,1),(2,2),(2,3),(2,4),(2,5),(2,6),\\ (3,1),(3,2),(3,3),(3,4),(3,5),(3,6),\\ (4,1),(4,2),(4,3),(4,4),(4,5),(4,6),\\ (5,1),(5,2),(5,3),(5,4),(5,5),(5,6),\\ (6,1),(6,2),(6,3),(6,4),(6,5),(6,6),\}$$



$$X(\omega) = \omega_1 \quad Y(\omega) = \omega_2$$

$$\rho_{X,Y}(x,y) = \begin{cases} \frac{x+y}{252} & \text{if } (x,y) \in \Omega \\ 0 & \text{otherwise} \end{cases}$$

Marginal Probability

$$p(X = x, Y = y) = \rho_{X,Y}(x, y)$$

$$p(X = x) = \sum_{y' \in \mathcal{Y}} p(X = x, Y = y')$$

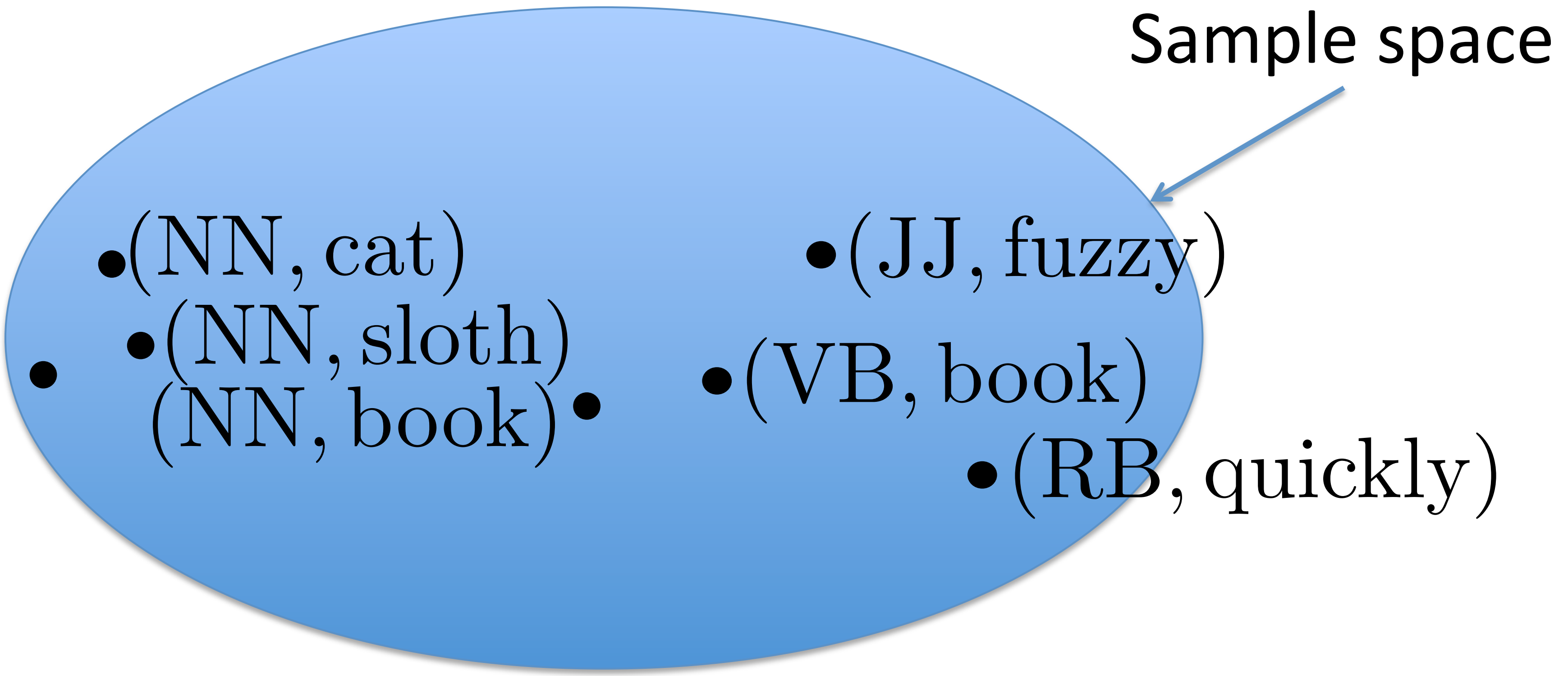
$$p(Y = y) = \sum_{x' \in \mathcal{X}} p(X = x', Y = y)$$

$$\begin{aligned}\Omega = \{&(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),\\ &(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),\\ &(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),\\ &(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),\\ &(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),\\ &(6,1),(6,2),(6,3),(6,4),(6,5),(6,6),\}\end{aligned}$$

