Efficient Algorithms for Path Problems in Weighted Graphs

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Thesis Proposal

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Thesis Committee:

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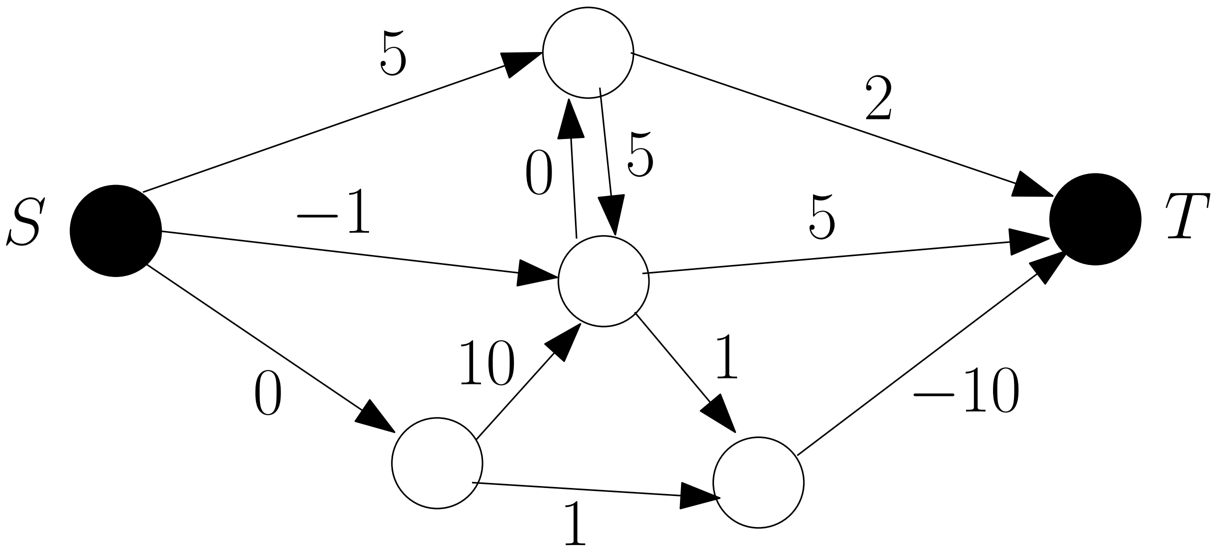
Weighted Graph Path Problems – Introduction



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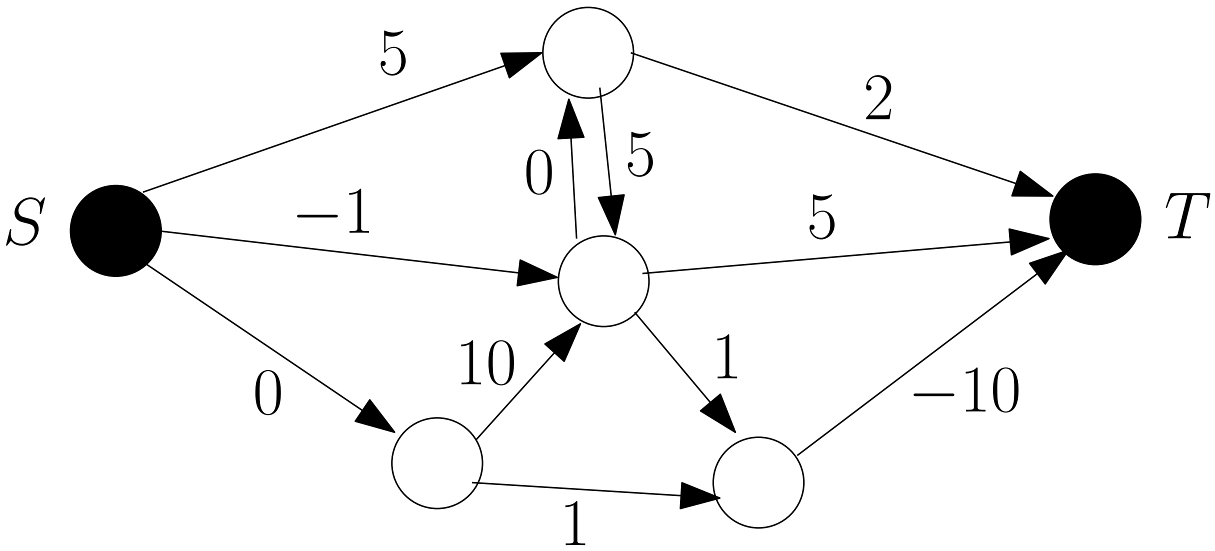
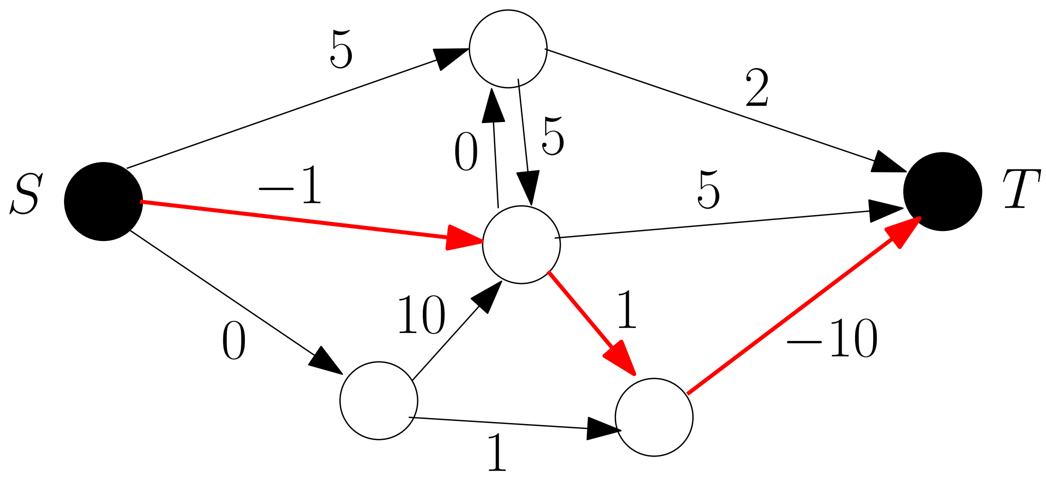


