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* ## Bregman's Theorem

$A = (a_{ij}) \in \{0,1\}^{n \times n}$

$$\mathrm{Perm}(A) := \sum_{\substack{\sigma:[n]\to[n] \\ \text{bijection}}} \prod_{i\in[n]} a_{i\,\sigma(i)}$$

A can be thought of as the adjacency matrix of a bipartite graph $G=(L,R,E)$, where $L=R=[n]$, and $(i,j)\in E$ iff $a_{ij}=1$.

A perfect matching in G is a ~~bijection~~ set ~~$\sigma:[n]\to[n]$ s.t. $\forall i$, $(i,\sigma(i$~~
$M \subseteq E$ s.t. every vertex is incident to exactly one edge in M.

Equivalently, a perfect matching is a bijection $\sigma:[n]\to[n]$ s.t. $\forall i\in[n]$, $(i,\sigma(i))\in E$.

$\Rightarrow$ $\mathrm{Perm}(A) = \#$ perfect matchings in G.

<u>Def:</u> $d_i :=$ deg. of the left vertex i.

Obviously, $\mathrm{Perm}(A) \le \prod_{i\in[n]} d_i$.

<u>Bregman's Theorem:</u> $\mathrm{Perm}(A) \le \prod_{i\in[n]} (d_i!)^{1/d_i}$.

* Entropy proof by Jaikumar Radhakrishnan:

Let $\sigma : [n] \to [n]$ be a uniform random perfect matching.

Then Bregman's Theorem is equivalent to:

$$H(\sigma) \leq \sum_{i \in [n]} \frac{\log(d_i!)}{d_i} \quad (*)$$

Of course: $H(\sigma) = H(\sigma(1)) + H(\sigma(2) \mid \sigma(1)) + \cdots +$

$$H(\sigma(n) \mid \sigma(1), \ldots, \sigma(n-1))$$

$$\leq \sum_{i \in [n]} H(\sigma(i)) \leq \sum_{i \in [n]} \log d_i ,$$

which proves $\text{Perm}(A) \leq 2^{\sum \log d_i} = \prod_{i \in [n]} d_i$,

the trivial bound!

Trick: Introduce more randomness.

Let $\tau : [n] \to [n]$ be ~~uniform random~~ any given bijection. Then,

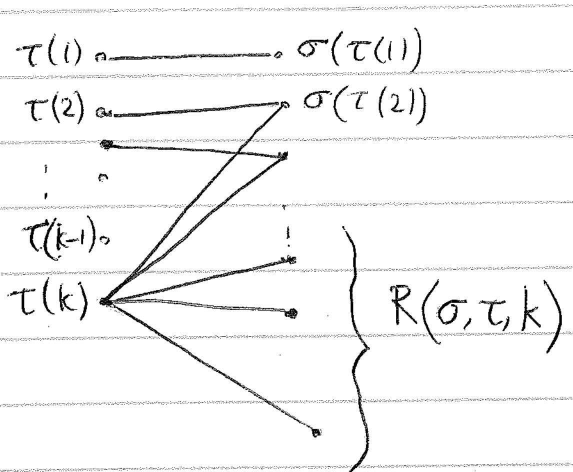
$$H(\sigma) = H(\sigma \cdot \tau), \text{ where}$$

$\sigma\tau(i) \triangleq \sigma(\tau(i))$.

Suppose we have revealed

$$\sigma\tau(1), \sigma\tau(2), \ldots, \sigma\tau(k-1),$$

How much information remains in $\sigma\tau(k)$?



[at most log of support size of $R(\sigma, \tau, k)$ under uniform σ and conditioned on observations]

Lemma: For random vars X, Y, suppose support(X) can be partitioned into sets $A_1, \ldots, A_d$ s.t. $\forall i$, $|\mathrm{Supp}(Y \mid X \in A_i)| \leq i$.

