A Dominance Approach to Weighted Graph Problems

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Theory Lunch

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Introduction

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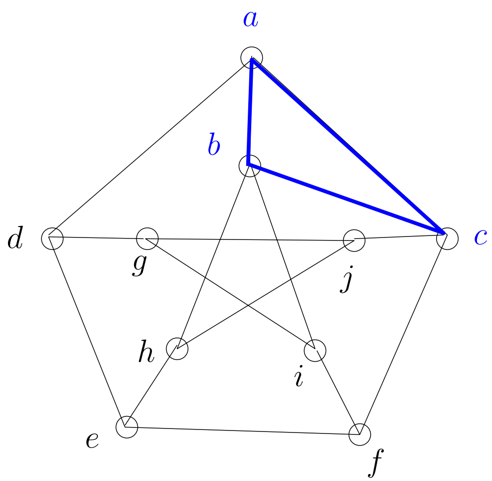

$$G^3 = \begin{pmatrix} 2 & \cdot & \cdot & \cdots \\ \cdot & 2 & \cdot & \cdots \\ \cdot & \cdot & 2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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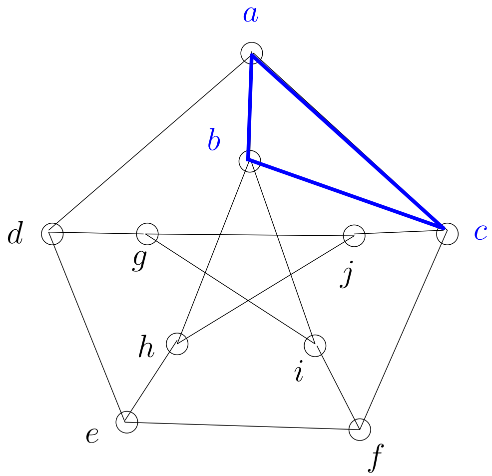

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Other examples: *LP, exact algorithms for NP-hard problems, graph perfect matching, unweighted APSP.*

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In general it is not clear how to speed-up weighted versions of problems in a similar way.

Example open problems include: *maximum weighted matching, finding minimum weighted triangles and other patterns, weighted APSP*.

Our approach [VW06]

Instead of matrix multiplication we use the so called dominance product to speed-up weighted problems.

We demonstrate the approach on *finding minimum weighted triangles, computing bits of the distance product, all pairs bottleneck paths*.

Talk outline

1. Some definitions
2. Dominance product in subcubic time
3. Maximum weighted triangle
4. Computing bits of the distance product
5. All pairs bottleneck paths
6. Open problems

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Dominance Product:

$$\begin{aligned} C[i,j] = (A \odot B)[i,j] &= |\{k \;:\; A[i,k] \leq B[k,j]\}|. \\ &= \sum_k (A[i,k] \leq B[k,j]). \end{aligned}$$

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Thm. (Matousek) Dominance Product can be computed in $n^{(3+\omega)/2}$ time.

We sketch the elegant algorithm in the next few slides.

It uses fast matrix multiplication.

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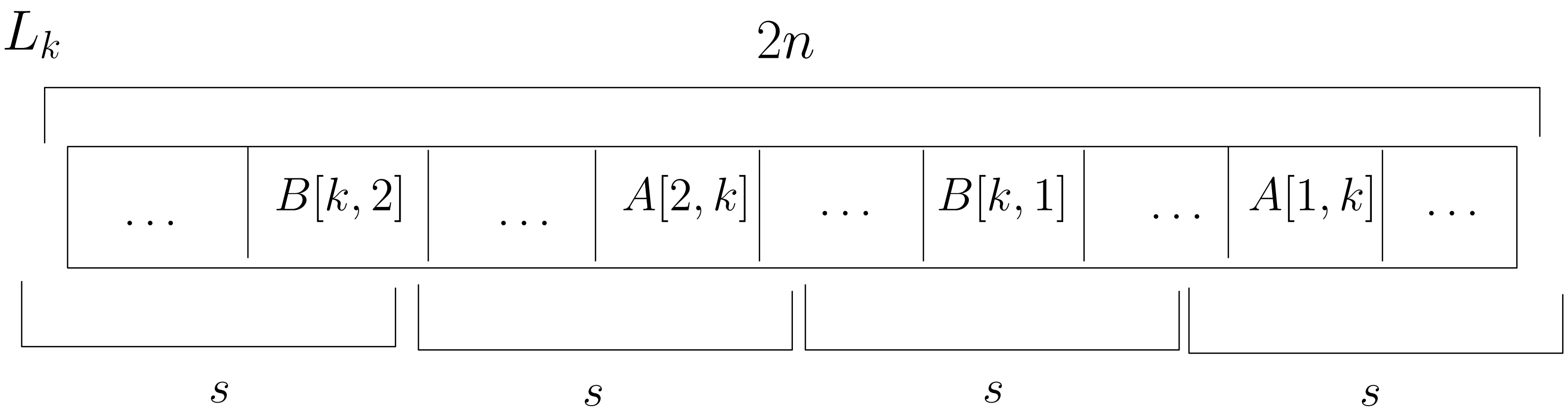
Partition each L_k into "buckets" with s elements in each bucket



Dominance Product in $n^{(3+\omega)/2}$, Cont.