Our Problem: find a path $S = v_0 \longrightarrow v_1 \longrightarrow v_2 \longrightarrow \ldots \longrightarrow v_k = T$ optimizing a given measure.

Path Measures

Shortest Paths: Find $S = v_0 \to v_1 \to v_2 \to \ldots \to v_k = T$ minimizing

$$\sum_{i=0}^{k-1} w(v_i, v_{i+1}).$$



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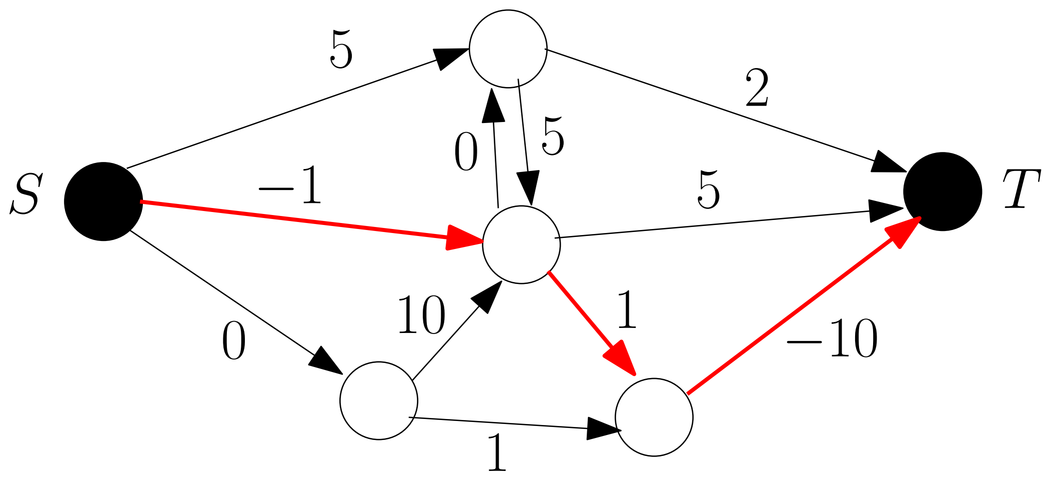

Application: find shortest road distance between two cities on a map.

Path Measures Cont.

Maximum Bottleneck Paths:

Find $S = v_0 \longrightarrow v_1 \longrightarrow v_2 \longrightarrow \ldots \longrightarrow v_k = T$ maximizing

