Today: $R(DISJ)$.

$DISJ(x,y) = NAND(x_1,y_1) \land NAND(x_2,y_2) \land \cdots \land NAND(x_n,y_n)$

Let $M: x_1, \ldots, x_n, y_1, \ldots, y_n \text{ iid } \sim \text{Bernoulli}(\frac{1}{Tn})$.

Goal: $D_M^\mu(DISJ) \geq \frac{1}{100}$ large.

NB: We need $M$ to be "balanced" $(M(DISJ^{-1}(\frac{1}{2})) \approx \frac{1}{2})$.

(Our choice satisfies this) $\Rightarrow \text{Prob}_M(DISJ(x,y) = 1) \approx \frac{1}{2}$.

Theorem: Under $M$, $D_M^\mu(DISJ) = \Omega(\sqrt{n})$.

(Babai-Frankl-Simon '86) $\Omega(\frac{1}{100})$ (in fact $\Theta(\sqrt{n})$).

Corollary: $R(DISJ) \geq \Omega(\sqrt{n})$.

Proof: Let $\Pi_0$ be a deterministic protocol s.t.

$$P_x(\text{DISJ}(x,y) = \Pi_0(x,y)) \geq 0.99$$

$(x,y) \sim M$ the r.v.

Let $\Pi(x,y)$ be the transcript of $\Pi_0$ on $(x,y) \sim M$. 

We know: $\text{CC}(\Pi_0) \geq \log_2 \left(1 - \sup_{(X,Y)} \left(\frac{\text{supp}(\Pi(X,Y))}{\Pi(X,Y)}\right)\right)$

\[
\geq H(\Pi(X,Y)) - I(X,Y; \Pi(X,Y))
\]

(in fact =)

\[
= I(X_1, \ldots, X_n; Y_1, \ldots, Y_n; \Pi(X,Y))
\]

independence

\[
\geq \sum_{i=1}^{n} I(X_i; Y_i; \Pi(X,Y))
\]

Def. $\Pi^i_{a,b} = \Pi(X,Y)$ conditioned on $X_i = a, Y_i = b$

P.S. 1, Problem 6 $\Rightarrow I(X_i; Y_i; \Pi(X,Y))$

\[
\geq \mathbb{E}_{(a,b) \sim \text{Beta}(k_1)} \left[ \frac{\Delta^2_{TV}(\Pi^i_{a,b}, \Pi(X,Y))}{\Delta^2_{TV}(\Pi^i_{0,0}, \Pi(X,Y))} \right]
\]

(follows from Pinsker's inequality)

Recall: $\Delta^2_{TV}(A, B) = \frac{1}{2} \sum_{l=1}^{L} |Pr(A=l) - Pr(B=l)|$

\[
I(X_i; Y_i; \Pi(X,Y)) \geq \frac{1}{\sqrt{n}} \left(1 - \frac{1}{\sqrt{n}}\right) \left[ \Delta^2_{TV}(\Pi^i_{0,0}, \Pi(X,Y)) + \Delta^2_{TV}(\Pi^i_{1,1}, \Pi(X,Y)) \right]
\]

\[
\geq \frac{1}{4 \sqrt{n}} \left(\Delta^2_{TV}(\Pi_{0,0}^{1}, \Pi) + \Delta^2_{TV}(\Pi_{1,1}^{1}, \Pi)^2\right)
\]

triangle ineq

\[
\leq \frac{1}{4 \sqrt{n}} \cdot \Delta^2_{TV}(\Pi_{0,0}^{1}, \Pi)
\]
\[\Rightarrow \frac{CC(\Pi_0)}{n} \geq \frac{1}{n} \sum_i \text{TV}^2(\Pi_{10}^i, \Pi_{01}^i)\]

\[\geq \frac{1}{4\sqrt{n}} \sum_i \text{TV}^2(\Pi_{10}^i, \Pi_{01}^i)\]

\[\text{(Goal:} \geq \text{Const)}\]

\[\geq \frac{1}{4\sqrt{n}} \left(\text{TV}(\Pi_{00}^i, \Pi_{11}^i)\right)^2.\]

Two remaining parts:

1. Argue, using the fact that \(\Pi\) is getting \(\text{DISJ}\) right w.p. \(\frac{99}{100}\), that

\[\text{TV}(\Pi_{00}^i, \Pi_{11}^i) = \Omega(1).\]