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1. Pairs $(A[i,k], B[k,j])$ such that $A[i,k] \leq B[k,j]$, but $A[i,k]$ and $B[k,j]$ fall in **the same** bucket of L_k
   - Only $O(n^2 s)$ possible pairs of this form
   - Can compute these in $O(1)$ amortized time

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2. Pairs $(A[i,k], B[k,j])$ such that $A[i,k] \leq B[k,j]$, but $A[i,k]$ and $B[k,j]$ fall in **different** buckets of L_k

- Can count these using $2n/s$ matrix multiplications (One matrix multiply for each bucket)

Dominance computation step 2

For every $t = 1, \ldots, 2n/s$, create matrices A_t and B_t such that

$$A_t[i,k] = \begin{cases} 1 & \text{if } A[i,k] \text{ in bucket } t \text{ of } L_k \\ 0 & \text{otherwise} \end{cases} \quad B_t[k,j] = \begin{cases} 1 & \text{if } B[k,j] \text{ in bucket } s > t \text{ of } L_k \\ 0 & \text{otherwise} \end{cases}$$

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$\sum_t A_t B_t$ gives the pairs $A[i,k], B[k,j]$ such that $A[i,k] \leq B[k,j]$ and they are in *different* buckets of L_k.

This can be done in $n/s \cdot n^\omega$ time.

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Overall Runtime: Pick s : $\quad n^2 s = n/s \cdot n^\omega \iff s = n^{\frac{\omega-1}{2}}$.

The final running time is $O(n^{\frac{3+\omega}{2}}) = O(n^{2.69})$.

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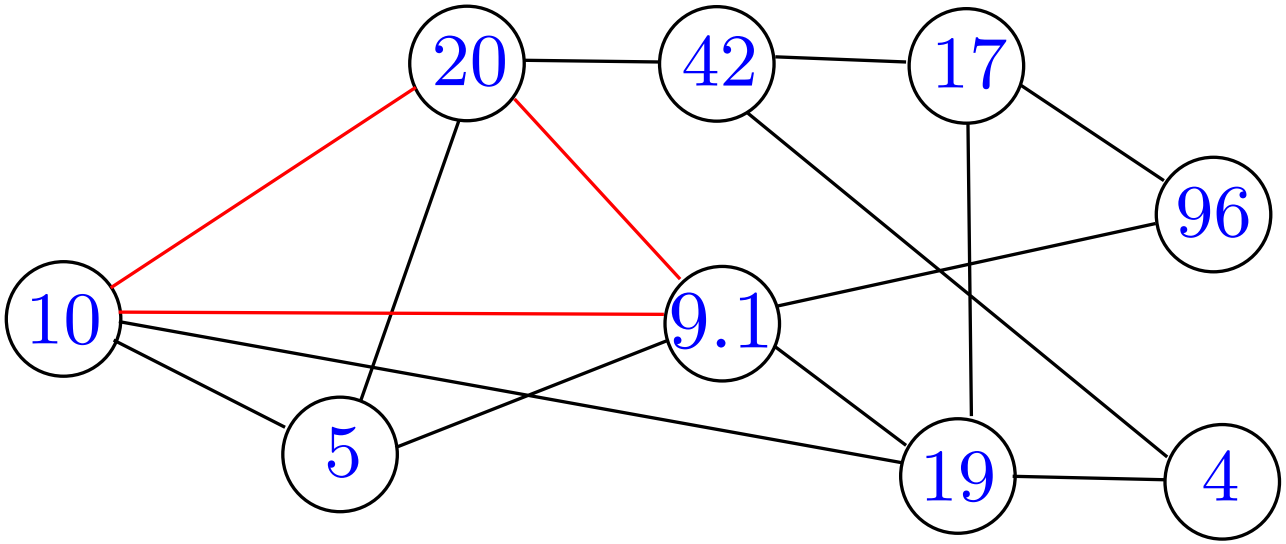

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(Reduce Node-Weighted Triangle to Edge-Weighted Triangle):

Push weights from nodes to edges: $w(u, v) = (w(u) + w(v))/2$

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Observation: Distance Product can solve Max Weighted Triangle

$\longrightarrow$ Compute $\quad MAX_{i,j}\{((-A) \star (-A))[i,j] - A[i,j]\}$

(Min Weight Triangle: $MIN_{i,j}\{(A \star A)[i,j] + A[i,j]\}$)

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Truly Sub-Cubic Algorithm?