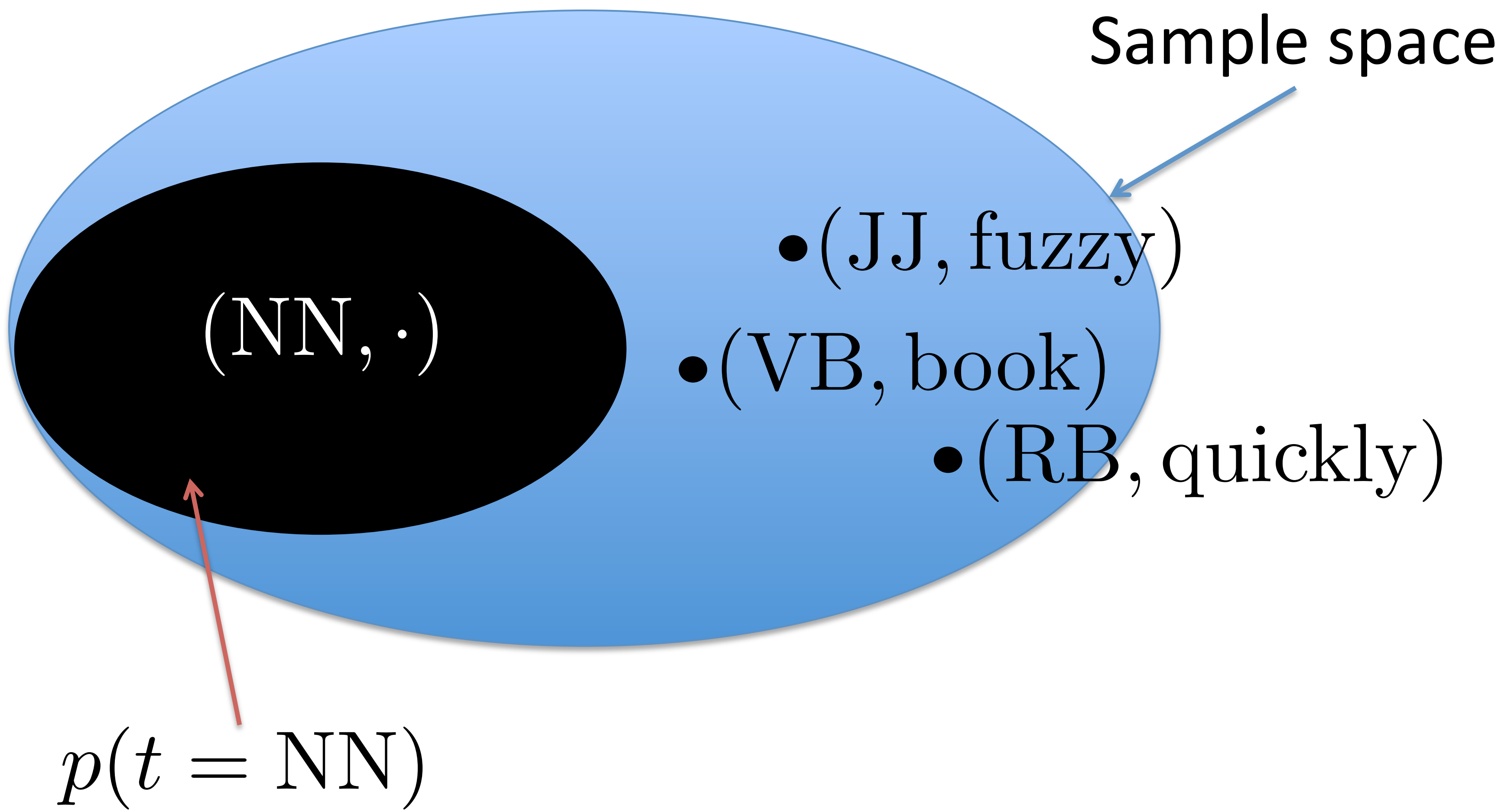
$$p(X = 4) = \sum_{y' \in [1,6]} p(X = 4, Y = y')$$