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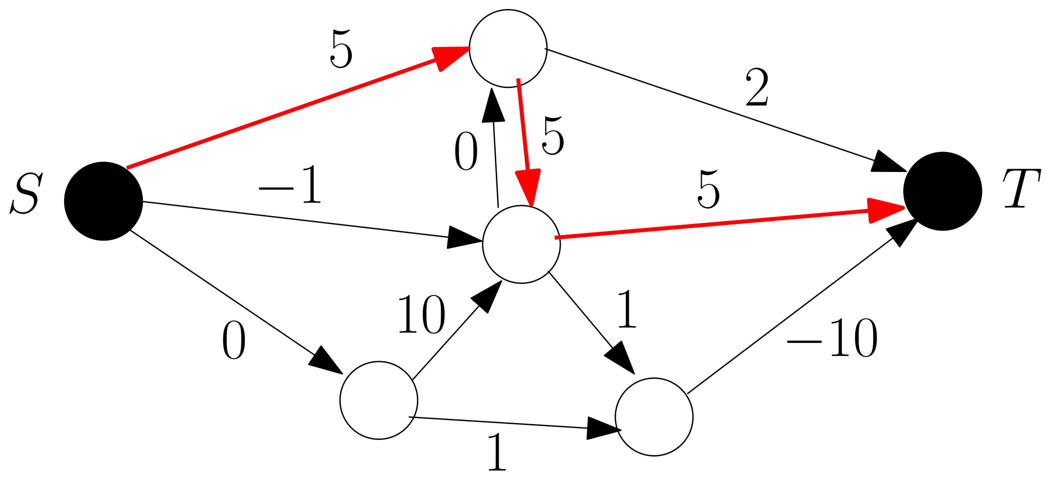

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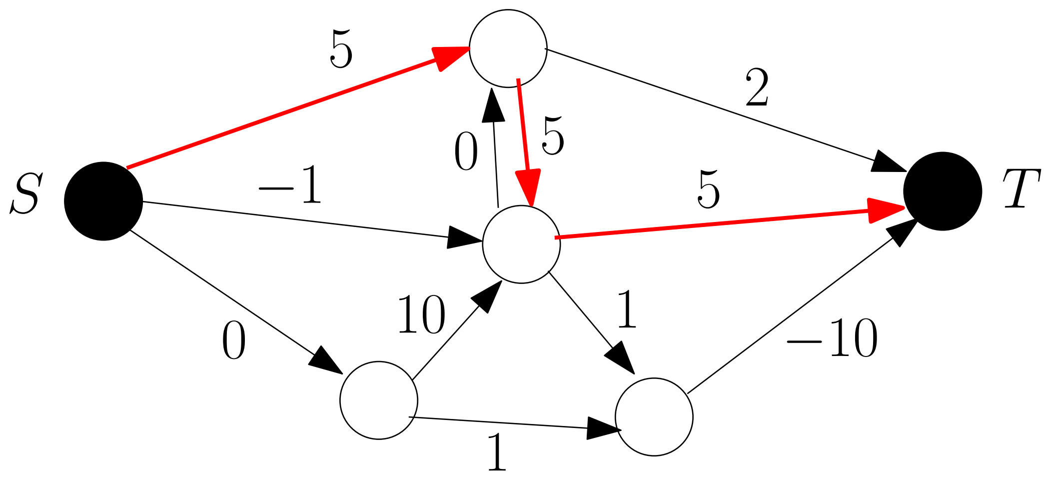
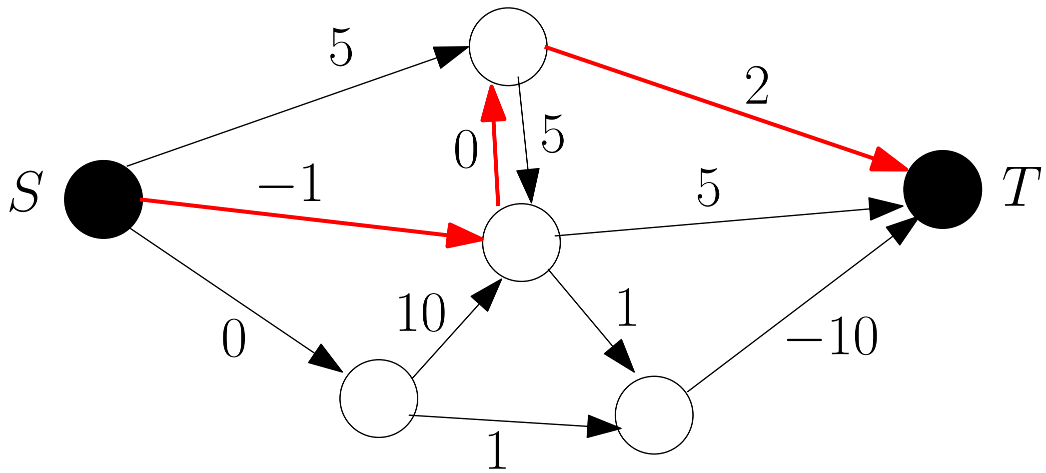


Application: find road path of highest tunnel clearance between two cities.

Path Measures Cont.

Minimum Nondecreasing Paths:

Find $S = v_0 \longrightarrow v_1 \longrightarrow v_2 \longrightarrow \ldots \longrightarrow v_{k-1} \longrightarrow v_k = T$ such that $w(v_i, v_{i+1}) \leq w(v_{i+1}, v_{i+2})$ for all i, minimizing $w(v_{k-1}, T)$.