Using the dominance product we get:

- Deterministic Algorithm [VW06]

  $O(B \cdot n^{(3+\omega)/2}) \leq O(B \cdot n^{2.688})$, where B is the bit precision

- Randomized (Strongly Polynomial) Algorithm [VW06]

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Aside: It is already known how to find a max node weighted triangle in $O(n^{\omega})$ [CzumajLingas07].

We can get for *all edges* the max node weighted triangle including the edge in $O(n^{2.58})$ time [VWY06].

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- row vector $A[i,;] = (A[i,1], \ldots, A[i,n])$ s.t.

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- column vector $B[;,i] = (B[1,i], \ldots, B[n,i])$ s.t.

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$A[i, j] \leq B[j, k] \iff K \leq w(i) + w(k) + w(j)$ and $(i, j), (j, k) \in E$

Step 1 cont.

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Hence to check whether there is a triangle of weight at least K, compute $C = A \odot B$ and check for an entry $C[i,j] \neq 0$ such that $(i,j) \in E$.

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But this algorithm is not strongly polynomial because of the binary search.

Can use random sampling of weighted triangles to obtain a $O(n^{\frac{3+\omega}{2}} \log n)$ strongly polynomial randomized algorithm.

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- by Han in $O(n^3(\log\log n/\log n)^{5/4})$.

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Compute $D(K) = (A(K) \odot B)$

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Then $C(K)[i,j] = 1 \iff \min_k(A[i,k] + B[k,j]) \geq K$.

Most significant bit is then $C(\frac{W}{2})$ where W is the *smallest power of* 2 *larger than the largest distance*.

Computing bits of the distance product

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$$C(K)[i,j] = 1 \iff \min_k(A[i,k] + B[k,j]) \geq K$$

The second most significant bit of $(A \star B)[i,j]$ is

$$(\neg C(W)[i,j] \wedge C(\frac{3W}{4})[i,j]) \vee (\neg C(\frac{W}{2})[i,j] \wedge C(\frac{W}{4})[i,j]).$$

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The ℓth bit is

$$\bigvee_{s=0}^{2^{\ell-1}-1} [\neg C(W(1 - \frac{s}{2^{\ell-1}}))[i,j] \wedge C(W(1 - \frac{s}{2^{\ell-1}} - \frac{1}{2^{\ell}}))[i,j]].$$

Here need $O(2^{\ell})$ dominance products.

Computing bits of the distance product

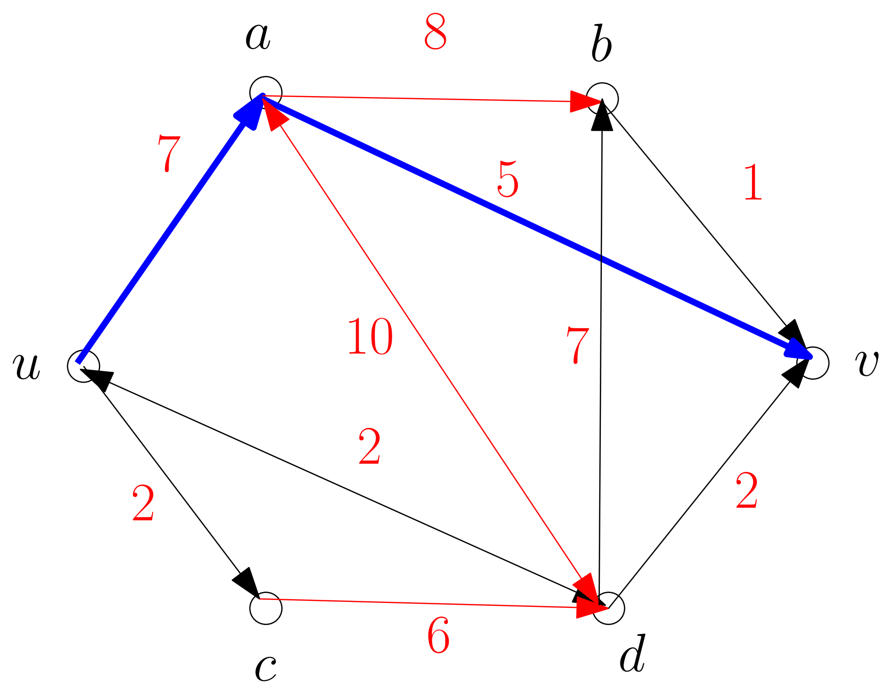
Thm. The first $\mathcal{B}$ most significant bits of the distance product of two $n \times n$ matrices can be computed in $O(2^{\mathcal{B}} n^{\frac{3+\omega}{2}})$ time.

