All Pairs Bottleneck Paths in Truly Subcubic Time

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STOC

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joint work with Ryan Williams and Raphael Yuster

Introduction

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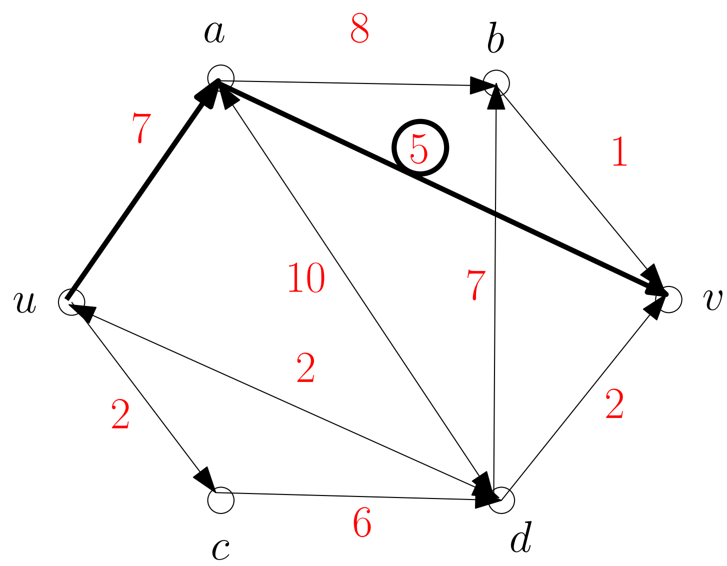
This talk: truly subcubic algorithm for APBP – studied alongside APSP.

Bottleneck paths - definitions

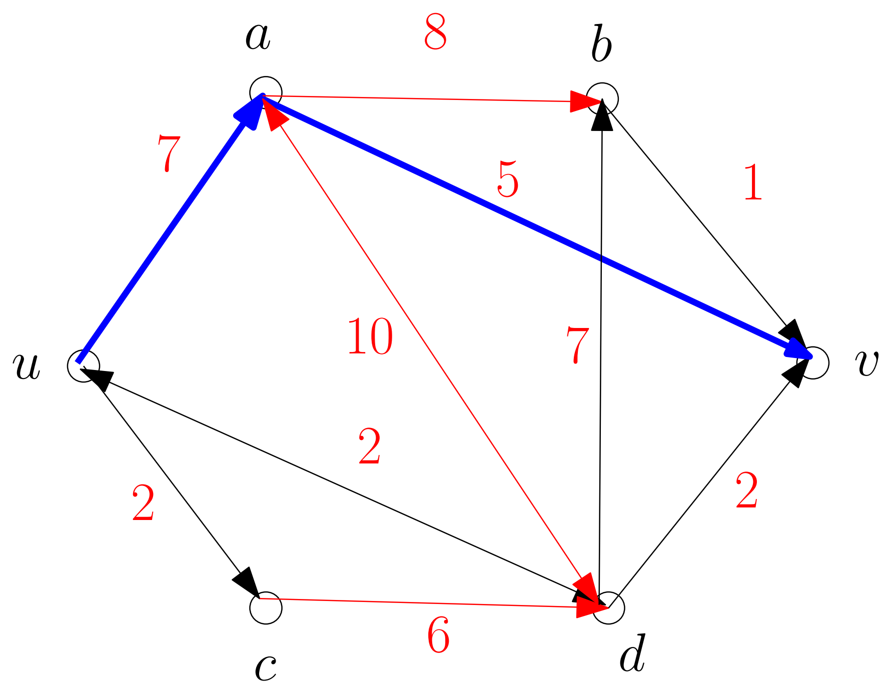
Given: graph $G = (V, E)$ with arbitrary edge weights $w : E \longrightarrow \mathbb{R}$.

The bottleneck edge of a path in G from vertex u to vertex v is the edge of smallest weight on the path.



Maximum bottleneck paths

In many applications (e.g. max flow), the path of maximum bottleneck is needed.



$ab : 8$	$bu : -\infty$
$ac : 2$	$ub : 7$
$ad : 10$	$cd : 6$
$au : 2$	$dc : 2$
$av : 5$	$ud : 7$
$bc : -\infty$	$du : 2$
$cb : 6$	$uv : 5$

In this talk we will consider the all pairs max bottlenecks problem: for all pairs of vertices s and t in the graph, find the weight of the maximum bottleneck edge on a path from s to t.

All pairs bottleneck paths – related work

- Pollack 1960: introduced APBP and showed a cubic algorithm.
- Hu 1961: undirected, edge weighted – max spanning tree. $\leftarrow O(n^2)$
- Shapira, Yuster, Zwick 2007: directed, node weighted in $O(n^{2.58})$.
- this work: directed, edge weighted in $O(n^{2.79})$.

MaxMin product

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Adjacency matrix for weighted graph $G = (V, E, w)$:

$A[i,j] = w_{ij}$, $w_{ii} = \infty$, $w(i,j) = -\infty$ if $(i,j) \notin E$.