Then, $H(Y|X) \leq \sum_{i \in [d]} \Pr(X \in A_i) \log i$

$$H(\sigma)=\sum_{k\in[n]} H\left(\sigma\tau(k) \,\middle|\, \sigma\tau(1),\dots,\sigma\tau(k-1)\right)$$

$$= \mathop{\mathbb{E}}_{\tau} \sum_{k} \quad ''$$

$$= \sum_{k} \mathop{\mathbb{E}}_{\tau} \quad '' \quad (i=\tau(k))$$

$$= \sum_{i\in[n]} \mathop{\mathbb{E}}_{\tau} H\left(\sigma(i) \,\middle|\, \sigma\tau(1),\dots,\sigma\tau\left(\tau^{-1}(i)-1\right)\right)$$

lemma

$$\le \sum_{i\in[n]} \mathop{\mathbb{E}}_{\tau} \sum_{j\in[d_i]} \Pr_{\sigma}\left(|\underbrace{R(\sigma,\tau,\tau^{-1}(i))}_{R}|=j\right)\log j$$

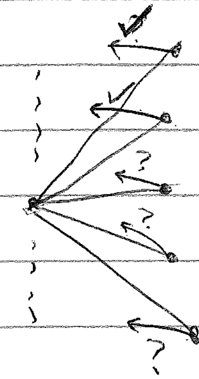
$$= \sum_{i,j} \mathop{\mathbb{E}}_{\tau} \quad ''$$

$$= \sum_{i,j} \Pr_{\sigma,\tau}\left(|R|=j\right)\log j \qquad (**)$$

Claim: $(\forall j)(\forall\sigma)\ \Pr_{\tau}\left(|R|=j\right)=\frac{1}{d_i}$.

~~random~~

$\sigma\tau(1),\dots,\sigma\tau(k-1)$ revealed $\quad \tau(k)=i$



τ random $\Rightarrow$ k random.

$|R|$ = # of unrevealed neighbors of i.

R only depends on the ordering of neighbors of i under τ.

⑤

$|R| = d_i - l + 1$, $l :=$ order of $\sigma(i)$ ~~under~~ among $\Gamma(i)$ under τ = uniform random.

$\Rightarrow$ $|R|$ = uniform random

$\Rightarrow$ $\Pr(|R| = j) = \frac{1}{d_i}$ $(\forall \sigma, j)$

$\square$

By Claim $\Rightarrow$ $(**) = \sum_{i,j} \frac{1}{d_i} \log j$

$i \in [n]$, $j \in [d_i]$

$= \sum_{i \in [n]} \frac{1}{d_i} \log(d_i!) \Rightarrow (*)$. $\square$

$\left(H(\sigma) \le \sum_{i \in [n]} \frac{1}{d_i} \log(d_i!) \right)$

* Shearer's Lemma:

$X = (X_1, \ldots, X_n)$. S a r.v. on subsets of $[n]$,

s.t. $\forall i,\ \Pr(i \in S) \geq \mu$. Then,

$$\mathop{\mathbb{E}}_{S} H(X_S) \geq \mu \cdot H(X).$$

e.g., S uniform on singletons $\Rightarrow \mu = \frac{1}{n} \Rightarrow \mathop{\mathbb{E}}_{S} H(X_S) \geq \frac{H(X)}{n}$

$$\Leftrightarrow \quad \sum H(X_i) \geq H(X).$$

(sub additivity of entropy).

* Proof (due to Jaikumar Radhakrishnan):

Fix $T = \{i_1, \ldots, i_k\}$, $i_1 < i_2 < \cdots < i_k$.

$$H(X_T) = H(X_{i_1}) + H(X_{i_2} \mid X_{i_1}) + \cdots + H(X_{i_k} \mid X_{i_1}, \ldots, X_{i_{k-1}})$$

[(more conditioning)]

$$\geq H(X_{i_1} \mid X_{<i_1}) + H(X_{i_2} \mid X_{<i_2}) + \cdots + H(X_{i_k} \mid X_{<i_k}).$$

Now, $\mathop{\mathbb{E}}_{S}(H(X_S)) \geq \mathop{\mathbb{E}}_{S} \sum_{i \in S} H(X_i \mid X_{<i})$

$$= \sum_{i \in [n]} \Pr(i \in S) \cdot H(X_i \mid X_{<i})$$

$$\geq \mu \sum_{i \in [n]} H(X_i \mid X_{<i}) = \mu\, H(X). \qquad \square$$

★ Application: Counting embeddings of Graphs.

Let T be a fixed graph with m edges.