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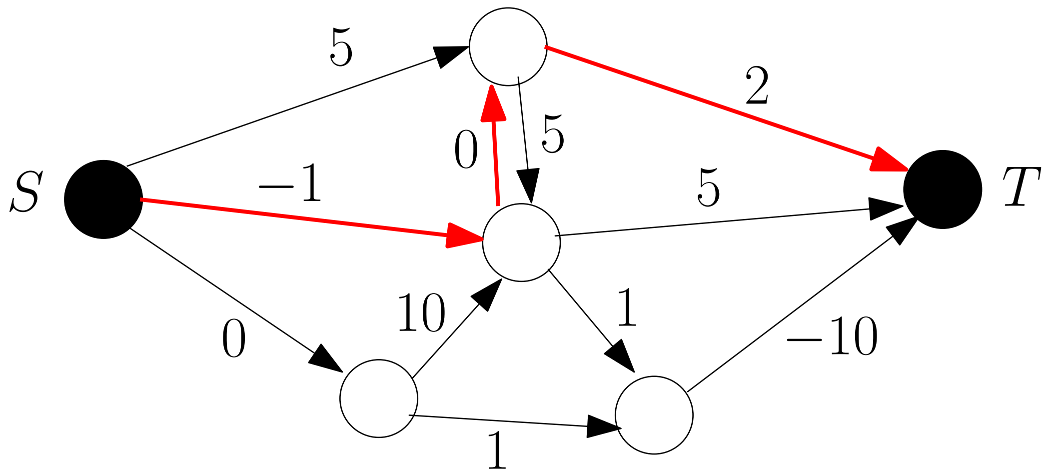
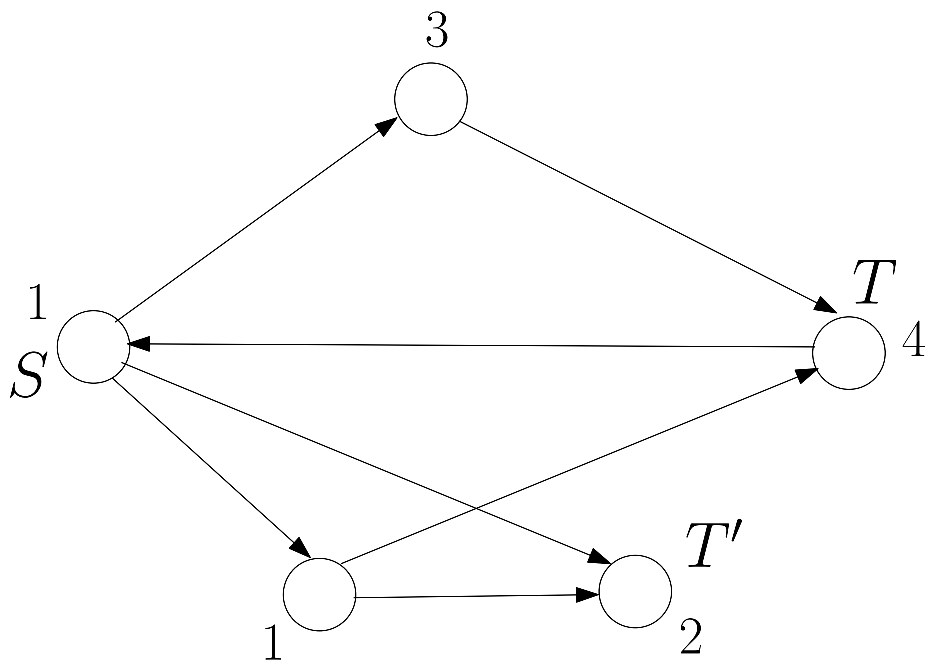


Application: compute train itinerary which gets you from one city to another as early as possible.

Path Measures Cont.

Maximum Node Weighted Triangle:

In a node-weighted graph, if (T, S) is an edge, find $S \rightarrow v \rightarrow T$ maximizing $w(S) + w(v) + w(T)$.