One can compute $(\frac{3-\omega}{2} - \varepsilon) \log n$ bits in $O(n^{3-\varepsilon})$ time.

Bottleneck paths

The bottleneck edge of a path in a graph from vertex u to vertex v is the edge of smallest weight.

In many applications (e.g. max flow), the path of maximum bottleneck is needed.



$ab : 8$	$bu : -\infty$
$ac : 2$	$ub : 7$
$ad : 10$	$cd : 6$
$au : 2$	$dc : 2$
$av : 5$	$ud : 7$
$bc : -\infty$	$du : 2$
$cb : 6$	$uv : 5$

In this talk we will consider the all pairs max bottlenecks problem.

Bottleneck paths – related work

single source:

- Folklore: in $O(m + n \log n)$ by Dijkstra.

all pairs:

- Folklore: undirected edge weighted in $O(n^2)$ using min spanning tree.
- Shapira, Yuster, Zwick 2007: directed node weighted in $O(n^{2.58})$.
- VW: directed edge weighted in $O(n^{2.79})$.

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The MaxMin product is used to compute all pairs maximum bottleneck paths (APBP), similar to how one uses distance product for APSP.

Computing the MaxMin product of two $n \times n$ matrices takes the same time as computing all pairs bottleneck distances in an n vertex graph.

Computing the MaxMin product faster

$C = (A \bullet B)[i,j] = \max_k \min\{A[i,k], B[k,j]\}$

We use the dominance product again:

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We will proceed as follows:

Computing the MaxMin product faster

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3. set for all i, j, $C[i,j] = \max\{a_{ij}, b_{ij}\}$.

Computing the MaxMin product faster

We want $a_{ij} = \max_k \{A[i,k] \mid A[i,k] \leq B[k,j]\}$.

1. Take the rows of A and sort the entries of each row.
2. Bucket the entries of each row of A, in their sorted order into s roughly equal buckets.

$$A = \begin{pmatrix} 10 & -1.1 & 5.1 & 3.2 \\ 2 & 3 & 7 & 1 \\ 0 & -1 & -2 & -3 \\ 7 & 2.1 & 4 & 2.1 \end{pmatrix}$$

row $1:$ $A[1,2]$, $A[1,4]$, $A[1,3]$, $A[1,1]$

row $2:$ $A[2,4]$, $A[2,1]$, $A[2,2]$, $A[2,3]$

row $3:$ $A[3,4]$, $A[3,3]$, $A[3,2]$, $A[3,1]$

row $4:$ $A[4,4]$, $A[4,2]$, $A[4,3]$, $A[4,1]$

Computing the MaxMin product faster

3. For each bucket b create a matrix $A(b)$ containing only the elements in bucket b and ∞ in all other entries.

$$A(1) = \begin{pmatrix} \infty & -1.1 & \infty & 3.2 \\ 2 & \infty & \infty & 1 \\ \infty & \infty & -2 & -3 \\ \infty & 2.1 & \infty & 2.1 \end{pmatrix} \quad A(2) = \begin{pmatrix} 10 & \infty & 5.1 & \infty \\ \infty & 3 & 7 & \infty \\ 0 & -1 & \infty & \infty \\ 7 & \infty & 4 & \infty \end{pmatrix}$$

Computing the MaxMin product faster

4. Compute $A(b) \odot B$ for each bucket b.

$$A(2) \odot A = \begin{pmatrix} 10 & \infty & 5.1 & \infty \\ \infty & 3 & 7 & \infty \\ 0 & -1 & \infty & \infty \\ 7 & \infty & 4 & \infty \end{pmatrix} \odot \begin{pmatrix} 10 & -1.1 & 5.1 & 3.2 \\ 2 & 3 & 7 & 1 \\ 0 & -1 & -2 & -3 \\ 7 & 2.1 & 4 & 2.1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 2 & 1 & 2 & 2 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

This tells us for every bucket b and each i, j, the number of coords k such that $A[i, k]$ is in bucket b and $A[i, k] \leq B[k, j]$.

This step takes $O(sn^{\frac{3+\omega}{2}})$.

Computing the MaxMin product faster

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5. For each i, j we know the largest bucket b in which there is an entry $A[i,k]$ such that $A[i,k] \leq B[k,j]$.

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   For each i, j, search that bucket for k - there are at most $O(n/s)$ entries we have to go through for each pair i, j.

   This step takes $O(n^3/s)$ and explicitly finds witnesses.

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7. You can do slightly better by using sparse dominance $\longrightarrow O(n^{2.79})$.

Open Problems

1. dominance product in n^ω? (VW Conjecture)

2. truly subcubic distance product using dominance product?

3. generalize the technique for some class of problems?

Thank You!