$$p(Y = 3) = \sum_{x' \in [1,6]} p(X = x', Y = 3)$$

Marginal Probability



Marginal Probability



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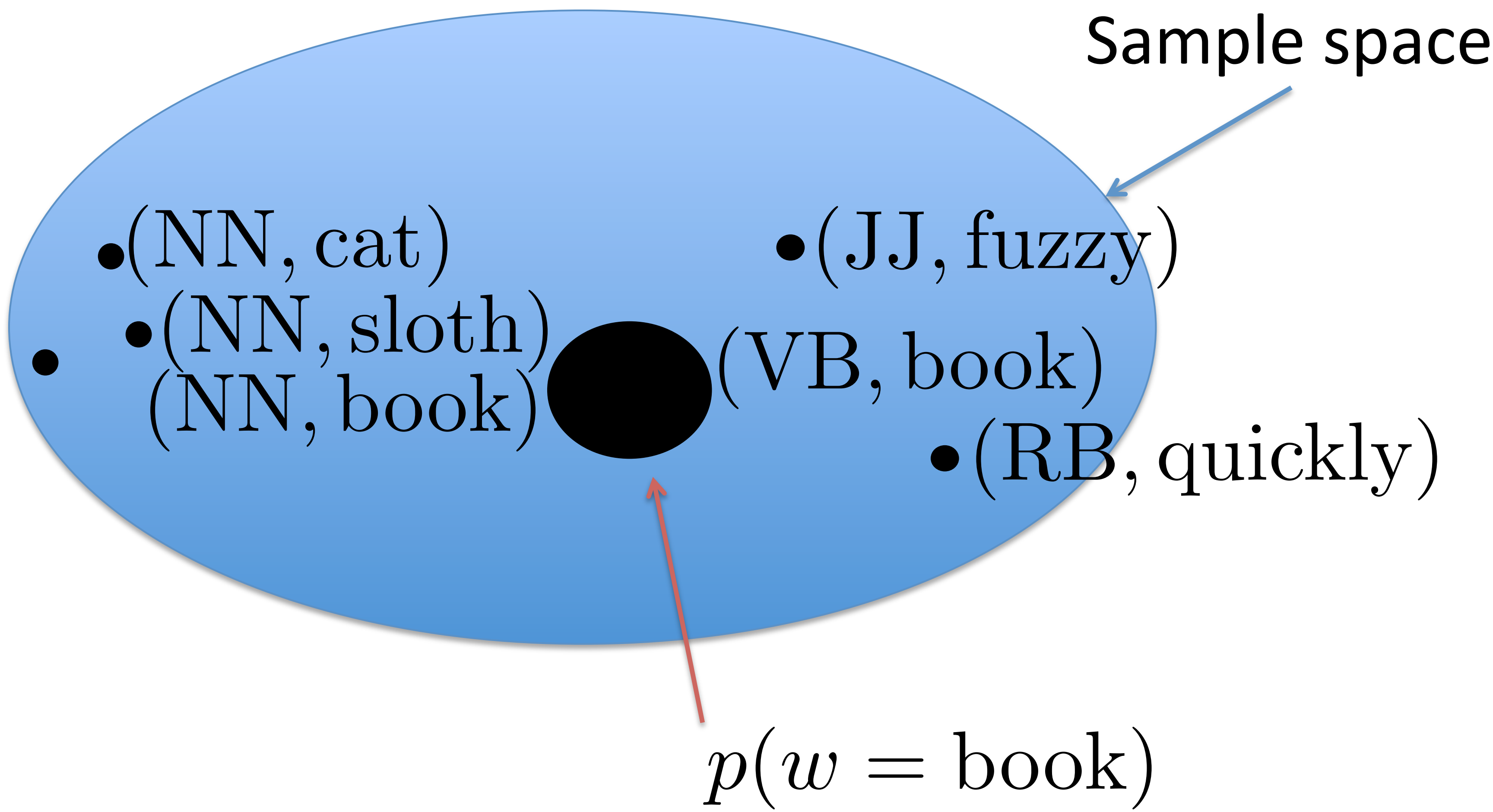
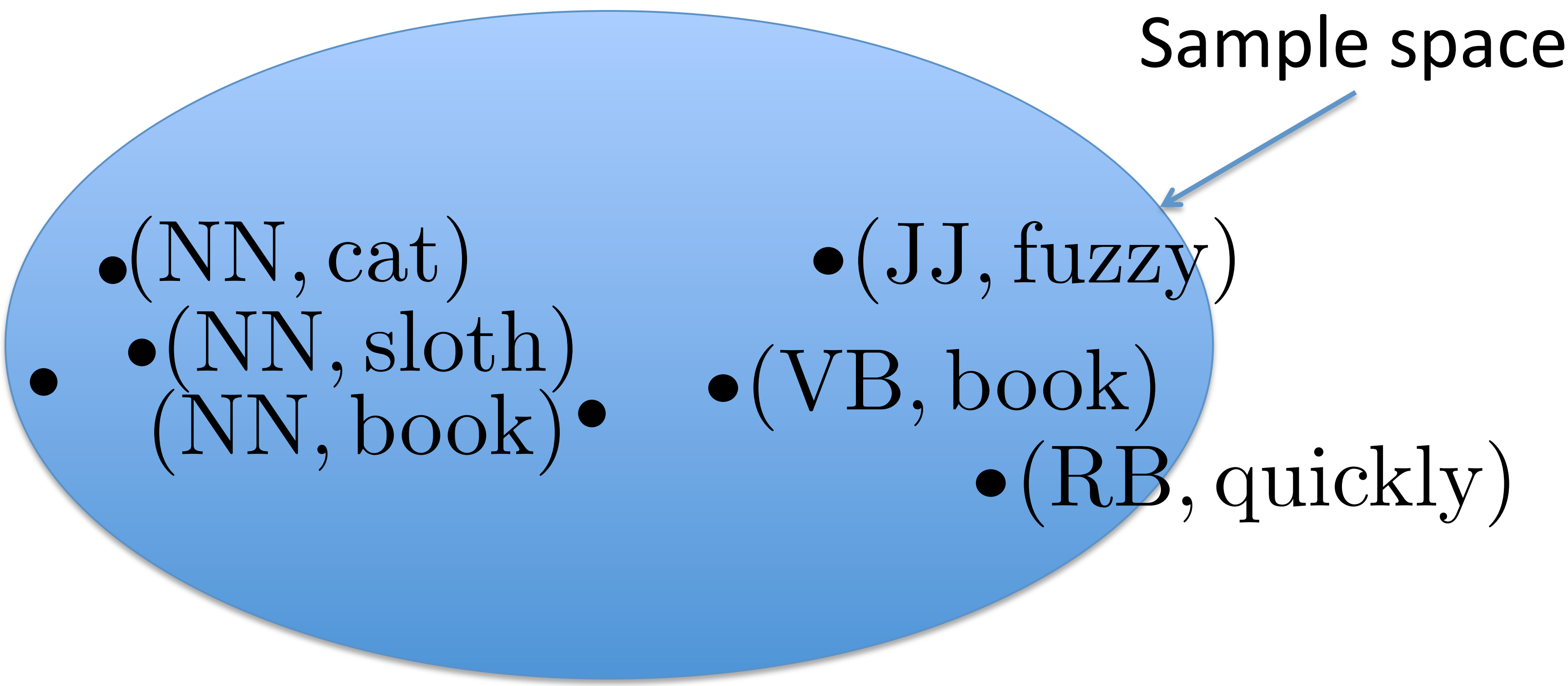


Marginal Probability



Marginal Probabilities

- In a joint model of word and tag sequences p(**w**,**t**)
  - The probability of a word sequence p(**w**)
  - The probability of a tag sequence p(**t**)
  - The probability of a word sequence with the word "cat" somewhere in it
  - The probability of a tag sequence containing three verbs in a row

Conditional Probability

The **conditional probability** is defined as follows:

$$p(X = x \mid Y = y) = \frac{p(X = x, Y = y)}{p(Y = y)} = \frac{\text{joint probability}}{\text{marginal}}$$

This assumes $p(Y = y) \neq 0$

We can construct joint probability distributions out of conditional distributions:

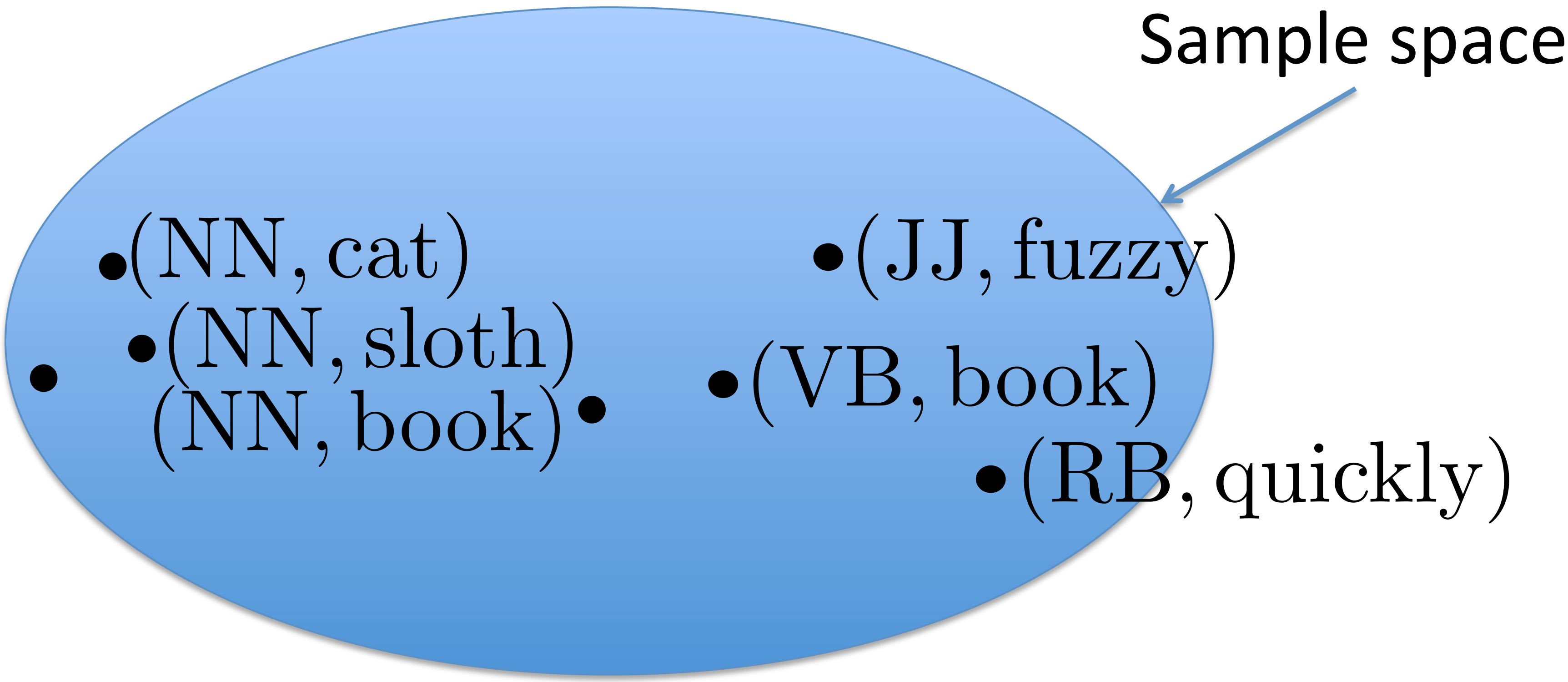
$$p(x \mid y)p(y) = p(x, y) = p(y \mid x)p(x)$$

Conditional Probability Distributions

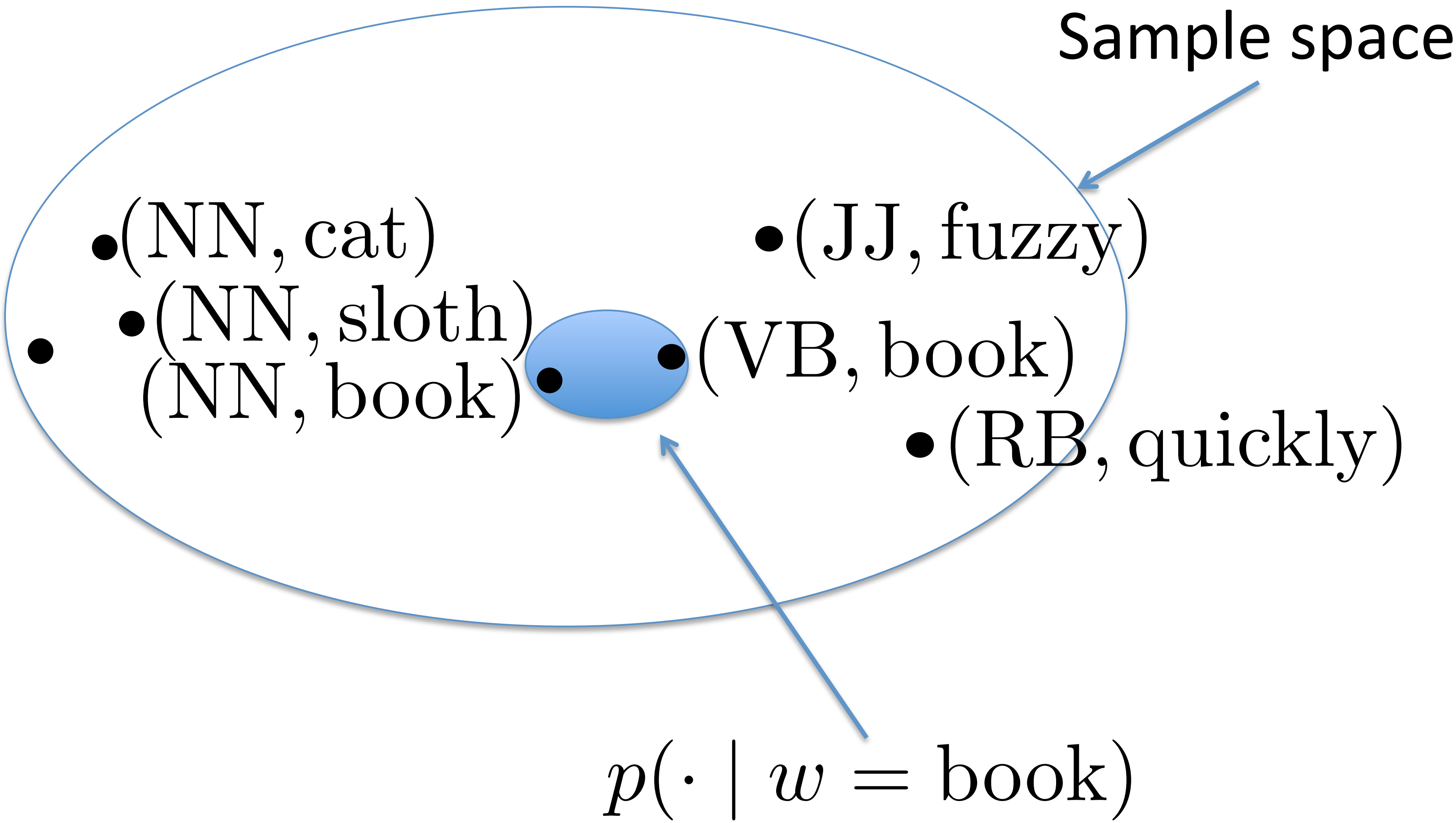
The **conditional probability distribution** of a variable X given a variable Y has the following properties:

$$\forall\ y \in Y,\ \sum_{x \in X} p(X = x \mid Y = y) = 1$$

Conditional Probability



Conditional Probability



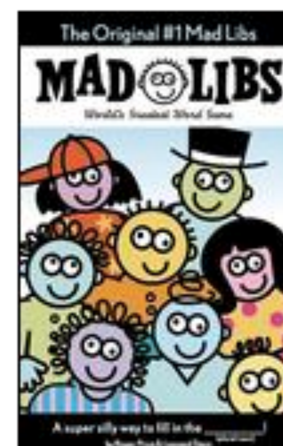
Conditional Probabilities

- In a joint model of word and tag sequences p(**w**,**t**)
  - The probability of a tag sequence given a word sequence p(**t** | **w**)
  - The probability of a word sequence given a tag sequence p(**w** | **t**)

Joint and Marginal Probabilities

- In a joint model of word and tag sequences p(**w**,**t**)
  - The probability that the 3rd tag is VERB, given **w** = “Time flies ***like*** an arrow”
    p(t_3 = VERB| **w** = Time flies like an arrow)

  - The probability that the 3rd word is ***like***, given **w** = “Time flies ______ an arrow”, t_3 = VERB
    p(t_3 = ***like*** | **w** = Time flies _____ an arrow, t_3 = VERB)



Chain Rule

$$\begin{aligned} p(a, b, c, d, \ldots) = & \, p(a) \times \\ & p(b \mid a) \times \\ & p(c \mid a, b) \times \\ & p(d \mid a, b, c) \times \\ & \vdots \end{aligned}$$

Bayes Rule

Likelihood

Posterior

Prior

$$p(x \mid y) = \frac{p(y \mid x)p(x)}{p(y)} \quad \left(= \frac{p(y \mid x)p(x)}{\sum_{x'} p(y \mid x')p(x')} \right)$$

Evidence

Independence

Two r.v.'s are **independent** iff

$$p(X = x, Y = y) = p(X = x) \times p(Y = y)$$

Equivalently (prove with def. of cond. prob.)

$$p(X = x \mid Y = y) = p(X = x)$$

Alternatively,

$$p(Y = y \mid X = x) = p(Y = y)$$

Conditional Independence

Two equivalent statements of conditional independence:

$$p(a, c \mid b) = p(a \mid b)p(c \mid b)$$

and:

$$p(a \mid b, c) = p(a \mid b)$$

"If I know B, then C doesn't tell me about A"

$$p(a \mid b, c) = p(a \mid b)$$

$$\begin{aligned} p(a, b, c) &= p(a \mid b, c)p(b, c) \\ &= p(a \mid b, \cancel{c})p(b \mid c)p(c) \end{aligned}$$

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$$\begin{aligned} p(a, b, c) &= p(a \mid b, c) p(b, c) \\ &= p(a \mid b \cancel{, c}) p(b \mid c) p(c) \\ &= p(a \mid b) p(b \mid c) p(c) \end{aligned}$$

Conditional Independence

- Useful thing to assume when designing models
  - Limit the variables that influence distributions
  - Classical example: Markov assumption
- Questions
  - Does conditional independence imply marginal independence?
  - Does marginal independence imply conditional independence?

Expected Values

$$\mathbb{E}_{p(X=x)}\left[f(x)\right] \doteq \sum_{x\in\mathcal{X}} p(X=x) \times f(x)$$

Some special expectations:

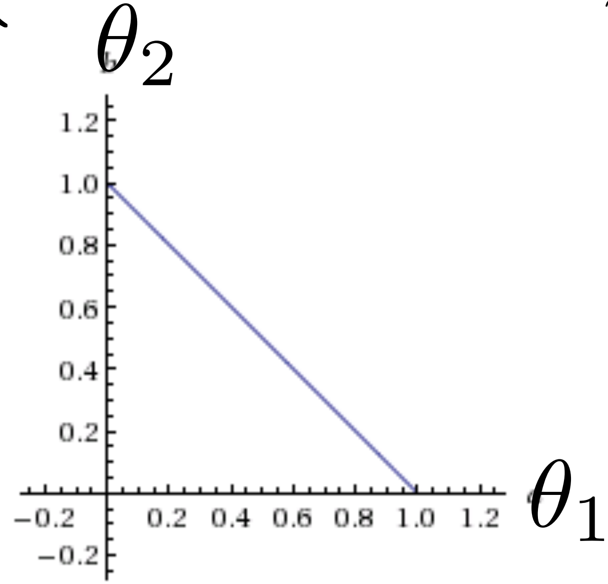
$$p(X=y) = \mathbb{E}_{p(X=x)}\left[\mathbb{I}_{x=y}\right]$$

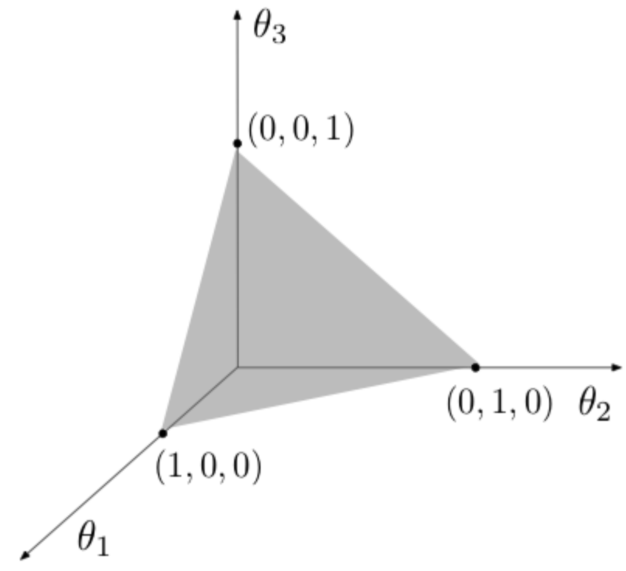
$$H(X) = \mathbb{E}_{p(X=x)}[-\log_2 x]$$