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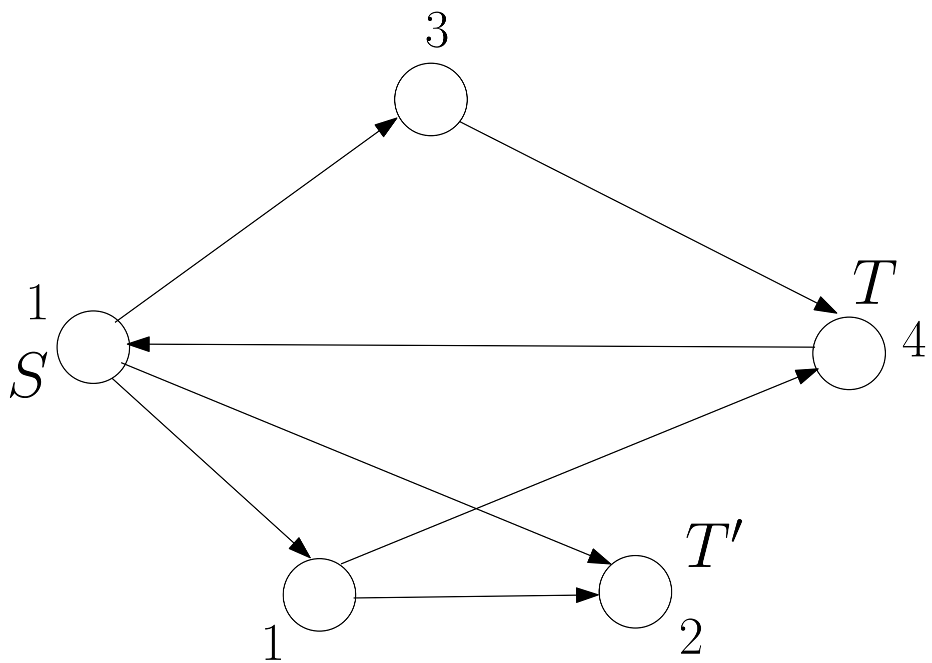

Application: Find important clusters in a database.

Problem Versions

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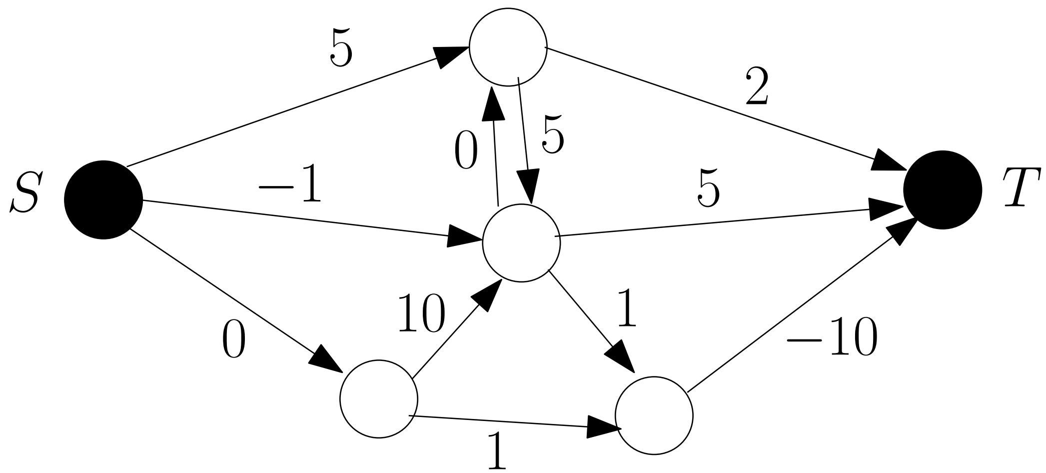
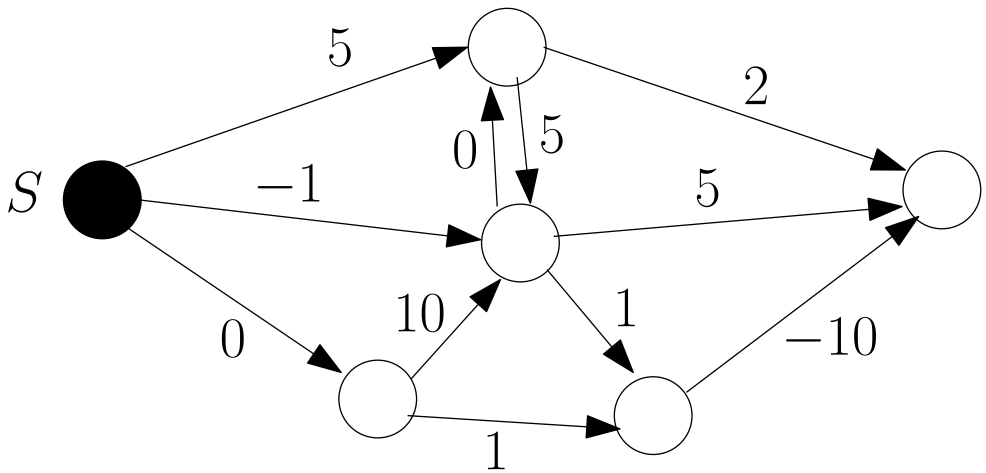
Single Source, Single Destination (S-T Best Path)



