Hybrid Algorithms for Graph Problems

Joint work with Ryan Williams and Maverick Woo
Hybrid Algorithms for Graph Problems
Defying Hardness using Graph Minors and Separators

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Introduction

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- **Running Time**, or
- **Space**, or
- **Simultaneous Time and Space**, or
- **Approximation Ratio and Time**, etc.
Hybrid Algorithms
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$h_1$ approximates the optimal solution within a factor of $\alpha$ and runs in polynomial time.

$h_2$ solves the problem exactly but runs in subexponential time $2^{o(n)}$. 
Hybrid Algorithms cont.
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A selector $S$ which on each instance selects in polynomial time the best heuristic.
Hybrid Algorithms cont.
Defying Hardness: Some NP-Hard problems are known or conjectured to be *hard* on several complexity measures $m_i$ separately.
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Defying Hardness: Some NP-Hard problems are known or conjectured to be *hard* on several complexity measures $m_i$ separately.

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There exist hybrid algorithms for NP-Hard problems which for each $h_i$ (on the instances on which $S$ chooses $h_i$ to be run) do *strictly better* than the corresponding known hardness guarantees $m_i$. 
Max Cut

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No better than $1/2$-approximation is known without using SDP.
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If $|M| < \varepsilon \frac{m}{2}$,

try all $2^{\varepsilon m}$ cuts of the vertices in $M$ to find the maximum. Add the vertices from the independent set $V - M$ so that the cut is maximized.
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If $|M| \geq \varepsilon \frac{m}{2}$,

for each edge in $M$, with probability $1/2$ choose which of its endpoints to put in $A$. Put the other endpoint in $B$;
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If $|M| \geq \varepsilon \frac{m}{2}$,

for each edge in $M$, with probability $1/2$ choose which of its endpoints to put in $A$. Put the other endpoint in $B$;

for each vertex $v$ not covered by $M$, with probability $1/2$ choose whether to place it in $A$ or $B$. 

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Max Cut cont.
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If $|M| < \varepsilon \frac{m}{2}$, $M$ has at most $\varepsilon m$ vertices, and the rest of the vertices form an independent set $I$. Placing the vertices of $I$ so that the cut is maximized, given an arrangement of $M$ is easy.

We get an exact solution in $\tilde{O}(2^{\varepsilon m})$ time.
Max Cut cont.

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The probability that an edge not in $M$ crosses the cut is $1/2$. Hence we get a cut of expected size at least $(\varepsilon \frac{m}{2}) + \frac{1}{2}(m - \varepsilon \frac{m}{2}) = (\frac{1}{2} + \frac{\varepsilon}{4})m$.

We get a $(\frac{1}{2} + \frac{\varepsilon}{4})$-approximation with no semidefinite programming.
The Longest Path Problem
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Karger, Motwani and Ramkumar, 1993: Longest Path is hard to approximate within $2^{O\left(\frac{\log n}{\log \log n}\right)}$, unless $\text{NP} \subseteq \bigcap_{\delta > 0} \text{DTIME}\left(2^{O(n^\delta)}\right)$. 
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We give a hybrid algorithm which for any $\ell(n)$

- either finds a path of length $\ell$, or
- solves the Longest Path exactly in time $2^{O(\ell \log L \log \frac{n}{\ell})}$. 
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Notice that for $\ell = n/\text{polylog}(n)$ we get subexponential exact running time and a polylog approximation.
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3. Else, remove $f$ from $P$ and add it to $A$.
   
   If $|A| = n/2$, stop and output $P$ as a separator.
   
   Otherwise, attempt to continue $P$ with vertices from $V - P - A$ until $f$ with no neighbors is reached. Go to 2.
Tree Width

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The width of a tree decomposition is the maximum size of a bag $W_i$, minus 1. The *tree width* of a graph $G$ is the minimum width of a tree decomposition of $G$. 
Towards a Hybrid Algorithm
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A result by Matousek and Thomas implies: if $G$ has treewidth at most $K$, then there is a $O(L^{K+1}n)$ algorithm to find a path of length $L$ in $G$, or to determine that no such exists.
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We show: If $G$ has a separator decomposition with separator size $\ell$, then $G$ has a tree decomposition of width at most $O(\ell \log \frac{n}{\ell})$. 
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4. Recurse on $G_L$ and $G_R$ to obtain either a path of length $\ell$, or a separator decomposition.
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4. Recurse on $G_L$ and $G_R$ to obtain either a path of length $\ell$, or a separator decomposition.

5. Run the Matousek and Thomas algorithm on the tree decomposition obtained from the separator tree, on successive powers of 2 for the path length, to obtain the longest path in $\tilde{O}(2^{\ell \log L \log \frac{n}{\ell}})$. 
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1. there are no edges between $A$ and $B$, and
2. $|A| \leq |B| \leq \alpha|V(G)|$.