Categorical (Multinomial) Distributions

- Generalized model of a di to *k* dimensions
- Option 1: Parameters lie on the ***k*-simplex**

$$\Delta^k = \left\{ (\theta_1, \theta_2, \ldots, \theta_k) \;\middle|\; \sum_{i=1}^{k} \theta_i = 1 \wedge \theta_i \geq 0 \;\forall\; i \in [0, k] \right\}$$





Log-linear Parameterization

Weight vector

Feature vector function

$$p(x) = \frac{\exp \boldsymbol{w}^\top \boldsymbol{f}(x)}{Z}$$

$$\text{where } Z = \sum_{x' \in \mathcal{X}} \exp \boldsymbol{w}^\top \boldsymbol{f}(x)$$

Assumption: Z converges

Categorical (Multinomial) Distributions

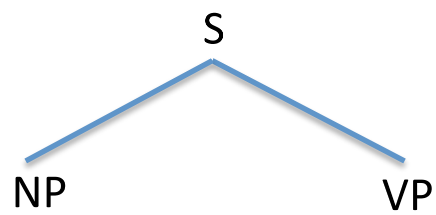
- "Naïve" parameterization
  - k outcomes, k(-1) independent parameters
  - Model as tables of (conditional) probabilities
  - MLE estimation (given fully observed data) is easy
- Log-linear parameterization
  - k outcomes, n, possibly overlapping parameters
    - Share statistical strength across "related" events
    - How are elements related? Depends how you define f

Locally Normalized Models

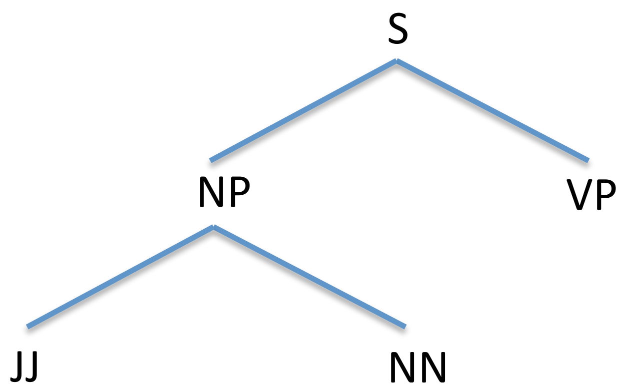
- Structure as the result of a **discrete time branching process**
  - Start in a known initial state, carry out stochastic steps (parameterized using multinomials) until some termination condition is met
  - Steps are (conditionally) independent of one another: probabilities multiply
  - *Total probability is the probability of the steps*
- Usually for joint (generative) models
  - not always though (see Appendix D.2)

S

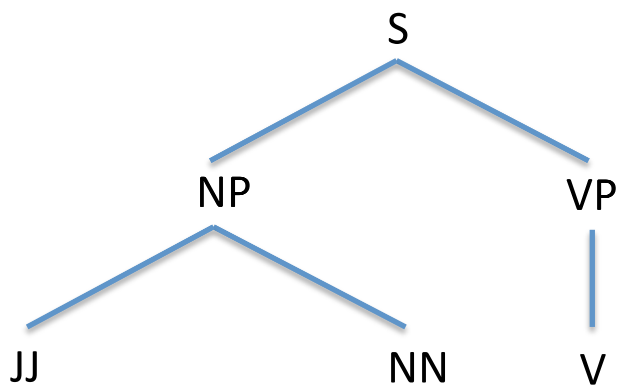
1.0



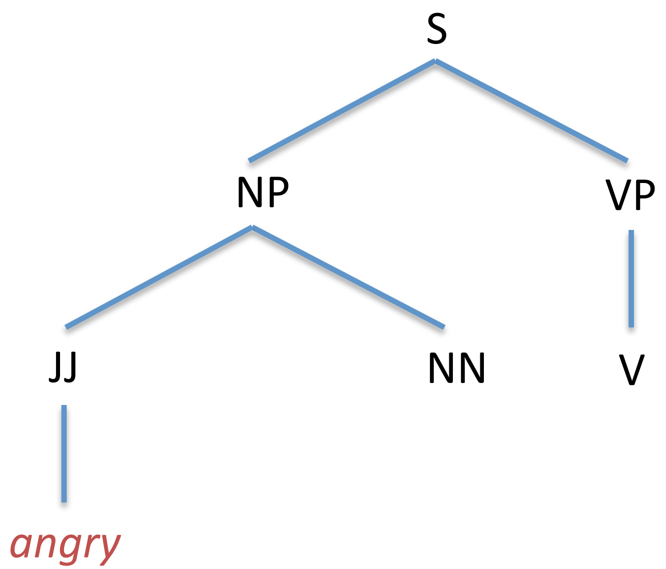
1.0 x p(NP VP | S)



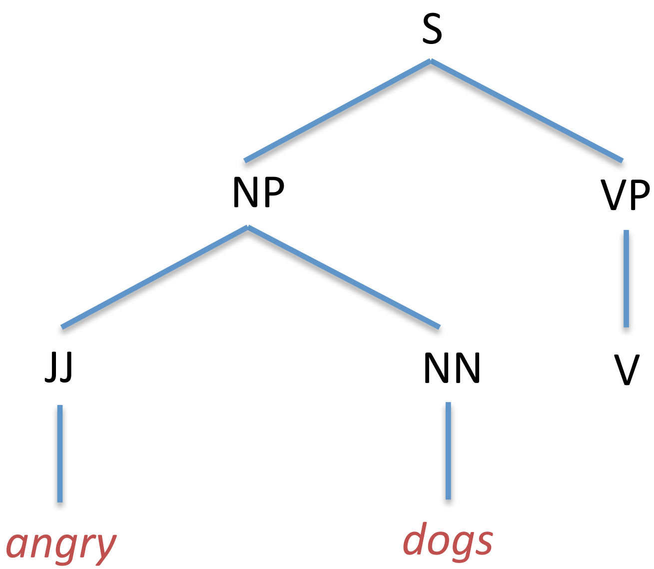
1.0 x p(NP VP | S)
x p(JJ NN | NP)