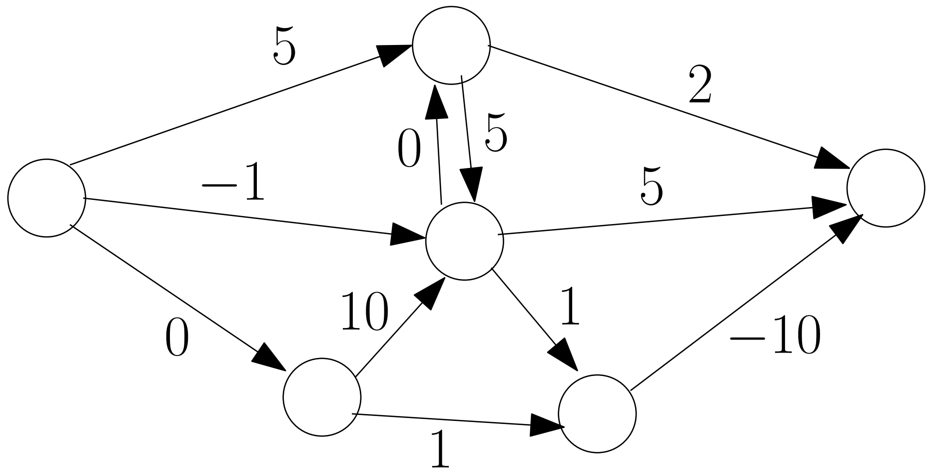
Problem Versions

Single Source (every destination) Best Path



Problem Versions

All Pairs Best Path



Talk Outline

1. Algorithms for All Pairs Best Paths

2. Algorithms for Single Source Best Paths

3. Directions for Further Research

All Pairs Path Problems – Results

n-number of vertices, m-number of edges

Problem	Previous Best	Our Results
AP Max Triangle	n^3	$n^{2.58}$ (VWY 2006)
AP Max Bottleneck Paths	n^3	$n^{2.79}$ (VWY 2007)
AP Min Nondecreasing Paths	n^3	$n^{2.9}$ (V 2008)
k Bits of Distance Product	$n^3/\log n$ (Chan 2005)	$2^k n^{2.69}$ (VW 2006).

All Pairs Path Problems – Outline

1. Path Problems and Matrix Products
2. Properties and Algorithms
3. Techniques
4. Example
5. Summary of results

Path Problems and $\otimes$ Products

For all these problems, the length of $v_1 \longrightarrow v_2 \longrightarrow \ldots \longrightarrow v_k$ is

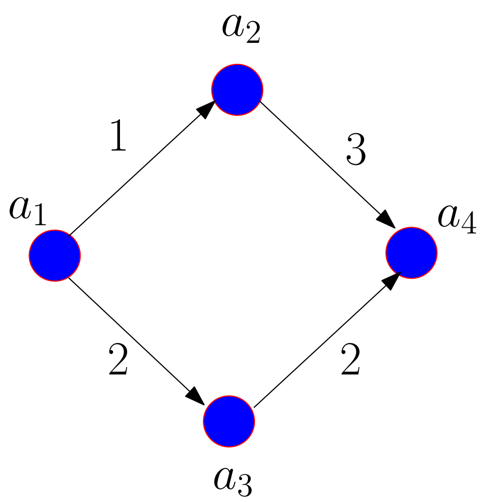
$$\ell(v_1, \ldots, v_k) = \ell(v_1, \ldots, v_{k-1}) \otimes w(v_{k-1}, v_k), \;\; \ell(v_1, v_2) = w(v_1, v_2),$$

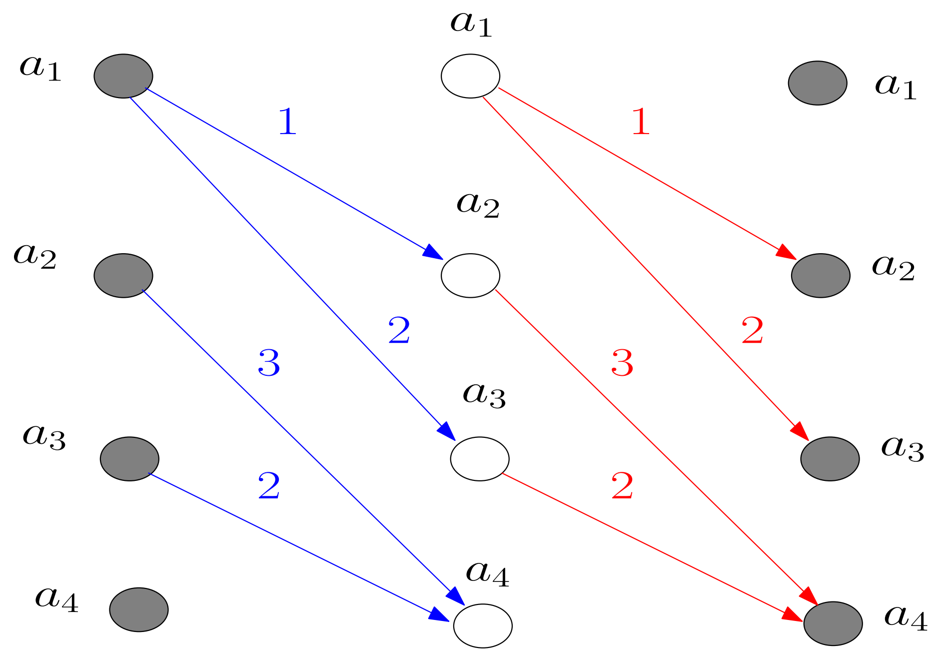
where $\otimes$ is different for each problem.

