Finding a Maximum Weight Triangle in $O(n^{3-\delta})$ Time, With Applications

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Task: Find a triangle of maximum weight sum

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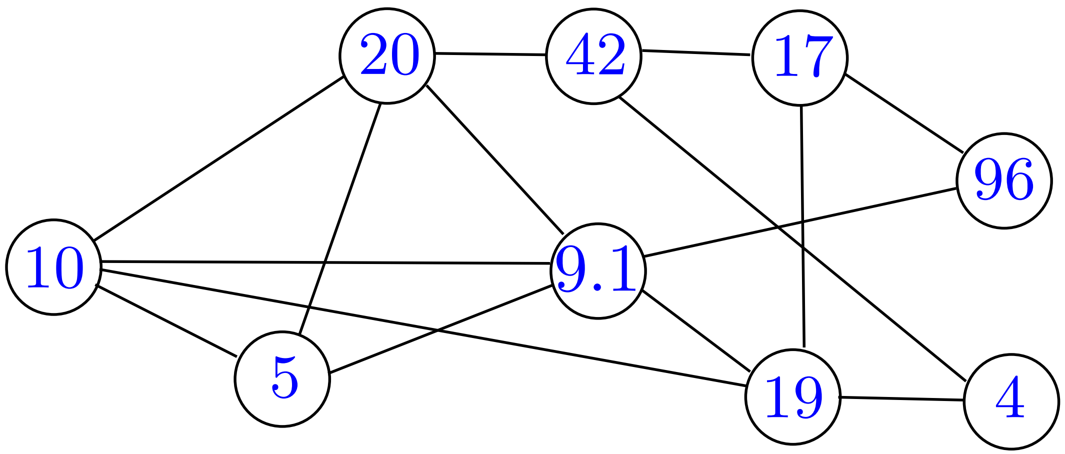

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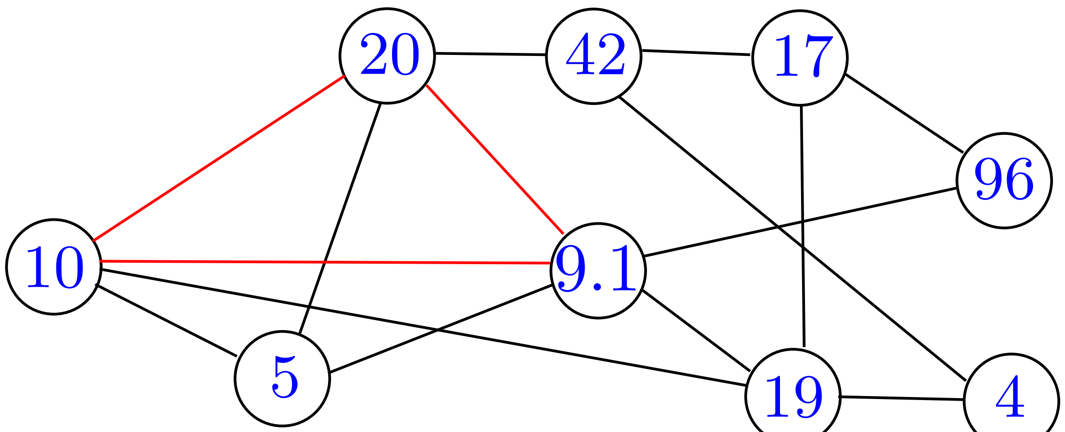

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Check if $(A \wedge (A \times A)) \neq 0$

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Their paper ends with:

"A related problem is finding a minimum weighted circuit in a weighted graph. It is unclear to us whether our methods can be modified to answer this problem too."

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Observation: Distance Product can solve Max Weight Triangle

1. Push weights from nodes to edges: $w(u,v) = (w(u) + w(v))/2$

   (Reduce Node-Weighted Triangle to Edge-Weighted Triangle)

2. Compute $MAX_{i,j}\{((-A) \star (-A))[i,j] - A[i,j]\}$

   (Min Weight Triangle: $MIN_{i,j}\{(A \star A)[i,j] + A[i,j]\}$)

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Truly Sub-Cubic Algorithm?

