1

Thu 4/11

Today: $R(\mathrm{DISJ})$.

$$\mathrm{DISJ}(x,y) = \mathrm{NAND}(x_1,y_1) \wedge \mathrm{NAND}(x_2,y_2) \wedge \cdots \wedge \mathrm{NAND}(x_n,y_n)$$

Let μ: $x_1, \dots, x_n$ $y_1, \dots, y_n$ iid $\sim$ Bernoulli$\left(\frac{1}{\sqrt{n}}\right)$.

Goal: $D^{\mu}_{1/100}(\mathrm{DISJ}) \geq$ large.

NB. We need μ to be "balanced" $\left(\mu(\mathrm{DISJ}^{-1}(0/1)) = \Omega(1)\right)$

(Our choice satisfies this) $\longrightarrow$ $\mathrm{Prob}_{\mu}(\mathrm{DISJ}(x,y)=1) \approx 1/e$.

Theorem: Under μ, $D^{\mu}_{1/100}(\mathrm{DISJ}) = \Omega(\sqrt{n})$
(Babai–Frankl–Simon '86)

(in fact $\Theta(\sqrt{n})$)

Coro: $R(\mathrm{DISJ}) \geq \Omega(\sqrt{n})$.

Proof: Let Π_0 be a deterministic protocol s.t.

$$\Pr_{(x,y)\sim\mu}\left(\mathrm{DISJ}(x,y) = \Pi_0(x,y)\right) \geq 0.99$$

Let the r.v. $\Pi(X,Y)$ be the transcript of Π_0 on $(X,Y)\sim\mu$.

②

We Know: $CC(\Pi_0) \geq \log_2 \left| \text{supp}(\Pi(X,Y)) \right|$

$$\geq H(\Pi(X,Y)) \geq I(X,Y;\Pi(X,Y))$$

(in fact $=$)

$$= I(X_1 - X_n\, Y_1 - Y_n ; \Pi(X,Y))$$

$$\overset{\text{independence}}{\geq} \sum_{i=1}^{n} I(X_i, Y_i ; \Pi(X,Y))$$

<u>Def:</u> $\Pi^{i}_{a,b} = \Pi(X,Y)$ conditioned on $\left| \begin{matrix} X_i = a \\ Y_i = b \end{matrix} \right.$

<u>P.S. 1, Problem 6</u> $\Longrightarrow I(X_i, Y_i ; \Pi(X,Y))$

$$\geq \mathop{\mathbb{E}}_{\substack{(a,b) \\ \sim (Ber(\frac{1}{\sqrt{n}}))^2}} \left[\Delta^2_{TV} \left(\Pi^{i}_{a,b}, \Pi(X,Y) \right) \right].$$

Δ_{TV}: total var. distance

(follows from Pinsker's ineq.)

<u>Recall:</u> $\Delta_{TV}(A,B) \triangleq \frac{1}{2} \sum_{l} \left| Pr(A=l) - Pr(B=l) \right|$

$$\Rightarrow I(X_i, Y_i ; \Pi(X,Y)) \geq \overbrace{\frac{1}{\sqrt{n}}\left(1 - \frac{1}{\sqrt{n}}\right)}^{\geq 1/2\sqrt{n}} \left[\Delta^2_{TV}\left(\Pi^{i}_{10}, \Pi(X,Y)\right) + \Delta^2_{TV}\left(\Pi^{i}_{01}, \Pi(X,Y)\right) \right]$$

$$\geq \frac{1}{4\sqrt{n}} \left(\Delta_{TV}(\Pi^{i}_{10}, \Pi) + \Delta_{TV}(\Pi^{i}_{01}, \Pi) \right)^2$$

$$\overset{\text{triangle ineq}}{\geq} \frac{1}{4\sqrt{n}} \cdot \Delta^2_{TV}(\Pi^{i}_{10}, \Pi^{i}_{01})$$

③

$$\Rightarrow \frac{CC(\Pi_0)}{n} \geq \mathbb{E}_i\left(I(X_i, Y_i ; \Pi(X,Y))\right)$$

$$\geq \frac{1}{4\sqrt{n}} \underbrace{\mathbb{E}_i\, \Delta^2_{TV}\left(\Pi^i_{10}, \Pi^i_{01}\right)}_{(\text{Goal: } \geq \text{Const})}.$$

$$\geq \frac{1}{4\sqrt{n}}\left(\mathbb{E}_i\, \Delta_{TV}(\Pi^i_{10}, \Pi^i_{01})\right)^2.$$

Two remaining parts:

① Argue, using the fact that Π is getting DISJ right w.p. $99/100$, that $\mathbb{E}_i\left(\Delta_{TV}(\Pi^i_{00}, \Pi^i_{11})\right) = \Omega(1)$.