$(A \bullet A)[i,j]$ is the maximum bottleneck edge weight over all paths of length ≤ 2 from i to j.

$\underbrace{A \bullet A \bullet \ldots \bullet A}_{n \text{ times}}$: the maximum bottleneck weights for all vertex pairs.

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Moreover: computing the MaxMin product of two $n \times n$ matrices takes the same time (asymptotically) as computing all pairs bottleneck weights in an n vertex graph. [AhoHopcroftUllman74]

This work: first truly subcubic algorithm for the MaxMin product.

MaxMin product in subcubic time

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3. set for all i, j, $C[i,j] = \max\{a_{ij}, b_{ij}\}$.

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Thm. (Matousek) Dominance Product can be computed in $O(n^{(3+\omega)/2})$ time, where ω is the exponent of fast matrix multiplication. $\leftarrow O(n^{2.69})$

MaxMin product in subcubic time

We want $a_{ij} = \max_k\{A[i,k] \mid A[i,k] \leq B[k,j]\}$.

1. Take the rows of A and sort the entries of each row.
2. Bucket the entries of each row of A, in their sorted order into s roughly equal buckets.

$$A = \begin{pmatrix} 10 & -1.1 & 5.1 & 3.2 \\ 2 & 3 & 7 & 1 \\ 0 & -1 & -2 & -3 \\ 7 & 2.1 & 4 & 2.1 \end{pmatrix}$$

$$\begin{array}{lllll} \text{row } 1: & A[1,2], & A[1,4], & A[1,3], & A[1,1] \\ \text{row } 2: & A[2,4], & A[2,1], & A[2,2], & A[2,3] \\ \text{row } 3: & A[3,4], & A[3,3], & A[3,2], & A[3,1] \\ \text{row } 4: & A[4,4], & A[4,2], & A[4,3], & A[4,1] \end{array}$$

MaxMin product in subcubic time

3. For each bucket b create a matrix $A(b)$ containing only the elements in bucket b and ∞ in all other entries.

$$A(1) = \begin{pmatrix} \infty & -1.1 & \infty & 3.2 \\ 2 & \infty & \infty & 1 \\ \infty & \infty & -2 & -3 \\ \infty & 2.1 & \infty & 2.1 \end{pmatrix} \quad A(2) = \begin{pmatrix} 10 & \infty & 5.1 & \infty \\ \infty & 3 & 7 & \infty \\ 0 & -1 & \infty & \infty \\ 7 & \infty & 4 & \infty \end{pmatrix}$$

MaxMin product in subcubic time

Recall, $(A \otimes B)[i,j] = |\{k : A[i,k] \leq B[k,j]\}|$.

4. Compute $A(b) \otimes B$ for each bucket b.

$$A(2) \otimes A = \begin{pmatrix} 10 & \infty & 5.1 & \infty \\ \infty & 3 & 7 & \infty \\ 0 & -1 & \infty & \infty \\ 7 & \infty & 4 & \infty \end{pmatrix} \otimes \begin{pmatrix} 10 & -1.1 & 5.1 & 3.2 \\ 2 & 3 & 7 & 1 \\ 0 & -1 & -2 & -3 \\ 7 & 2.1 & 4 & 2.1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 2 & 1 & 2 & 2 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

This tells us for every bucket b and each i, j, the number of coords k such that $A[i,k]$ is in bucket b and $A[i,k] \leq B[k,j]$.

This step takes $O(sn^{\frac{3+\omega}{2}})$.

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   For each i, j, search that bucket for k - there are at most $O(n/s)$ entries we have to go through for each pair i, j.

   This step takes $O(n^3/s)$ and explicitly finds witnesses.

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6. The overall runtime is maximized for $s = n^{\frac{3-\omega}{4}}$ and the runtime is then $O(n^{\frac{9+\omega}{4}}) = O(n^{2.85})$.

7. You can do slightly better by using sparse dominance $\longrightarrow O(n^{2.79})$.

Sparse dominance

Theorem: Let A and B be $n \times n$ matrices with entries from a totally ordered set. Let $S \subseteq [n] \times [n]$ such that $|S| = m \geq n$. Let C be the matrix such that

$$C[i,j] = |\{k \mid (i,k) \in S \text{ and } A[i,k] \leq B[k,j]\}|.$$

There is an algorithm that, given A, B, and S, outputs C in $O(\sqrt{m} \cdot n^{\frac{1+\omega}{2}})$ time.

Intuition: The set S of coordinate pairs contains all entries of A we care about. Comparisons between entries of A not in S and entries of B are ignored.

MaxMin Product

Recall the matrices $A(b)$:

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$A(b)$ has $O(n^2/s)$ finite entries. Each of the s dominance products thus takes $O(n^{\frac{3+\omega}{2}}/\sqrt{s})$, and the running time for the entire algorithm is: $O(n^3/s + \sqrt{s}n^{\frac{3+\omega}{2}})$, minimized for $s = n^{1-\omega/3}$.

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Open Problems

1. dominance product, MaxMin product in $O(n^{\omega})$?
2. truly subcubic distance product using dominance product?

Thank You!