Often one says $S$ is a $1/3 - 2/3$–separator, meaning that in the worst case $|A| = \frac{1}{3}|V(G)|$ and $|B| = \frac{2}{3}|V(G)|$. 
A More General Minor – Separator Theorem

The following is a generalization of Plotkin, Rao, Smith, 1994:

Given graphs $G$, $H$ and some $\ell \geq 1$, there is a polynomial time algorithm which
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For large values of $\ell$ the above can be generalized to finding a minor, or finding a $1/2 - 1/2$–separator.
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We build a minor $M$ and a set $B$ with $M \cap B = \emptyset$ and $B$ having few neighbors in $W = (V(G) - M - B)$. 
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In the end either $M$ will be an $H$-minor of the desired size, or $B$ will be the larger partition of $V$ with $S = N(B)$ becoming the separator.
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At each stage $M$ is an $H'$-minor of $G$ where $H'$ is an induced subgraph of $H$. 
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$B$ is a subset of $V(G)$, $B \cap M = \emptyset$, and $|N(B) \cap W| \leq \frac{|B|}{\ell}$. 
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At each stage we look at a vertex $v$ from $H - H'$ and its neighbors $u_1, \ldots, u_k$ in $H'$. 
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If any of the supernodes in $M$ corresponding to the $u_i$ have no neighbors in $W$, we move them to $B$ (updating $M$ and thus $H'$).
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If $B$ became large, we have found a separator.
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Otherwise, we pick a node $w$ in $W$ and start doing a two-stage BFS in $W$. 
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If \( B \) became large, we have found a separator.

Otherwise, we pick a node \( w \) in \( W \) and start doing a two-stage BFS in \( W \). This BFS finds the new supernode corresponding to \( v \), or more nodes to add to \( B \). If \( W \) becomes smaller than \( 2n/3 \), we have found a separator.
Two Stage BFS in $W$

Recall the parameter $\ell \geq 1$. Let $R = \{w\}$. 
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2. If both $R$ did not expand too much ($|T| \leq |R|(1 + 1/\ell)$), and $W - R$ did not shrink too much ($|W - R| \leq (1 + 1/\ell)|W - T|$), stop.
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3. Otherwise, set $R = T$ and continue from 2.

4. If in the end $R$ contains a neighbor $n_i$ of each $u_i$, return a shortest path tree from all the $n_i$ to $w$. This is the new supernode for vertex $v$ and $M$ is a $H' \cup \{v\}$–minor.
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5. Otherwise, the *smaller* of $R' = R \cup N(R)$, and $R'' = V - R'$ has *few neighbors in* $W$ ($|N(R')| \leq \frac{|R'|}{\ell}$ and $|N(R'')| \leq \frac{|R''|}{\ell}$).

We add it to $B$. 

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Why?: When a supernode corresponding to $v \in V(H)$ is added it has size at most $O(\text{deg}_H(v)\ell \log n)$ since the BFS tree has depth at most $2 \log_{(1+1/\ell)} n \leq 2\ell \log n$, and since there are at most $\text{deg}_H(v)$ neighbors to be covered.
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Once a supernode is added, its size is never changed, unless the supernode is removed.
If we find a minor $M$, its size is $O(\ell h \log n)$.

**Why?**: When a supernode corresponding to $v \in V(H)$ is added it has size at most $O(\text{deg}_H(v) \ell \log n)$ since the BFS tree has depth at most $2 \log_{1+1/\ell} n \leq 2\ell \log n$, and since there are at most $\text{deg}_H(v)$ neighbors to be covered.

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Minor or Separator

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The separator consists of the (unfinished) minor $M$ and of the neighbors of $B$ in $W$. Since $|N_W(B)| \leq |B|/\ell \leq \frac{2n}{3\ell}$, the size of the separator is $O(n/\ell + \ell h \log n)$. 

19-e
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As in the case of Path-Separator we can obtain a separator tree, or an $H$-minor.
Minimum Bandwidth

Problem: Given a graph \( G \), give a permutation \( \pi \) on the vertices of \( G \) so that the maximum edge stretch \( \max_{(i,j) \in E(G)} |\pi(i) - \pi(j)| \) is minimized.

Best approximation: \( O(\log^3 n \sqrt{\log \log n}) \) by Dunagan and Vempala, 2001, \( O(\sqrt{\frac{n}{B}} \log n) \) by Avrim Blum et al. where \( B \) is the optimum bandwidth

Best Exact Algorithm: \( \tilde{O}(10^n) \) by Feige and Killian, 2000
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$G = (V, E)$ is an $\varepsilon$-expander iff for every $S \subseteq V$ with $|S| \leq |V|/2$ we have $|S \cup N(S)| \geq (1 + \varepsilon)|S|$.

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Gabber and Galil show how to construct $5$–regular $(\frac{2-\sqrt{3}}{4})$–expanders efficiently.
Graphs with Expander Minors have Large Bandwidth
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**Lemma.** Let $H$ be an $\varepsilon$-expander on $h$ nodes for some constant $\varepsilon > 0$. Let $G$ contain an $H$-minor $M$. Then the minimum bandwidth of $G$ is at least $\Omega(h)$. 
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Let $\pi$ be a linear arrangement of the nodes of $M$.