② Prove that if $\Delta_{TV}\left(\Pi^i_{00}, \Pi^i_{11}\right) \geq \Omega(1)$, then $\Delta_{TV}\left(\Pi^i_{01}, \Pi^i_{10}\right) \geq \Omega(1)$.

(using rectangle property)

Clearly, we are done assuming ① and ②.

4

For 1:

Note: $X_i = Y_i = 0 \Rightarrow DISJ(X, Y \mid X_i = Y_i = 0) = 1$ w.p. $\geq \frac{1}{4}$.

$\Rightarrow \Pr(\Pi_0 \text{ accepts given transcript } \Pi^i_{00}) \geq \frac{1}{5}$

$\left(\frac{1}{4} - \text{error prob.}\right)$

also: $X_i = Y_i = 1 \Rightarrow DISJ(X, Y) = 0$.

since i is also random (averaged),

we cover $\Omega(1)$ fraction of the prob. space

$\Rightarrow \mathbb{E}_i \Pr(\Pi_0 \text{ accepts given transcript } \Pi^i_{11}) \leq \frac{1}{6}$.

$\Rightarrow$ Protocol is a good distinguisher (on average) for Π^i_{00} and Π^i_{11}.

Now we prove 2:

Rectangle Property: Fix i. We show, $\exists A_0\ A_1\ B_0\ B_1$ non-negative functions s.t. (Alice: A; Bob: B)

s.t. $\Pr(\Pi^i_{ab} = \tau) = A_a(\tau) \cdot B_b(\tau)$.

Note: Inputs X^{-i}, Y^{-i} that lead to transcript τ form a rectangle $R_\tau = S_\tau \times T_\tau$.

5

$$\Rightarrow \Pr(\Pi_{ab}^{i} = \tau) = \Pr\left(X^{-i} \in S_\tau \wedge Y^{-i} \in T_\tau\right)$$

$$\overset{X \text{ indep } Y}{=} \underbrace{\Pr\left(X^{-i} \in S_\tau\right)}_{= A_a(\tau)} \underbrace{\Pr\left(Y^{-i} \in T_\tau\right)}_{B_b(\tau)}.$$

Using this "product property" and some calculations, one can show 2.

$$\text{2}: \quad \sum_\tau \left| A_0(\tau) B_0(\tau) - A_1(\tau) B_1(\tau) \right| \geq \Omega(1) \quad (P)$$

$$\Downarrow$$

$$\sum_\tau \left| A_0(\tau) B_1(\tau) - A_1(\tau) B_0(\tau) \right| \geq \Omega(1). \quad \left(\tfrac{P^2}{2}\right)$$

NB: A_0, A_1, B_0, B_1 are prob. distributions.

Def. $\Delta^2_{Hel} \triangleq 1 - \sum_\ell \sqrt{\Pr(A=\ell)\Pr(B=\ell)}$

(Hellinger distance)

In fact, this "dot product" is the Bhattacharya coefficient.

Exercise: (Using Cauchy-Schwarz)

$$\Delta^2_{Hel}(A,B) \leq \Delta_{TV}(A,B) \leq \sqrt{2}\, \Delta_{Hel}(A,B).$$

<u>*Lemma:*</u>

$$\Delta^2_{Hel}\left(\Pi^i_{00}, \Pi^i_{11}\right) = \Delta^2_{Hel}\left(\Pi^i_{01}, \Pi^i_{10}\right).$$

Clearly, having the Lemma we are done with 2.

<u>Proof of Lemma:</u> Suffices to show the dot product of distributions remain the same.

$$\Pr\left(\Pi^i_{00} = \tau\right) \Pr\left(\Pi^i_{11} = \tau\right)$$

$$= A_0(\tau)\, B_0(\tau)\, A_1(\tau)\, B_1(\tau)$$

$$= \underbrace{A_0(\tau)\, B_1(\tau)}\ \underbrace{A_1(\tau)\, B_0(\tau)}$$

$$= \Pr\left(\Pi^i_{01} = \tau\right) \Pr\left(\Pi^i_{10} = \tau\right).$$

$\square$

---

* B.F.S. showed that every μ that is a product distribution $\left(\mu(x,y) = \mu_{Alice}(x)\cdot\mu_{Bob}(y)\right)$,

$$D^\mu(\mathrm{DISJ}) = O\left(\sqrt{n}\,\log n\right).$$

So getting a better lower bound requires correlation.

7

Next time: $R(\mathrm{DISJ}) = \Omega(n)$.

* This was first proved by Kalyanasundaram–Schnitger '87,
* Razborov '90 "simplified" this.
* The proof we'll see is by Bar-Yossef–Jayram–Kumar–Sivakumar 2004 (Information-theory based).