Problem	$\otimes$
Shortest Paths	$+$
Bottleneck Paths	$\min$
Nondecreasing Paths	$\leq'$

$a \leq' b$ returns b if $a \leq b$ and ∞ otherwise.

Path Problems and Matrices





$$\begin{pmatrix} \infty & 1 & 2 & \infty \\ \infty & \infty & \infty & 3 \\ \infty & \infty & \infty & 2 \\ \infty & \infty & \infty & \infty \end{pmatrix}$$

shortest paths

$$\begin{pmatrix} -\infty & 1 & 2 & -\infty \\ -\infty & -\infty & -\infty & 3 \\ -\infty & -\infty & -\infty & 2 \\ -\infty & -\infty & -\infty & -\infty \end{pmatrix}$$

bottleneck paths

$$\begin{pmatrix} \varepsilon_0 & 1 & 2 & \varepsilon_0 \\ \varepsilon_0 & \varepsilon_0 & \varepsilon_0 & 3 \\ \varepsilon_0 & \varepsilon_0 & \varepsilon_0 & 2 \\ \varepsilon_0 & \varepsilon_0 & \varepsilon_0 & \varepsilon_0 \end{pmatrix}$$

in general

Path Problems and Matrix Products

Definition [$\{\oplus, \otimes\}$-Product]:

Given two $n \times n$ real matrices A and B, and two operations $\otimes$ and $\oplus$ on $\mathbb{R}$, such that $\oplus$ is commutative and associative, the $\{\oplus, \otimes\}$-product of A and B is the $n \times n$ matrix C given by

$$C[i,j] := \bigoplus_{k=1}^{n} (A[i,k] \otimes B[k,j]), \forall i,j = 1, \ldots, n.$$

For our path problems, $\oplus$ is always $\max$ or $\min$.

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Algebraic Product: (CoppersmithWinograd90)

$C[i,j] = (A \cdot B)[i,j] = \sum_k \{A[i,k] \cdot B[k,j]\}.$ $n^\omega = n^{2.376}$

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Properties and Path Algorithms

Let $OPT[x, y]$ be the best $x \longrightarrow y$ path length. $OPT[x, x] = \varepsilon_1$.

Edge-Padding Property:

There are operations $\oplus$ and $\otimes$, such that $\oplus$ is commutative and associative, and for all pairs of vertices x, y in the graph,

$$OPT[x, y] = \bigoplus_{z \in V} (OPT[x, z] \otimes w(z, y)).$$

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This holds when $\otimes$ is right-distributive over $\oplus$.

Properties and Path Algorithms Cont.

Midpoint Property:

If $x \longrightarrow v_1 \longrightarrow \ldots \longrightarrow v_k \longrightarrow y$ is an optimal path, then for all i, $x \longrightarrow \ldots \longrightarrow v_i$ and $v_i \longrightarrow \ldots \longrightarrow y$ are also optimal.

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When $(\mathbb{R}, \oplus, \otimes, \varepsilon_0, \varepsilon_1)$ form a semiring, then all pairs best paths can be done in $O(T[(\oplus, \otimes)\text{-product}])$ (AHU74).

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Hence, we can concentrate on matrix products.

Techniques for Computing Matrix Products

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1. **Bucketting:** Preprocess each input matrix and assign its entries in a 1-to-1 fashion to some number of buckets, each getting a small number of entries. For each input matrix X and each bucket b, create a new matrix X_b.

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2. **Bucket Processing:** For each bucket b, multiply A_b and B_b using (perhaps recursively) a different matrix product $(\oplus', \otimes')$.

3. **Exhaustive Search:** The bucket processing step provides information which allows us to choose a small number of buckets on which the problem is solved by exhaustive search.

Example: $(\min, \leq)$-Product

We want $a_{ij} = \min_k\{B[k,j] \mid A[i,k] \leq B[k,j]\}$.

We will use the dominance product: $(A \odot B)[i,j] = |\{k \; : \; A[i,k] \leq B[k,j]\}|$.