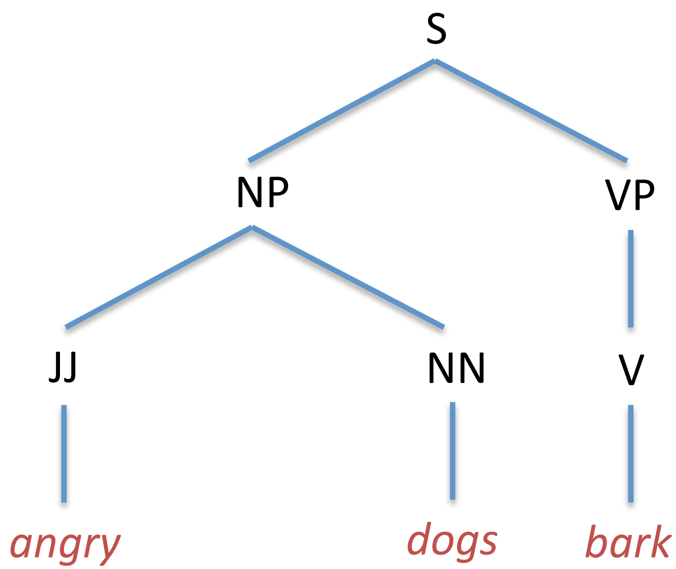
1.0 x p(NP VP | S)
x p(JJ NN | NP)
x p(V | VP)



1.0 x p(NP VP | S)
x p(JJ NN | NP)
x p(V | VP)
x p(*angry* | JJ)

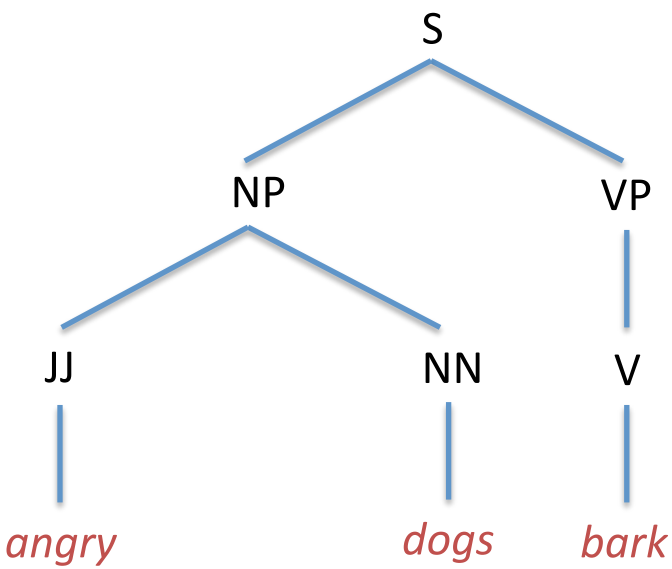


1.0 x p(NP VP | S)
x p(JJ NN | NP)
x p(V | VP)
x p(*angry* | JJ)
x p(*dogs* | NN)



1.0 x p(NP VP | S)
x p(JJ NN | NP)
x p(V | VP)
x p(*angry* | JJ)
x p(*dogs* | NN)
x p(*bark* | V)

$$p(\tau, \mathbf{x}) = \prod_{r \in \mathcal{G}} p(r \mid \mathcal{G})^{f(r \in \tau)}$$



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Here's an alternative way of building a tree and string:

S

1.0

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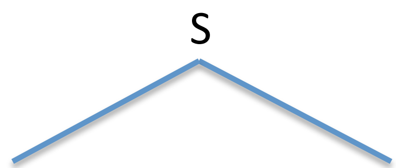

1.0 x p(2 kids | S)

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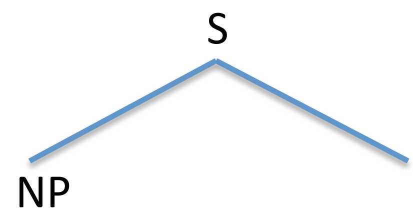

1.0 x p(2 kids | S)
x p(NP | S, n=1, total=2)

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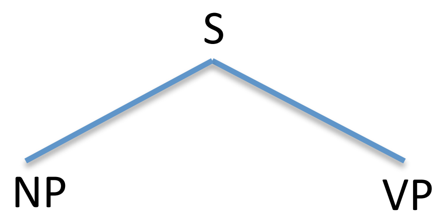

1.0 x p(2 kids | S)
x p(NP | S, n=1, total=2)
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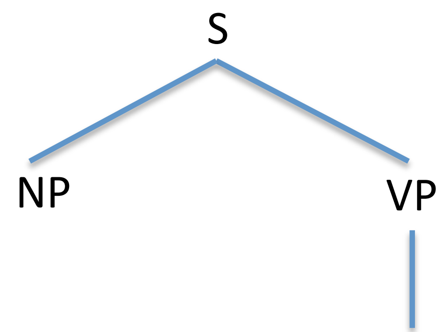

1.0 x p(2 kids | S)
x p(NP | S, n=1, total=2)
x p(VP | S, n=2, total=2)
x p(1 kid | VP)

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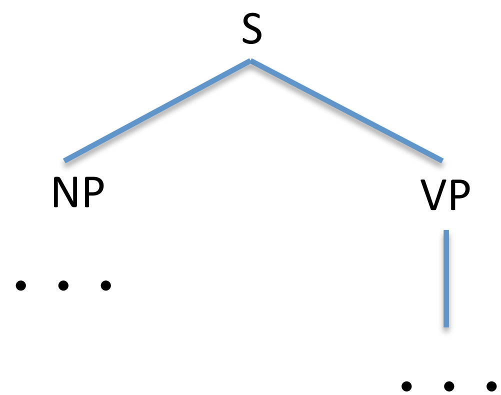