Talk Outline

- Deterministic Algorithm

$O(B \cdot n^{(3+\omega)/2}) \leq O(B \cdot n^{2.688})$, where B is the bit precision

- Randomized (Strongly Polynomial) Algorithm

$$O(n^{(3+\omega)/2} \log n) \leq O(n^{2.688})$$

- Some Applications

Deterministic Algorithm

Key Steps:

- Suffices to check if there's a triangle of weight $\geq K$
- Compute a matrix A' s.t.

$$A'[i,j] = |\{k : i \to k \to j, w(i) + w(k) + w(j) \geq K\}|.$$

- Check if $\exists i, j$ where $A[i,j]$ and $A'[i,j]$ are non-zero.

Computing C

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(Sketched in next few slides)

Thm. If Dominance Product is in $O(f(n))$ time, then can check if there's a triangle of weight $\ge K$, in $O(f(n) + n^2)$ time.
(Virginia will prove this)

Dominance Product in $n^{(3+\omega)/2}$

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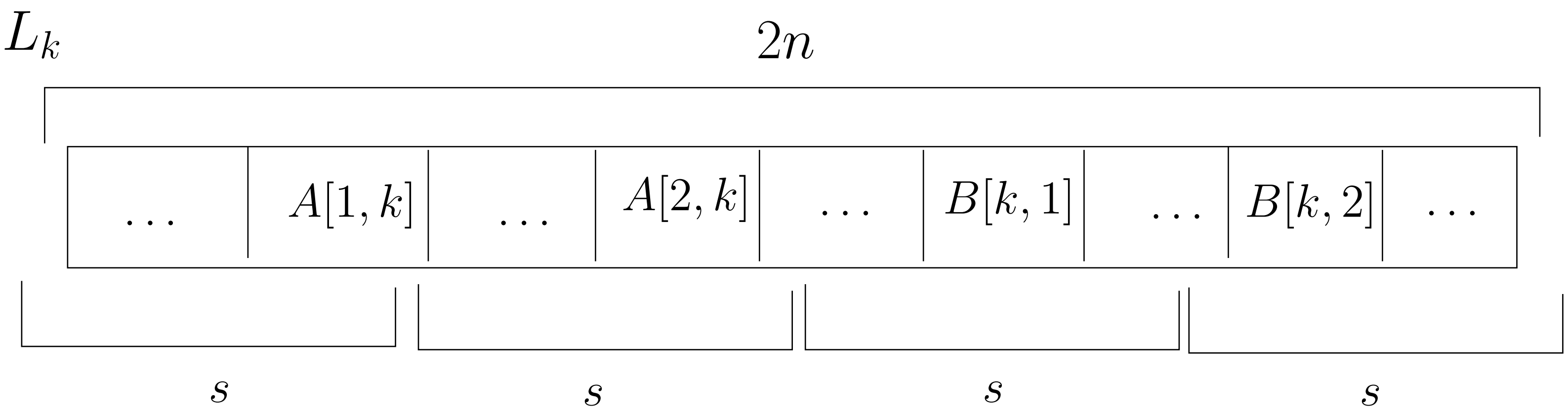
Partition each L_i into “buckets” with s elements in each bucket (roughly $2n/s$ buckets in total)



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Idea 2: Two types of data are counted in C:

1. Pairs $(A[i,k], B[k,j])$ such that $A[i,k] \leq B[k,j]$, but $A[i,k]$ and $B[k,j]$ fall in **the same** bucket of L_k
   - Only $O(n^2 s)$ possible pairs of this form
   - Can compute these in $O(1)$ amortized time

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2. Pairs $(A[i,k], B[k,j])$ such that $A[i,k] \leq B[k,j]$, but $A[i,k]$ and $B[k,j]$ fall in **different** buckets of L_k

   - Can count these using $2n/s$ matrix multiplications
     (One matrix multiply for each bucket)

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2. Do binary search on K to find the maximum weight W of a triangle.
3. Find a triangle of weight W.

Step 1: Given K, reduce to dominance product instance.