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1. Take the columns of B and sort the entries of each column.

$$B = \begin{pmatrix} 10 & 2 & 0 & 7 \\ -1.1 & 3 & -1 & 2.1 \\ 5.1 & 7 & -2 & 4 \\ 3.2 & 1 & -3 & 2.1 \end{pmatrix}$$

column 1 : $A[2,1], \quad A[4,1], \quad A[3,1], \quad A[1,1]$

column 2 : $A[4,2], \quad A[1,2], \quad A[2,2], \quad A[3,2]$

column 3 : $A[5,3], \quad A[3,3], \quad A[2,3], \quad A[1,3]$

column 4 : $A[4,4], \quad A[2,4], \quad A[3,4], \quad A[1,4]$

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1. Take the columns of B and sort the entries of each column.
2. (Technique 1) Bucket the entries of each column of B, in their sorted order into s roughly equal buckets.

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column 4 : $A[4,4]$, $A[2,4]$, $A[3,4]$, $A[1,4]$

Example: $(\min, \leq)$-Product cont.

3. (Technique 1 - **Bucketting**) For each bucket b create a matrix $B(b)$ containing only the elements in bucket b and $-\infty$ in all other entries.

$$B(1) = \begin{pmatrix} -\infty & 2 & -\infty & -\infty \\ -1.1 & -\infty & -\infty & 2.1 \\ -\infty & -\infty & -2 & -\infty \\ 3.2 & 1 & -3 & 2.1 \end{pmatrix} \quad B(2) = \begin{pmatrix} 10 & -\infty & 0 & 7 \\ -\infty & 3 & -1 & -\infty \\ 5.1 & 7 & -\infty & 4 \\ -\infty & -\infty & -\infty & -\infty \end{pmatrix}$$

Example: $(\min, \leq)$-Product cont.

We want $a_{ij} = \min_k \{B[k,j] \mid A[i,k] \leq B[k,j]\}$.

We will use the dominance product: $(A \odot B)[i,j] = |\{k \,:\, A[i,k] \leq B[k,j]\}|$.

4. (Technique 2 - **Bucket Processing**) Compute $A \odot B(b)$ for each b.

$B \odot B(2) =$

$$\begin{pmatrix} 10 & 2 & 0 & 7 \\ -1.1 & 3 & -1 & 2.1 \\ 5.1 & 7 & -2 & 4 \\ 3.2 & 1 & -3 & 2.1 \end{pmatrix} \odot \begin{pmatrix} 10 & -\infty & 0 & 7 \\ -\infty & 3 & -1 & -\infty \\ 5.1 & 7 & -\infty & 4 \\ -\infty & -\infty & -\infty & -\infty \end{pmatrix} = \begin{pmatrix} 2 & 2 & 0 & 1 \\ 2 & 2 & 1 & 2 \\ 2 & 1 & 0 & 2 \\ 2 & 2 & 0 & 2 \end{pmatrix}$$

This tells us for every bucket b and each i, j, the number of coords k such that $B[k,j]$ is in bucket b and $A[i,k] \leq B[k,j]$.

This step takes $O(sn^{\frac{3+\omega}{2}})$ since dominance product takes $O(n^{\frac{3+\omega}{2}})$.

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7. The overall runtime is minimized for $s = n^{\frac{3-\omega}{4}}$ and the runtime is then $O(n^{\frac{9+\omega}{4}}) = O(n^{2.85})$.

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All Pairs Bottleneck Paths - compute $(\max, \min)$-Product using $(\min, \leq)$-Product. Best Running Time: $O(n^{2.792})$. (VWY STOC07)

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Single Source Bottleneck Paths - A $O(m+n)$ algorithm is known for the undirected single source, single target version (Punnen91). In the general case: Dijkstra's with Fibonacci Heaps $O(m+n\log n)$.

Directions for Further Research

1. More single source algorithms - better single source algorithm for bottleneck paths?
2. Parallel Algorithms
3. Combinatorial Algorithms

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Hence our all pairs algorithms running sequentially in $O(n^c)$ time can be done in parallel on $O(n^c)$ processors and $O(\mathrm{poly}\log\mathrm{n})$ time.

Parallel Algorithms Cont.

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Conjecture: for K-source bottleneck / nondecreasing paths reduce preprocessing work to $O((n + s(n)^\alpha)\log n)$–work where:

$\alpha = 2.792$ for bottleneck and $\alpha = 2.896$ for nondecreasing paths.

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We want similar runtimes for AP bottleneck paths, and AP nondecreasing paths.

Purely Combinatorial Algorithms Cont.

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The somewhat sparse case – $O(mn \log(n^2/m)/\log^2 n)$ algorithm for matrix multiplication, transitive closure and max weight triangle (BVW 08).

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Similar algorithms for AP shortest, AP nondecreasing, or AP bottleneck paths?

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- April - thesis defense?

Thank you!