Let $h_{LHS}$ and $h_{RHS}$ be the *number* of supernodes completely contained among the first $k/2$ nodes (respectively, last $k/2$ nodes) in $\pi$. 
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Let $h_S = h - h_{LHS} - h_{RHS}$. 
Lemma Proof cont.
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If $h_S \geq \varepsilon \cdot h$ for some $\varepsilon > 0$, then the bandwidth is at least $\varepsilon \cdot h$:

Each supernode is disjoint from other supernodes and is connected, so the arrangement has $\varepsilon \cdot h$ nodes in the first half that connect to distinct nodes in the second half.

Any arrangement with this property has bandwidth at least $\varepsilon \cdot h$. 
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If \( h_{LHS} < h/3 \) or \( h_{RHS} < h/3 \) then \( h_S \geq 2h/3 \), so the bandwidth is \( \Omega(h) \) in this case.
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- \( h_S \geq \varepsilon h/6 \), which by the above implies the bandwidth is at least \( \varepsilon \cdot h/6 \), or
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If $h_S \geq \varepsilon \cdot h$ for some $\varepsilon > 0$, then the bandwidth is at least $\varepsilon \cdot h$:

Each supernode is disjoint from other supernodes and is connected, so the arrangement has $\varepsilon \cdot h$ nodes in the first half that connect to distinct nodes in the second half.

Any arrangement with this property has bandwidth at least $\varepsilon \cdot h$.

If $h_{LHS} < h/3$ or $h_{RHS} < h/3$ then $h_S \geq 2h/3$, so the bandwidth is $\Omega(h)$ in this case.

If $h_{LHS} \geq h/3$, then the supernodes contained in the first half have at least $\varepsilon h/3$ supernodes as neighbors, by the expansion condition. Thus either

- $h_S \geq \varepsilon h/6$, which by the above implies the bandwidth is at least $\varepsilon \cdot h/6$, or
- there are at least $\varepsilon h/6$ first half neighbors in the second half, in which case there are $\varepsilon h/6$ edges crossing from nodes in the first half to distinct nodes in the second half, so again the bandwidth is at least $\Omega(h)$. 
Hybrid Algorithm Idea
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- Either find a large constant degree expander as a minor of $G$.

  This guarantees that the bandwidth of $G$ is large, and hence the $O(\sqrt{\frac{n}{B} \log n})$–approximation algorithm by Avrim et al. gives a good approximation.
Hybrid Algorithm Idea

• Either find a large constant degree expander as a minor of $G$. This guarantees that the bandwidth of $G$ is large, and hence the $O(\sqrt{\frac{n}{B}} \log n)$—approximation algorithm by Avrim et al. gives a good approximation.

• Otherwise use the separator tree to get a good exact algorithm for bandwidth.
How to use the separator tree to solve Minimum Bandwidth
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- a $\log n$ bit index for the position of each separator node in the current allowed set of indices,

- a length $n$ bit string specifying whether left or right subtree nodes go at the corresponding position. We recurse on the left and right subtree separately, using the positions specified by the corresponding bits.
How to use the separator tree to solve Minimum Bandwidth, cont.
For example, for $\ell = 3, n = 5$, we may specify $(0, 2, 4)$ and $(0011)$. If the allowed positions are $3, 6, 7, 9, 10$, then
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For example, for $\ell = 3, n = 5$, we may specify $(0, 2, 4)$ and $(00111)$. If the allowed positions are $3, 6, 7, 9, 10$, then

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How to use the separator tree to solve Minimum Bandwidth, cont.

For example, for $\ell = 3$, $n = 5$, we may specify $(0, 2, 4)$ and $(00111)$. If the allowed positions are $3, 6, 7, 9, 10$, then

- the first, second and third separator nodes are in positions $3, 7, 10$, respectively,
- a node from the left subtree in position $6$, a node from the right subtree in position $9$. 

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- The recursive call is for position 6 on the left and position 9 on the right.
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At the end, the best linear arrangement is returned.
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The recurrence for the running time is (assuming a 1/2-1/2-separator):

$$T(n) \leq 2^{n+\ell \log n} \cdot 2T(n/2) + poly(n)$$
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$$T(n) = \tilde{O}(4^n \cdot n^{\ell \log(n/\ell)})$$, and if $\ell$ is chosen to be small, say $o\left(\frac{n}{(\log n \log \log n)}\right)$, then

$$T(n) = 4^n + o(n).$$
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We gave a hybrid algorithm for *Longest Path* which either finds a path of length $\ell$, or solves the problem exactly in time $2^\ell \log L \log \frac{n}{\ell}$.
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For $\ell = o\left(\frac{n}{\log n \log \log n}\right)$ we obtain either a $\log n \log \log n$ approximation, or a subexponential $2^{o(n)}$ exact solution. This beats the known conventional algorithms on both accounts. It also beats the inapproximability $2^{O\left(\frac{\log n}{\log \log n}\right)}$ by a huge margin.
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We gave a hybrid algorithm for *Minimum Bandwidth* which either approximates within $\alpha(n) \log^{2.5} n \log \log n$ (for unbounded $\alpha(n)$) or solves exactly in $4^{n+o(n)}$ time. This also beats the best known conventional algorithms on both accounts.
Thank You!