\[ \text{Total } \# \text{ of 0-monochromatic rectangles} \leq 2n - 1 \]

\[ \Rightarrow \text{dim}(A) + \text{dim}(B) \leq n \Rightarrow r + s \leq n \]

\[ \Rightarrow \text{Span}(A), \text{Span}(B) \]

\[ \Rightarrow \text{Span}(A') = \text{Span}(A) \]

\[ A' = \{ a \}, \quad B' = \{ b \} \]

\[ |A'| = 2, \quad |B'| = 2 \]

\[ |A| = 2r, \quad |B| = 2s \]

\[ \text{rank}(A) \times \text{rank}(B) \]

\[ \text{Let } A \times B \text{ be such a rectangle.} \]

\[ \text{Let's analyze how big a 0-monochromatic rectangle.} \]

\[ \text{DP}(x,y) = \langle x,y \rangle = \sum x_i y_i \text{ (mod 2)} \]

\[ r = \text{rank}(A), \quad s = \text{rank}(B) \]
Variant: If $\exists$ dist $\mu$ on $X \times X$ s.t. for all monochromatic rectangles $R$, $\mu(R) \leq \frac{1}{2^5}$, then $D(f) \geq \left\lceil \log_2 \frac{17}{8} \right\rceil$.

(above, $\mu$ was uniform).

* Fooling set method is weaker than this variant (since $\mu$ can be uniform on the fooling set).

**Exercise:** Prove $D(\text{DISJ}) \geq \Omega(n)$ by bounding size of monochromatic rectangles by $2n$.

**Rank Method**

**Theorem:** If $D(f) \leq c$, then $\text{rank}(M_f) \leq 2^c$ (rank over field $F$).

$M_f$

```
1 0 1
0 0 1
1 1 0
```

[In particular, to get the maximal rank, take $F = \mathbb{R}$]

**Corollary:** $D(f) \geq \log_2 \text{rank}_F(M_f)$

**Comment:** In fact, $D(f) \geq \log_2 \left(2 \text{rank}_F(M_f) - 1\right)$

**Proof:** $\exists \leq 2^c$ monochromatic 1-rectangles that partition all the 1's in $M_f$ ($R_1, \ldots, R_l$)

For each rectangle $R_i$, define a matrix $M_R$

$$M_i(x, y) = \begin{cases} 1 & \text{if } (x, y) \in R_i \\ 0 & \text{else} \end{cases}$$
\[ M_f = \sum_{i \in [d]} M_i, \text{ and } \text{rk}(M_i) = 1. \]

\[ \Rightarrow \text{using subadditivity of rank, } \text{rk}(M) \leq \sum \text{rk}(M_i) = 1 \leq 2^c. \]

**Cor:** \( \mathcal{D}(\text{EQ}) \geq n. \) (since \( M_{eQ} = I \),

(Exercise: Prove \( \text{rank}(M_{\text{DISJ}}) \geq 2^n \). \( \text{rk}(M_{\text{EQ}}) = 2^n \)).

**Example:** Dot product \( M_{DP}. \)

\[ \text{rk}(M_{DP}) \text{ over } \mathbb{F}_2 \text{ is small.} \]

So we need to work over \( \mathbb{R} \). \( \text{rank}_{\mathbb{R}}(M_{DP}) \geq 2^n - 1. \)

\( \widehat{\text{DP}} \triangleq (-1)^{x \cdot y} \) (Hadamard matrix)

\[ \text{rk}_{\mathbb{R}}(\widehat{\text{DP}}) = 2^n \] (orthogonal matrix)

\[ M_{\text{DP}} = \mathbf{J} - 2M_{DP} \]

Lovasz-Saks \( \Rightarrow \text{rk}(M_{DP}) \geq 2^n - 1. \)

Conjecture: \( \mathcal{D}(f) \leq \left\lfloor \log(\text{rk}(M_f)) \right\rfloor. \) \( \text{best known: } O(\text{rank}). \)
Randomized Communication Complexity

private coins:

\[ \text{Alice} \quad \text{Bob} \]

private randomness \[ r_A \quad r_B \]

\[ r_A \sim \pi_A \quad r_B \sim \pi_B \]

\[ \Pr \{ A_r(x, r_A) = \text{the only difference!} \} \]

with deterministic

\[ b_u(y, r_B) \]

need: Whp end up at a correct leaf.

Protocol $\Pi$ solves the problem with error $\epsilon$ if

\[ \forall (x, y), \quad \Pr \left[ \Pi(x, y) = f(x, y) \right] \geq 1 - \epsilon. \]

Cost of $\Pi$ on $(x, y) = \max \left( \text{cost of the protocol} \right)_{r_A, r_B}

\[ \text{CC} (\Pi) = \max_{x, y} \left( \text{cost of } \Pi \text{ on } (x, y) \right) \]

Def: $f : X \times Y \to \{0, 1\}, \quad R_e (f) = \min_{\Pi} \left( \text{CC}(\Pi) \right)$

$\Pi$ solves $f$ with error $\epsilon$. 
Other variants: one-sided error, zero error randomized, public randomness.

\[ R(f) \triangleq R_{1/3}^2(f) \]
(you can repeat and take majority to improve the confidence).

**Lemma:** \( R(EQ) = O(\log n) \).

**Idea:** Hashing. Alice has \( a \), Bob has \( b \).

Define:
- \( A(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} \)
- \( B(z) = b_0 + b_1 z + \cdots + b_{n-1} z^{n-1} \).

\( a = b \) iff \( A(z) \equiv B(z) \).

Pick prime \( p \in [n^2, 2n^2] \).

Alice: Pick \( \Theta \in \{0, \ldots, p-1\} \) at random:
- \( (\Theta - \Theta') \mod p \)
- (not needed)

Send \( A(\Theta) \) to Bob, along with \( p, \Theta \).

Bob: See if \( A(\Theta) \equiv B(\Theta) \). Send the result.
If $a = b$, $A = B \Rightarrow$ Bob always says yes.

If $a \neq b$, $(A-B) \neq 0$, $\deg(A-B) \leq n-1$,

$\Rightarrow \Pr_{\theta}( (A-B)(\theta) = 0 ) \leq \frac{n}{p} \leq \frac{1}{n}$

$\Rightarrow$ error $= O\left(\frac{1}{n}\right)$.

Also, # bits sent $= O(\log n)$.