1. (a) For a (simple, undirected) graph $G = (V, E)$, define its “square” $G^2$ as follows. The vertices of $G^2$ consist of ordered pairs of vertices of $G$, i.e., the vertex set is $V \times V$. Two pairs $(u_1, u_2)$ and $(v_1, v_2)$ are adjacent in $G^2$ if any of the following hold:

i. Both $(u_1, v_1) \in E$ and $(u_2, v_2) \in E$.

ii. $u_1 = v_1$, $(u_2, v_2) \in E$.

iii. $u_2 = v_2$, $(u_1, v_1) \in E$.

Is the following statement true or false: For every graph $G$, the size of the largest clique in $G^2$ is equal to the square of the size of the largest clique is $G$. Prove your answer.

(b) Suppose that there is a polynomial time algorithm $A$ that on any input graph $G$, finds a clique of size at least 1% of the largest clique in $G$. Show how can one use $A$ as a subroutine and design a polynomial time algorithm $B$ that finds a clique of size at least 99% of the largest clique in any input graph.

Hint: Use the previous part.

2. In this exercise, you will see a general form of the reduction from 3-SAT to CLIQUE that works with any constraint satisfaction problem in the place of 3-SAT.

Let $P : \{0, 1\}^k \rightarrow \{0, 1\}$ be a predicate and CSP($P$) be the associate constraint satisfaction problem. An instance $I$ of CSP($P$) consists of a set of variables $V$ and a collection $C$ of $m$ constraints (for some positive integer $m$ and indexed by $j \in \{1, 2, \ldots, m\}$) of the form $P(\tau^{(j)}_1, \tau^{(j)}_2, \ldots, \tau^{(j)}_k)$ for some tuple $\tau^{(j)} \in V^k$ of $k$ variables from $V$. For any assignment $\sigma : V \rightarrow \{0, 1\}$, we can count the number of constraints, call it $N(I, \sigma)$, of $I$ that are satisfied by the values assigned by $\sigma$ to its variables. Let OPT($I$) be the maximum over all assignments $\sigma : V \rightarrow \{0, 1\}$ of $N(I, \sigma)$. 
We now map an instance $\mathcal{I}$ of CSP($P$) to a graph $H = (W, E)$ as follows. Suppose $\mathcal{I}$ has $m$ constraints. The vertex set $W$ will consist of $m$ disjoint parts $W_1, W_2, \ldots, W_m$, one corresponding to each of the $m$ constraints of $\mathcal{I}$.

The vertices in $W_j, 1 \leq j \leq m$, will correspond to assignments to the $k$-tuple $\tau^{(j)}$ of variables that satisfy the $j$'th constraint of $\mathcal{I}$. (So all $W_j$'s will have equal size, equal to $|P^{-1}(1)|$, the number of assignments in $\{0, 1\}^k$ that satisfy $P$.)

There will be no edges amongst vertices in the same $W_j$, i.e., each $W_j$ is an independent set. A vertex $a \in W_j$ and $b \in W_{j'}$ for two parts $j \neq j'$ are adjacent in $H$ if the assignments corresponding to $a$ and $b$ are consistent on the variables that belong to both the tuples $\tau^{(j)}$ and $\tau^{(j')}$. (In particular, if the tuples $\tau^{(j)}$ and $\tau^{(j')}$ are disjoint, then all edges between $W_j$ and $W_{j'}$ are present in $H$.)

Prove that the size of the largest clique in $H$ equals $\text{OPT}(\mathcal{I})$. 