2. Prove that if \(\text{TV}(\Pi_{00}^i, \Pi_{11}^i) \geq \Omega(1),\) then

\[\text{TV}(\Pi_{01}^i, \Pi_{10}^i) \geq \Omega(1).\]

(Using rectangle property)

Clearly, we are done assuming 1 and 2.
For 1:

Note: \( X_i = Y_i = 0 \Rightarrow \text{DISJ}(X,Y | X_i = Y_i = 0) = 1 \)

w. p. \( \geq \frac{1}{4} \).

\[ \Rightarrow \text{Pr} ( \Pi_0 \text{ accepts given transcript } \Pi_{00}^i ) \geq \frac{1}{5} \]

\( (\frac{1}{4} \text{ - error prob.}) \)

also: \( X_i = Y_i = 1 \Rightarrow \text{DISJ}(X,Y) = 0 \).

Since \( i \) is also random (averaged), we cover \( \Omega(1) \) fraction of the prob. space

\[ \Rightarrow \text{Pr} ( \Pi_0 \text{ accepts given transcript } \Pi_{11}^i ) \leq \frac{1}{6} \]

\[ \Rightarrow \text{Protocol is a good distinguisher (on average)} \]

for \( \Pi_{00} \) and \( \Pi_{11}^i \)

Now we prove 2:

We show Alice Bob

Rectangle Property: Fix \( i \). \( \exists A_0, A_1, B_0, B_1 \)

non-negative functions s.t.

\[ \text{Pr} ( \Pi_{ab}^i = \tau ) = A_a (\tau) \cdot B_b (\tau) \]

Note: Inputs \( X^i, Y^i \) that lead to transcript \( \tau \) form a rectangle \( R_\tau = S_\tau \times T_\tau \).
\[ \Pr(\Pi_{ab}^i = \tau) = \Pr\left( X^i \in S_\tau \land Y^{-i} \in T_\tau \right) \]

\[
= \Pr\left( X^i \in S_\tau \right) \Pr\left( Y^{-i} \in T_\tau \right)
\]

\[ = A_a(\tau) \quad B_b(\tau) \]

Using this "product property" and some calculations, one can show \(2\).

\[ \sum_{\tau} \left| A_0(\tau) B_0(\tau) - A_1(\tau) B_1(\tau) \right| \geq \Omega(1) \]  \(\rho\)

\[ \sum_{\tau} \left| A_0(\tau) B_1(\tau) - A_1(\tau) B_0(\tau) \right| \geq \Omega(1) \]  \(\frac{\beta^2}{2}\)

NB: \(A_0, A_1, B_0, B_1\) are prob. distributions.

Def. \[ \Delta^2_{\text{Hel}} \triangleq 1 - \sum_l \sqrt{\Pr(A = l) \Pr(B = l)} \]

(Hellinger distance)

In fact, this "dot product" is the Bhattacharyya coefficient.

Exercise: (Using Cauchy-Schwarz)

\[ \Delta^2_{\text{Hel}}(A, B) \leq \Delta^2_{TV}(A, B) \leq \sqrt{2} \Delta_{\text{Hel}}(A, B) \]
Lemma: \[ \Delta^2_{\text{Hel}} (\Pi_{00}^i, \Pi_{11}^i) = \Delta^2_{\text{Hel}} (\Pi_{01}^i, \Pi_{10}^i). \]

Clearly, having the Lemma we are done with (2).

**Proof of Lemma:** Suffices to show the dot product of distributions remain the same.

\[ \Pr(\Pi_{00}^i = \tau) \Pr(\Pi_{11}^i = \tau) = A_0(\tau) B_0(\tau) A_1(\tau) B_1(\tau) \]

\[ = A_0(\tau) B_1(\tau) A_1(\tau) B_0(\tau) \]

\[ = \Pr(\Pi_{01}^i = \tau) \Pr(\Pi_{10}^i = \tau). \]

\[ \square \]

* B.F.S. showed that every \( \mu \) that is a product distribution \( (\mu(x,y) = \mu_{\text{Alice}}(x) \cdot \mu_{\text{Bob}}(y)) \),

\[ D^M(\text{DISJ}) = O(\sqrt{n \log n}) \]

So getting a better lower bound requires correlation.
Next time: $R(D_{\text{DISJ}}) = \Omega(n)$.

* This was first proved by Kalyanasundaram-Schnitger '87

* Razborov '90 "simplified" this.

* The proof we'll see is by Bar-Yossef - Jayram-Kumar-Sivakumar 2004

  (Information-theory-based)