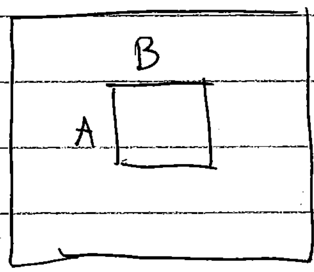
Tue 4/2

$DP(x,y) = \langle x, y \rangle = \sum x_i y_i \pmod 2$

Let's analyze how big a 0-monochromatic rectangle can be.



Let $A \times B$ be such a rectangle.

$\forall a \in A,\ b \in B,\ \langle a, b \rangle = 0$.

$r = \text{rank}(A)$, $s = \text{rank}(B)$.

$$|A| \leq 2^r, \quad |B| \leq 2^s.$$

$A' := \text{Span}(A)$, $\quad B' := \text{span}(B)$.

$|A'| = 2^r$, $\quad |B'| = 2^s$.

* $\forall a' \in A',\ \forall b' \in B',\ \langle a', b' \rangle = 0$.

$\Rightarrow \dim(A') + \dim(B') \leq n \Rightarrow r + s \leq n$.

$\Rightarrow |A||B| \leq 2^{r+s} \leq 2^n$.

* Total # of zeros in the matrix $\geq 2^{2n-1}$.

$\Rightarrow$ # of 0-monochromatic rectangles $\geq 2^{n-1}$.

$\Rightarrow D(DP) \geq n+1$

$\square$

②

Variant: If $\exists$ dist μ on $X \times Y$ s.t. for all monochromatic rectangles R,
$\mu(R) \leq \delta$, then $D(f) \geq \lceil \log_2 \frac{1}{\delta} \rceil$.

(above, μ was uniform).

* Fooling set method is weaker than this variant (since μ can be uniform on the fooling set).

* exercise: Prove $D(DISJ) \geq \Omega(n)$ by bounding size of monochromatic rectangles by 2^n.

Rank Method

Theorem: If $\mathcal{D}(f) \leq c$, then $\text{rank}(M_f) \leq 2^c$.

(rank over field $\mathbb{F}$)

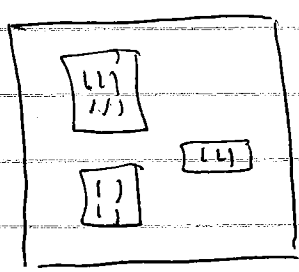
[In particular, to get the maximal rank, take $\mathbb{F} = \mathbb{R}$]

$$M_f \quad \begin{matrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ & \vdots & \end{matrix}$$

Coro: $\mathcal{D}(f) \geq \log_2 \text{rank}_{\mathbb{F}}(M_f)$.

Comment: In fact, $\mathcal{D}(f) \geq \log_2 \left(2 \text{rank}_{\mathbb{F}}(M_f) - 1\right)$

Proof: $\exists\, l \leq 2^c$ monochromatic 1-rectangles that partition all the 1's in M_f. $(R_1, \ldots, R_l)$



For each rectangle R, define a matrix M_R

$$M_i(x,y) \triangleq \begin{cases} 1 & \text{if } (x,y) \in R_i \\ 0 & \text{else} \end{cases}$$

$$M_f = \sum_{i \in [\ell]} M_i \text{, and } rk(M_i) = 1.$$

$\Rightarrow$ using subadditivity of rank, $rk(M) \leq \sum rk(M_i)$
$= \ell \leq 2^c.$

$\square$

Cor: $\mathcal{D}(EQ) \geq n$. (since $M_{EQ} = I$, $rk(M_{EQ}) = 2^n$).

$\boxed{\text{Exercise: Prove } rank(M_{DISJ}) \geq 2^n.}$

Example: Dot product M_{DP}.

$$\begin{array}{cc} \quad y & \\ \begin{bmatrix} \\ \langle x, y \rangle \\ \\ \end{bmatrix} & \leftarrow x \end{array}$$

$rk(M_{DP})$ over $\mathbb{F}_2$ is small, (n).

So we need to work over $\mathbb{R}$. $rank_{\mathbb{R}}(M_{DP}) \geq 2^n - 1$.

$\widetilde{DP} \triangleq (-1)^{\langle x, y \rangle}$ (Hadamard matrix)

$rk_{\mathbb{R}}(\widetilde{DP}) = 2^n$ (orthogonal matrix)

$$M_{\widetilde{DP}} = \underbrace{J}_{\text{all-ones } (rk=1)} - 2 M_{DP}$$

(Lovasz-Saks 88) $\Rightarrow rk(M_{DP}) \geq 2^n - 1$.

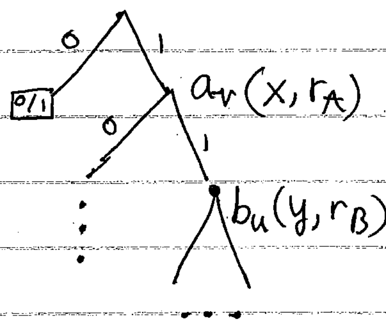
Conjecture: $\mathcal{D}(f) \leq [\log(rk(M_f))]^c$ $\left\{ \text{best known: } O(\text{rank}) \right.$. $\square$
$c \geq 1.631$

④

Randomized Communication Complexity

private coins:

	Alice	Bob
	x	y
private randomness →	r_A	r_B
	$r_A \sim \pi_A$	$r_B \sim \pi_B$



← the only difference! with deterministic

need: whp end up at a correct leaf.

Protocol Π solves the problem with error ε if

$$\forall (x,y), \quad \Pr\left[\Pi(x,y) = f(x,y)\right] \geq 1-\varepsilon.$$

$$\text{Cost of } \Pi \text{ on } (x,y) = \max_{r_A, r_B} (\text{cost of the protocol})$$

$$CC(\Pi) = \max_{x,y} \left(\text{Cost of } \Pi \text{ on } (x,y)\right)$$

Def: $f: X \times Y \to \{0,1\}$, $R_\varepsilon(f) = \min\limits_{\substack{\Pi \\ \Pi \text{ solves } f \\ \text{error } \varepsilon}} \left(CC(\Pi)\right)$.

Other variants: one-sided error, zero error randomized, public randomness.

$$R(f) \triangleq R_{1/3}(f)$$

(you can repeat and take majority to improve the ~~error~~ confidence).

Lemma: $R(EQ) = O(\log n)$.

Idea: Hashing. Alice has a, Bob has b.

~~[illegible]~~ define $A(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}$

$B(z) = b_0 + b_1 z + \cdots + b_{n-1} z^{n-1}$.

$a = b$ iff $A(z) \equiv B(z)$.

Pick prime $p \in [n^2, 2n^2]$.

Alice: Pick $\theta \in \{0, \ldots, p-1\}$ at random.

Send $\underbrace{A(\theta)}_{(\text{mod } p)}$ to Bob, along with p (not needed), θ.

Bob: See if $A(\theta) \overset{p}{\equiv} B(\theta)$. Send the result.

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* If $a=b$, $A=B \Rightarrow$ Bob alway says yes.

* If $a \neq b$, $(A-B) \not\equiv 0$, $\deg(A-B) \leq n-1$

$$\Rightarrow \Pr_{\theta}\left((A-B)(\theta)=0 \right) \leq n/p \leq \frac{1}{n}.$$

$$\Rightarrow \text{error} = O\left(\frac{1}{n}\right).$$

also, # bits sent = $O(\log n)$. $\square$