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Vertex $i \in V \rightarrow$

- row vector $A[i,;] = (A[i,1], \ldots, A[i,n])$ s.t.

$$A[i,j] = \begin{cases} K - w(i) & \text{if there is an edge from } i \text{ to } j \\ \infty & \text{otherwise.} \end{cases}$$

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$A[i,j] \leq B[j,k] \iff K \leq w(i) + w(k) + w(j)$ and $(i,j), (j,k) \in E$

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But this algorithm is NOT strongly polynomial because of the binary search.

A Strongly Polynomial Randomized Algorithm: Outline

1. show how to sample a triangle of weight in any interval $[W_1, W_2]$ efficiently uniformly at random
2. search using the weights of triangles chosen at random

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- the number E^1_{ij} of k such that

  $(\ldots, w(i) + w(k), \ldots)$ dominated in coord. k by $(\ldots, W_2 - w(j), \ldots)$

  $i \rightarrow k \rightarrow j$ and $w(i) + w(j) + w(k) \leq W_2$

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- the number $E_{ij} = (E^1_{ij} - E^2_{ij})$ of k such that

$$i \to k \to j \text{ and } W_1 \leq w(i) + w(j) + w(k) \leq W_2$$

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Recall: E_{ij} is the number of k such that $i \longrightarrow k \longrightarrow j$ and $W_1 \leq w(i) + w(j) + w(k) \leq W_2$

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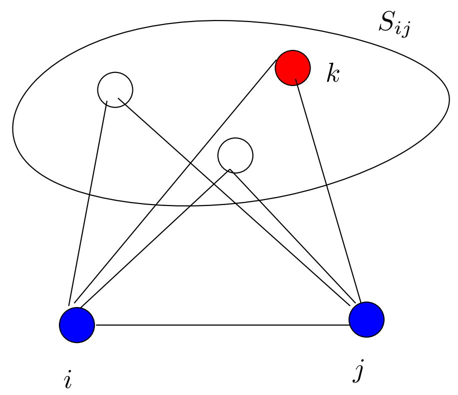

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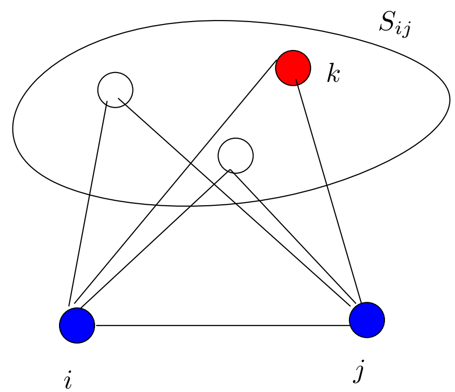

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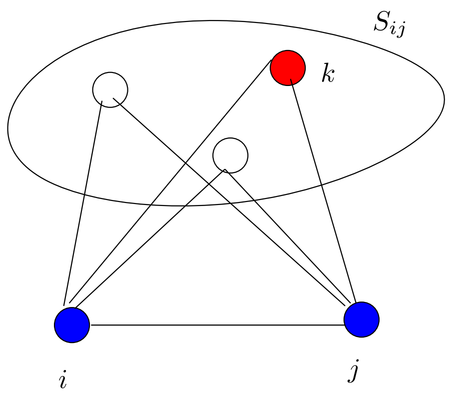

$\{i, j, k\}$ is a random triangle with weight in $[W_1, W_2]$.

Strongly Polynomial Algorithm

1. Let $M = 3 \cdot \max_{i \in V} w(i)$ and $K = 0$.
2. Pick a random triangle T in $[K, M]$. Set K to its weight.
3. Check if there exists a triangle of weight $> K$. If not, return T.
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The algorithm will terminate in $O(n^{\frac{3+\omega}{2}} \log n)$ expected worst case time.

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- generalized K-SUM
- computing K most significant bits of distance product in $O(2^K \cdot n^{\frac{3+\omega}{2}} \log W \log n)$ ...

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- Can we use our approach to *edge-weighted* triangle?

  Important stepping stone towards truly sub-cubic APSP?
- **Conjecture**:

  Dominance product can be computed in $O(n^{\omega+o(1)})$ time.

Thank You!