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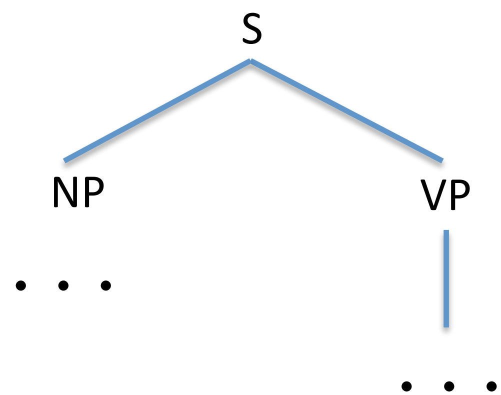

1.0 x p(2 kids | S)
x p(NP | ~~S,~~ n=1, total=2)
x p(VP | ~~S,~~ n=2, total=2)
x p(1 kid | VP)

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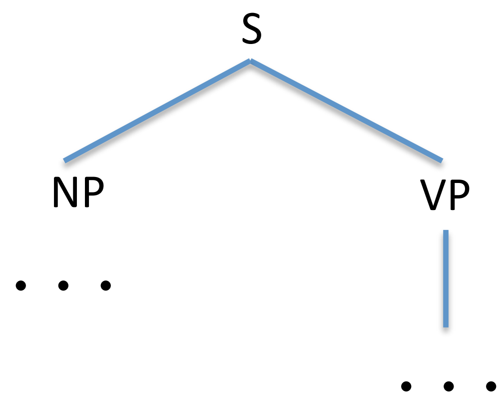


1.0 x p(2 kids | S)
x p(NP | S, n=1, total=2)
x p(VP | S, n=2, total=2)
~~x p(1 kid | VP)~~
x p(1 kid | VP, S)

Choosing a Model

- Independence is a property of distributions
  - Look at distributions in the wild, figure out what independence assumptions hold
- Dependence makes modeling more expensive
  - How big does your CKY chart have to be if you have “grandparent” annotation?

Parameterization

- For each step in the branching process
  - We have a multinomial distribution
  - We can use independent parameters (on simplex)
  - We can use log-linear models
    - “Locally normalized model” (cf. Appendix D.2)
    - Z is “local” to the decision being made

Globally Normalized Models

- Extension of the exponential parameterization to structured output spaces

$$p(\mathbf{x}) = \frac{\exp \mathbf{w}^\top \mathbf{F}(\mathbf{x})}{Z}$$

$$\text{where } Z = \sum_{\mathbf{x}' \in \mathcal{X}} \exp \mathbf{w}^\top \mathbf{F}(\mathbf{x}')$$

Conditional Random Fields

$$p(\mathbf{y} \mid \mathbf{x}) = \frac{\exp \mathbf{w}^\top \mathbf{F}(\mathbf{x})}{Z(\mathbf{x})}$$

$$Z(\mathbf{x}) = \sum_{\mathbf{y}' \in \mathcal{Y}_\mathbf{x}} \exp \mathbf{w}^\top \mathbf{F}(\mathbf{x})$$

Conditional Random Fields

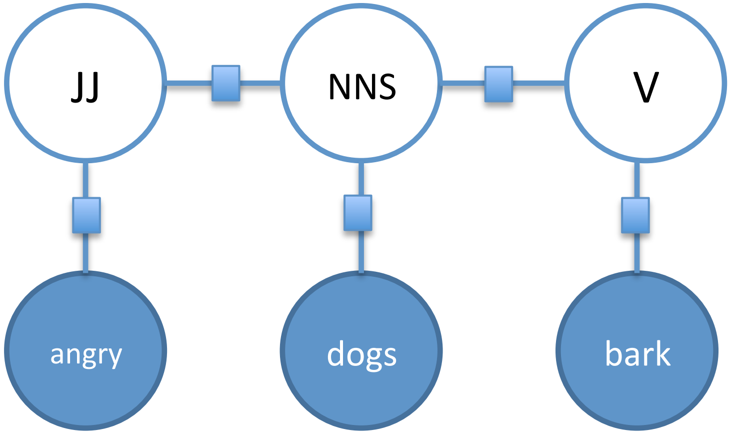
$$p(\mathbf{y} \mid \mathbf{x}) = \frac{\exp \mathbf{w}^\top \mathbf{F}(\mathbf{x}, \mathbf{y})}{Z(\mathbf{x})}$$

$$Z(\mathbf{x}) = \sum_{\mathbf{y}' \in \mathcal{Y}_\mathbf{x}} \exp \mathbf{w}^\top \mathbf{F}(\mathbf{x}, \mathbf{y})$$

Decoding is nice:

$$\begin{aligned}
\mathbf{y}^* &= \arg\max_{\mathbf{y} \in \mathcal{Y}_\mathbf{x}} \frac{\exp \mathbf{w}^\top \mathbf{F}(\mathbf{x}, \mathbf{y})}{Z(\mathbf{x})} \\
&= \arg\max_{\mathbf{y} \in \mathcal{Y}_\mathbf{x}} \exp \mathbf{w}^\top \mathbf{F}(\mathbf{x}, \mathbf{y}) \\
&= \arg\max_{\mathbf{y} \in \mathcal{Y}_\mathbf{x}} \mathbf{w}^\top \mathbf{F}(\mathbf{x}, \mathbf{y})
\end{aligned}$$

Conditional Random Fields



$$\mathbf{F}(\mathbf{x}, \mathbf{y}) = \sum_{C \in G} \mathbf{f}(C)$$

Comparison of Feature-Based Models

- Locally Normalized Models
  - Good joint models
  - Easy to training
  - Downside: decoding can be expensive
- Globally Normalized Models
  - Very popular conditional models (CRFs)
  - Challenge: computing Z / training
  - Advantage: decoding can be cheap