①

Thu 02/21 Polar Codes

$$u_0^{N-1} \longrightarrow \boxed{G_N} \longrightarrow x_0^{N-1} = G_N u_0^{N-1} \longrightarrow \boxed{W} \longrightarrow y_0 \ldots y_{N-1}$$

$$\xrightarrow[x]{0/1} \boxed{W} \longrightarrow y \qquad \text{Cap} = 1 - H(X|Y).$$

$$H(u_i \mid y_0^{N-1}, u_0^{i-1}) \approx \begin{cases} 0 & \text{for } i < RN \\ 1 & \text{"} \quad i > RN \end{cases}$$

but $H(x_i \mid y_0^{N-1}, x_0^{i-1}) = H(X_i \mid Y_i) = H(X|Y)$.

Insights in Polar Coding:

① Reversing the Connection, i.e., getting codes from such invertible matrix.
(by "freezing" input bits U_i when $H(u_i| \ldots) \approx 1$.

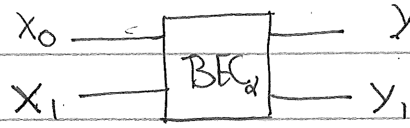
② A recursive construction of such "polarizing matrices".

* $\exists$ set F of "frozen" indices $F \subseteq \{0, \ldots, N-1\}$,
$|F| \approx H(X|Y)N$, s.t.

$$H(U_i \mid Y_0^{N-1}, U_0^{i-1}) \approx 1 \text{ if } i \in F$$
$$\approx 0 \text{ if } i \in \bar{F}.$$

Let us see how to get ~~x~~ u from X.

Think about BEC_α, $n=2$.



$$Y_0 = \begin{cases} X_0 & \text{w.p. } 1-\alpha \\ ? & \quad " \quad \alpha. \end{cases}$$

$* \quad H(X_i \mid Y_i) = \alpha.$

Goal: Find $(X_0, X_1) \rightarrow (U_0, U_1) = G_2 \cdot \underline{X}$

s.t. U_0 more uncertain (given Y_0, Y_1) and

U_1 less " (given Y_0, Y_1, U_0).

$$G_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{cases} U_0 = X_0 \oplus X_1 \\ U_1 = X_1 \end{cases}$$

$* \quad H(U_0 \mid Y_0 Y_1) = Pr($ ~~both~~ either Y_0 ~~&~~ or Y_1 are ~~not~~ $= ?)$

$$= 1 - (1-\alpha)^2 = 2\alpha - \alpha^2 \geq \alpha$$
$$(= \text{ iff } \alpha \in \{0,1\})$$

$* \quad H(U_1 \mid Y_0 Y_1 U_0) = Pr($both Y_0 and Y_1 are erased$)$

$$= \alpha^2 \leq \alpha$$
$$(= \text{ iff } \alpha \in \{0,1\})$$

③

Note:

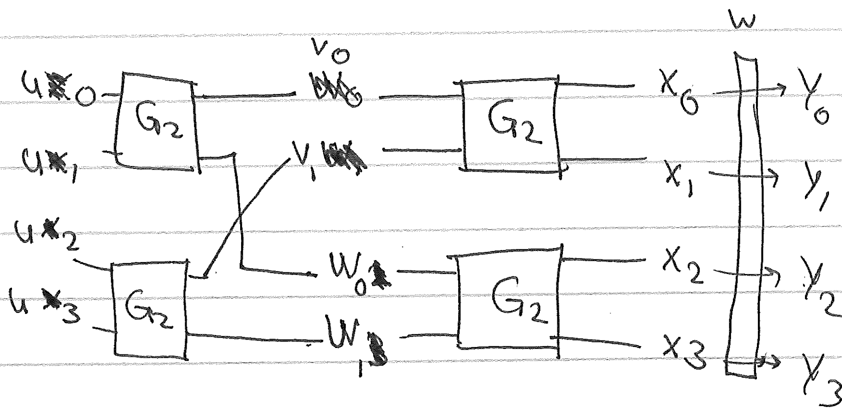
* $G_2 = G_2^{-1}$.

* $\Rightarrow$ If $\alpha \in (0,1)$, one channel is worse, other is better.

$(\alpha, \alpha) \longrightarrow (2\alpha - \alpha^2, \ \alpha^2)$

* Idea: iterate this.

Second step:



$$\begin{cases} u_0 = v_0 \oplus w_0 \\ u_1 = w_0 \\ u_2 = v_1 \oplus w_1 \\ u_3 = w_1 \end{cases}$$

* $H(u_0 \mid y_0^3) = Pr(u_0 \text{ can't be determined})$

$= Pr(\text{either } v_0 \text{ or } w_0 \text{ can't be determined})$

$= 1 - (1 - (2\alpha - \alpha^2))^2 = 1 - (1-\alpha)^4$.

* $H(u_1 \mid y_0^3, u_0) = Pr(v_0 \ \& \ w_0 \text{ are both unknown})$

$= (2\alpha - \alpha^2)^2$.

