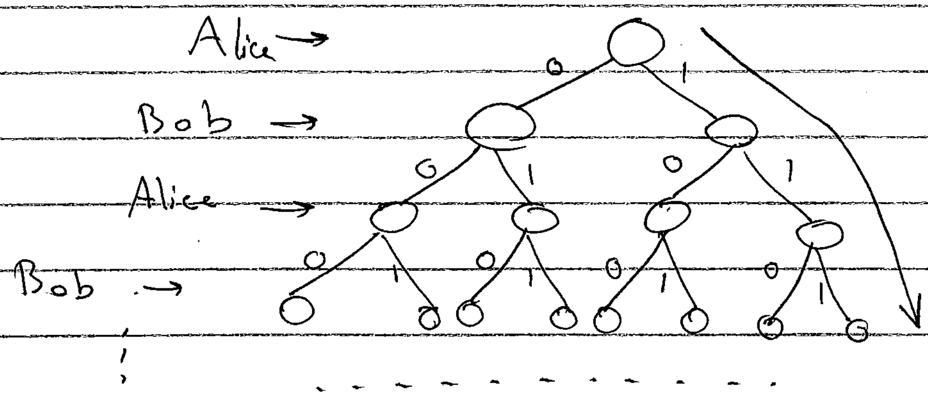
* Achieving Compression $\sim \sqrt{CC \cdot IC} \cdot \text{polylog}$.

[Inevitably we need to eliminate lots of rounds in communication]

First, assume we wish to simulate the behavior of π with <u>no</u> communication. What's the best we can achieve?



wlog, Alice & Bob alternate, each round transmits one bit and the tree is full.

* Node v at depth i is uniquely represented by a bit-string of length i corresponding to the transcript leading to v.

* Suppose Alice owns a node v.

Then, if π reaches v, Alice chooses the right (1) child w.p. p_v^A.

Bob doesn't know p_v^A, but he can produce a "best guess":

$$p_v^B = \text{Prob}\left(\text{Alice chooses } 1 \;\middle|\; \underbrace{Y = y,\ \pi \text{ reaches node } v}_{\text{(i.e., what Bob knows)}} \right)$$

In general, p_v^A and p_v^B may be very different, but if this is so, this means that ~~Alice~~ Bob learns substantially from Alice's message, and the bit sent by Alice isn't "redundant".

Without any communication, Alice and Bob can compute p_v^A and p_v^B for all nodes, and accordingly, sample a root-leaf path.

Problems: 1) • p_v^A and p_v^B different, means ~~the~~ distribution-wise the paths would be different.

2) even if prob. estimates are close, Alice and Bob must coordinate to sample "similar" paths.

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For "problem 2", Alice and Bob can use public randomness for ~~correlat~~ "correlated sampling".

<u>Correlated Sampling:</u> Alice and Bob are given parameters $\underset{\text{Alice}}{p}, \underset{\text{Bob}}{q} \in [0,1]$, and want to sample bits $\overset{\text{Alice}}{a}$ & $\overset{\text{Bob}}{b}$ s.t.

No communication!

$\Pr(a=1) = p, \quad \Pr(b=1) = q,$

$|p-q| = \text{small} \Rightarrow \Pr(a \neq b) = \text{small}.$

<u>Solution:</u> 1) Using public randomness, sample $\rho \in [0,1]$ u.a.r.

2) Alice outputs $a=1$ iff $\rho \leq p$

3) Bob " $b=1$ " $\rho \leq q$.

Observe: $\Pr(a \neq b) = |p-q|$.

$\square$

Alice and Bob use correlated sampling at each node (using their estimates) to create a root-leaf path.

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NB: * At each node, ~~one~~ either Alice or Bob samples "correctly".

* If all prob. estimates are the same, Alice & Bob both sample the correct path with no communication (i.e., perfectly simulate Π).

~~Goal: If Alice and Bob~~

After sampling root-leaf paths, Alice and Bob find the earliest point where the paths differ.

* Problem: Find the first disagreement.

Alice and Bob are given k-bit strings x and y.

Goal: Output the smallest i where $x_i \neq y_i$, or 0 if $x = y$. (error $\leq \varepsilon$)

* Binary search and hashing-based equality tests using public coin $\Rightarrow$ $CC = O\left(\log k \log\left(\frac{\log k}{\varepsilon}\right)\right)$.
* More careful idea leads to $CC = O\left(\log \frac{k}{\varepsilon}\right)$.

$\Rightarrow$ Each inconsistency can be found using $O\left(\frac{\log CC(\Pi)}{\varepsilon}\right)$ bits of communication.

* After fixing the first inconsistency, Alice and Bob resample the rest of the path using the updated information and repeat the disagreement test.

* The result follows if we show that, at any node v,

$$\Pr\left(\text{~~disagreement~~ Alice and Bob guess differently} \right) \le \sqrt{\frac{IC(\Pi)}{|\Pi|}}.$$

(idea: Information theoretic inequalities + Cauchy Schwarz)

<u>Def.</u> $\mathcal{E}_i = 1$ iff Alice and Bob disagree at level i.

NB: * $\mathcal{E}_i$ happens at node v exactly when

$$\min\{p_v^A, p_v^B\} \le \rho \le \max\{p_v^A, p_v^B\}.$$

(Fix public randomness for now)

$$\Rightarrow \mathbb{E}(\mathcal{E}_i) = \mathbb{E}_{XYV} \left| p^A_{v_{i-1}} - p^B_{v_{i-1}} \right|$$

(V = correct path of Π
$V = v_0 v_1 \dots v_{CC(\Pi)}$)

$$= \mathbb{E}_{XYV} \Delta_{TV}\left(V_i | X V_{<i}, \; V_i | Y V_{<i} \right)$$

$$\le \mathbb{E}_{XYV} \Delta_{TV}\left(V_i | X V_{<i}, \; V_i | XY V_{<i} \right) + \Delta_{TV}\left(V_i | Y V_{<i}, \; V_i | XY V_{<i} \right)$$

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(Note: $\Delta_{TV}(p,q) \le \sqrt{\mathcal{D}(p\|q)}$)

$$\le \mathop{\mathbb{E}}_{XYV} \sqrt{\cancel{\Delta_{TV}}\mathcal{D}\left(V_i \mid XYV_{<i} \,\|\, V_i \mid XV_{<i}\right) + \cancel{\Delta_{TV}}\mathcal{D}\left(V_i \mid XYV_{<i} \,\|\, V_i \mid \cancel{X}Y V_{<i}\right)}$$

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$$\le \sqrt{\mathop{\mathbb{E}}_{XYV} \cancel{\Delta_{TV}}\mathcal{D}(\sim\|\sim) + \cancel{\Delta_{TV}}\mathcal{D}(\sim\|\sim)}$$

$$\le \sqrt{I(V_i;Y;XV_{<i}) + I(V_i;X;YV_{<i})}$$

$$\Rightarrow \mathbb{E}(\#\text{ disagreements}) \le \sum_i \sqrt{I(\sim) + I(\sim)}$$

Cauchy-Schwarz

$$\le \sqrt{|\Pi| \sum_i \left(I(\sim) + I(\sim)\right)}$$

Chain-Rule

$$= \sqrt{|\Pi| \left(I(V;Y|X) + I(V;X|Y)\right)}$$

$$= \sqrt{|\Pi| \cdot IC(\Pi)}$$

[need to take $\mathbb{E}$ over public coins, to be precise]

$\square$