What is the max. copies of T that can be ~~embedded~~ packed in a graph with ℓ edges?

Def: Embedding of T in G:

$\sigma : V(T) \to V(G)$

$$\{i,j\} \in E(T) \Leftrightarrow \{\sigma(i), \sigma(j)\} \in E(G).$$

Def: $N(T,\ell)$ = max # of embedding of T in a graph with ℓ edges.

~~Theorem (Alon, Freidgut, Kahn):~~

$$\cancel{N(T,\ell) \le (2\ell)}$$

★ Fractional indep. sets:

$\psi : V(T) \to [0,1]$ s.t. $\forall \underset{\{u,v\}}{e} \in E(T)$,

$$\psi(u) + \psi(v) \le 1.$$

$$|\psi| := \sum_{v \in V} \psi(v).$$

$\psi^* := \psi$ that maximizes $|\psi|$.

* Fractional cover:

$\phi : E(G) \to [0,1]$, $\forall v \in V(T)$,

$$\sum_{\substack{e \in E(T) \\ v \in e}} \phi(e) \geq 1.$$

$$|\phi| = \sum_{e \in E(T)} \phi(e).$$

* $\phi^* := \phi$ that minimizes $|\phi|$

* By LP-duality, $|\phi^*| = |\psi^*|$.

* Example: Complete graph with k vertices,

ψ^* assigns $\frac{1}{2}$ to every vertex &

ϕ^* " $\frac{1}{k-1}$ to every edge

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Theorem: (Alon, Freidgut–Kahn)

$$\left(\frac{\ell}{m}\right)^{|\psi^*|} \le N(T, \ell) \le (2\ell)^{|\psi^*|}.$$

Proof: (upper bound)

Let σ be a uniform random embedding of $T \to G$, where G is any graph with ℓ edges.

All we need to bound is $H(\sigma)$.

$\star$ Let $S \in E(T)$ be random with

$$\Pr(S = e) \triangleq \frac{\phi^*(e)}{|\phi^*|}.$$

Apply Shearer's Lemma on r.v's

$\{\sigma(v) : v \in V(T)\}$ with respect to S.

$$\text{N.B: } \forall v,\ \Pr(v \in S) \ge \sum_{v \in e} \frac{\phi^*(e)}{|\phi^*|} \overset{\text{[def. cover]}}{\ge} \frac{1}{|\phi^*|}.$$

$$\overset{\text{Shearer}}{\Longrightarrow} \mathop{\mathbb{E}}_{S} H(\sigma_S) \ge \frac{H(\sigma)}{|\phi^*|}$$

On the other hand, $\forall e \in E(T)$, $H(\underbrace{\sigma_e}_{\text{[an edge of } G\text{]}}) \le \log(2\ell)$

$$\Rightarrow \mathop{\mathbb{E}}_{S} H(\sigma_S) \le \log(2\ell)$$

(10)

$$\Rightarrow N(T, \ell) \leq (2\ell)^{|\phi^*|} = (2\ell)^{|\psi^*|}.$$

<u>Lower bound:</u> (assuming $(\ell/m)^{|\psi^*|}$ is integer)

Construct G from T as follows:

1) replace every $v \in T$ with an indep. set of $\left(\frac{\ell}{m}\right)^{\psi^*(v)}$ vertices.

2) Connect every vertex in indep. set for u to every vertex in indep. set for v iff $\{u, v\} \in E(T)$.

* Every edge of T contributes to

$$\left(\frac{\ell}{m}\right)^{\psi^*(u)+\psi^*(v)} \text{ edges in } G \leq \left(\frac{\ell}{m}\right).$$

$\Rightarrow$ G has $\leq \ell$ edges.

* An embedding can choose any of the vertices in each indep. set $\Rightarrow$

$$\# \text{embeddings} \geq \left(\frac{\ell}{m}\right)^{\sum_v \psi^*(v)} = \left(\frac{\ell}{m}\right)^{|\psi^*|}.$$

$\square$

Corollary 1) $N(K_k, l) \leq (2l)^{k/2}$

Corollary 2) Any graph with l edges contains ~~has~~ $\leq \frac{1}{6}(2l)^{3/2}$ triangles.