④

$H(u_1 | y_0^3 u_0)$

$\overset{\|}{*}\ H(V_0 | y_0 y_1) \cdot H(W_0 | y_2 y_3)$

$*\ H(u_2 | y_0^3 u_0^1) = \Pr\Big(\text{either } V_1 \text{ or } W_1 \text{ is } \underline{\text{unknown}} \,\Big|\, y_0^3, V_0, W_0\Big)$

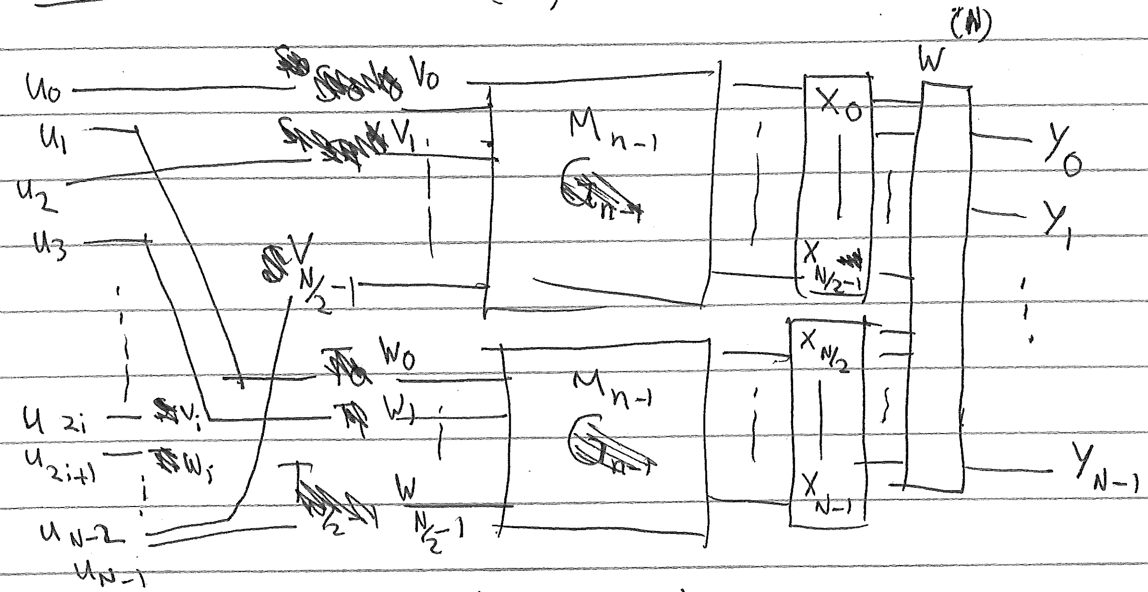
$= \Pr\Big(\big(V_1 = ? \,\big|\, y_0^1 \,\&\, V_0\big) \text{ OR } \big(W_1 = ? \,\big|\, y_2 y_3 W_0\big)\Big)$

$= 1 - (1-\alpha^2)^2 \quad \square$

$*\ H(u_3 | y_0^3 u_0^2) = \Pr\Big(\big(V_1 = ? \,\big|\, y_0^1 \,\&\, V_0\big) \text{ and } \big(W_1 = ? \,\big|\, y_2 y_3 W_0\big)\Big)$

$= \alpha^4.$

<u>Recursive Construction</u> (M_n): $N = 2^n$. , $M_1 := G_2$.



$$\begin{cases} u_{2i} = V_i \oplus W_i \\ u_{2i+1} = W_i \end{cases}$$

(3)

* M_n (ignoring reshuffling) $\begin{cases} M_{n-1} X_0^{N/2-1} \oplus M_{n-1} X_{N/2}^{N-1} \\ M_{n-1} \cdot X_{N/2}^{N-1} \end{cases}$

$$= \overbrace{\begin{pmatrix} M_{n-1} & M_{n-1} \\ 0 & M_{n-1} \end{pmatrix}}^{M_n} \begin{pmatrix} X_0^{N/2-1} \\ X_{N/2}^{N-1} \end{pmatrix}$$

$$\Rightarrow M_n = G_2 \otimes M_{n-1}.$$

* Exercise: $M_n = G_2^{\otimes n} \cdot B_n$, where

$B_n :=$ bit-reversal permutation matrix.

$$(\underbrace{B_n Z}_{\in \mathbb{R}^{2^n}})_{b_1 b_2 \cdots b_n} = Z_{b_n b_{n-1} \cdots b_1}.$$

⑥

$$\begin{cases} u_{2i} = \cancel{\ \ } \ v_i \oplus w_i \\ u_{2i+1} = w_i \end{cases}$$

$\star$ $\alpha_n(2i) := H\left(u_{2i} \mid y_0^{N-1} \, u_0^{2i-1}\right)$

$= \Pr\Big((v_i$ is undetermined given y_0^{N-1} & $v_0^{i-1}, w_0^{i-1})$

~~$(u_0 - u_{2i-1}) \Leftrightarrow (v_0 - v_{i-1})$~~

~~$(w_i - w_{i-1})$~~

OR $(w_i$ is undetermined given y_0^{N-1} & $v_0^{i-1}, w_0^{i-1})\Big)$

$= \Pr\Big((v_i = ?$ given $y_0^{N/2-1}$ & $v_0^{i-1})$ OR

$(w_i = ?$ given $y_{N/2}^{N-1}$ & $w_0^{i-1})\Big)$

$$= 1 - \left(1 - \alpha_{n-1}(i)\right)^2 = 2\alpha_{n-1}(i) - \alpha_{n-1}^2(i).$$

$\star$ $\alpha_n(2i+1) = H\left(u_{2i+1} \mid y_0^{N-1} \, u_0^{2i}\right)$

$= \Pr\Big((v_i = ? \mid y_0^{N/2-1}$ & $v_0^{i-1})$ AND

$(w_i = ? \mid y_{N/2}^{N-1}$ & $w_0^{i-1})\Big) = \alpha_{n-1}^2(i).$